Title: Scattering matrices for noncommutative instantons

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

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Abstract:

The assignment of scattering data to monopoles is a classical construction in gauge theory useful to understand the geometry of monopole moduli spaces; from the modern perspective, it is of interest because monopole scattering matrices are quasiclassical limits of R-matrices of shifted Yangians. I will describe an analogous construction of quasiclassical R-matrices of affine Yangians using moduli spaces of noncommutative instantons; this can be interpreted via defect intersections in a semiclassical limit of the 5d Chern-Simons theory. In the simplest possible case, corresponding R-matrices reproduce Miura operators, but otherwise they seem to be new. If time permits I will describe a (dual) construction of the quantization of these R-matrices using geometry of quasimap moduli spaces.

Instanton S-matrices

Motivation: Do an R-matrix formalism for Coulomb branches of general quiver gauge thy's

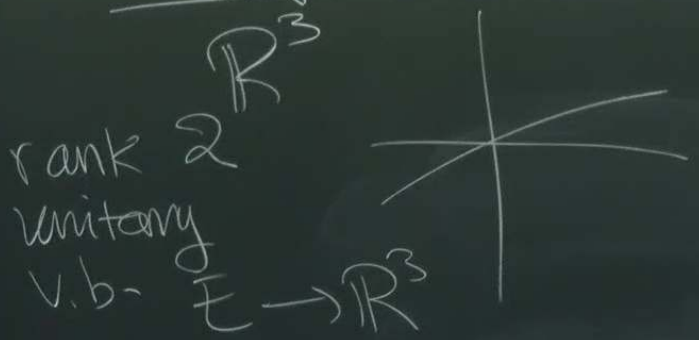
These will be R-matrices for $Y(g_{\mathcal{M}_C}, Q)$ shifted

- For g_Q = ADE type, corresponding R-matrices are scattering matrices for ADE monopoles/instanton defect of gen. affine Gr s/e.

Today: Describe analog for $\widehat{gl_n}$.

Monopole S-matrix

$g = gl_2$.



rank 2
unitary
v.b. $E \to \mathbb{R}^3$

A – unitary connection

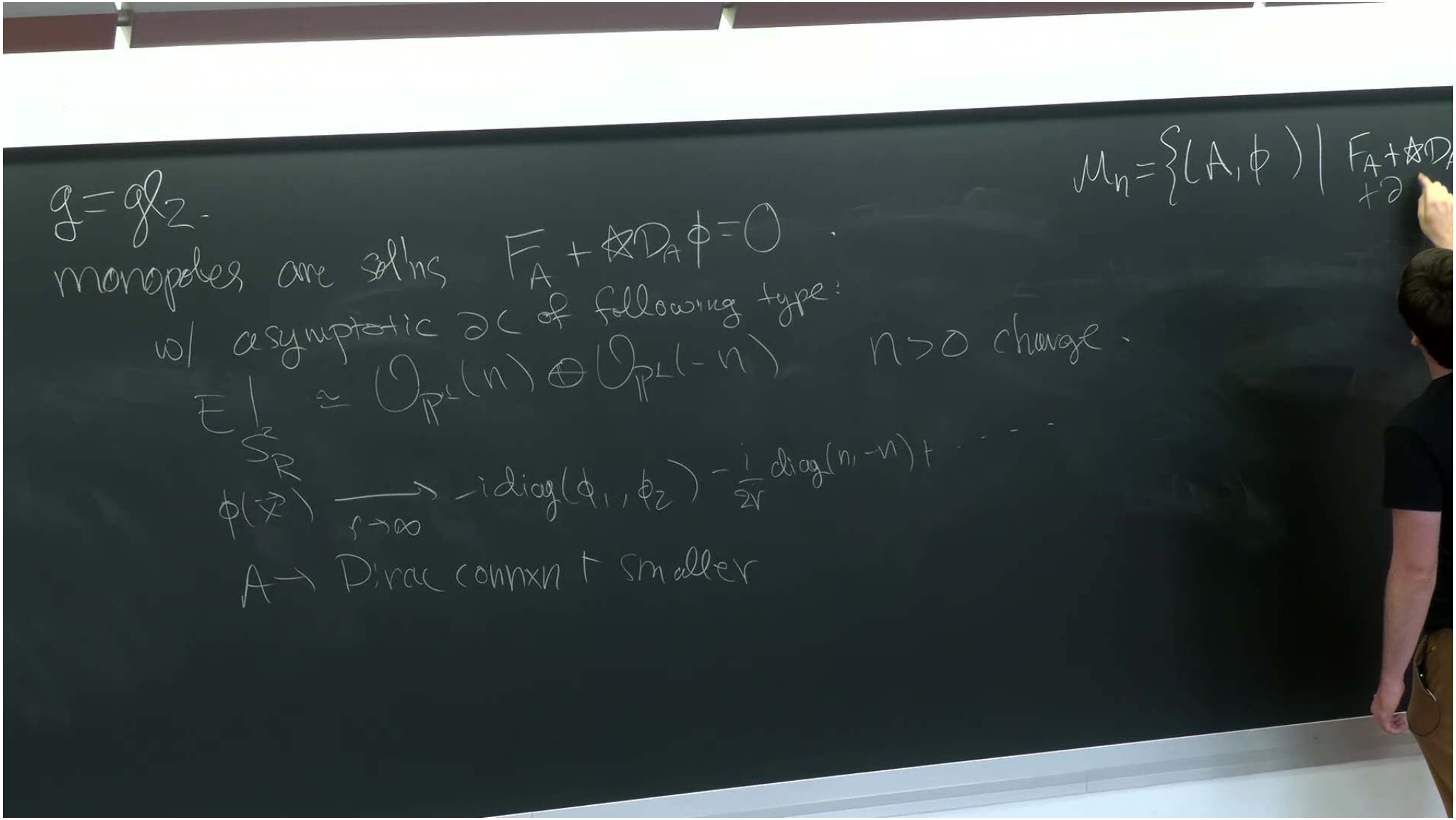
$\phi(\vec{x})$ – $\mathbb{R}$-valued $\mathrm{End}(E)$.

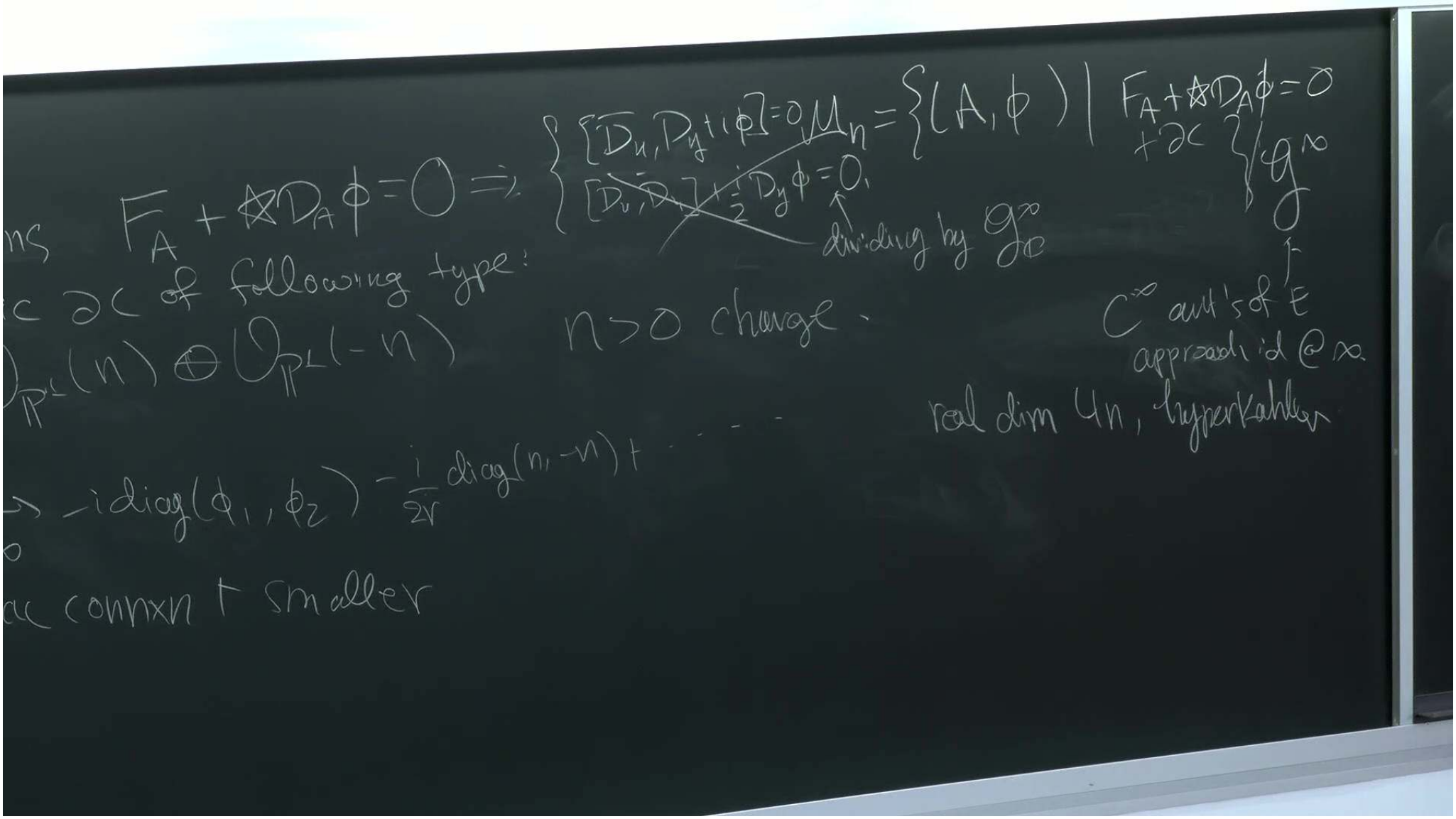
monopoles are solns $F_A + *D_A\phi = 0$.

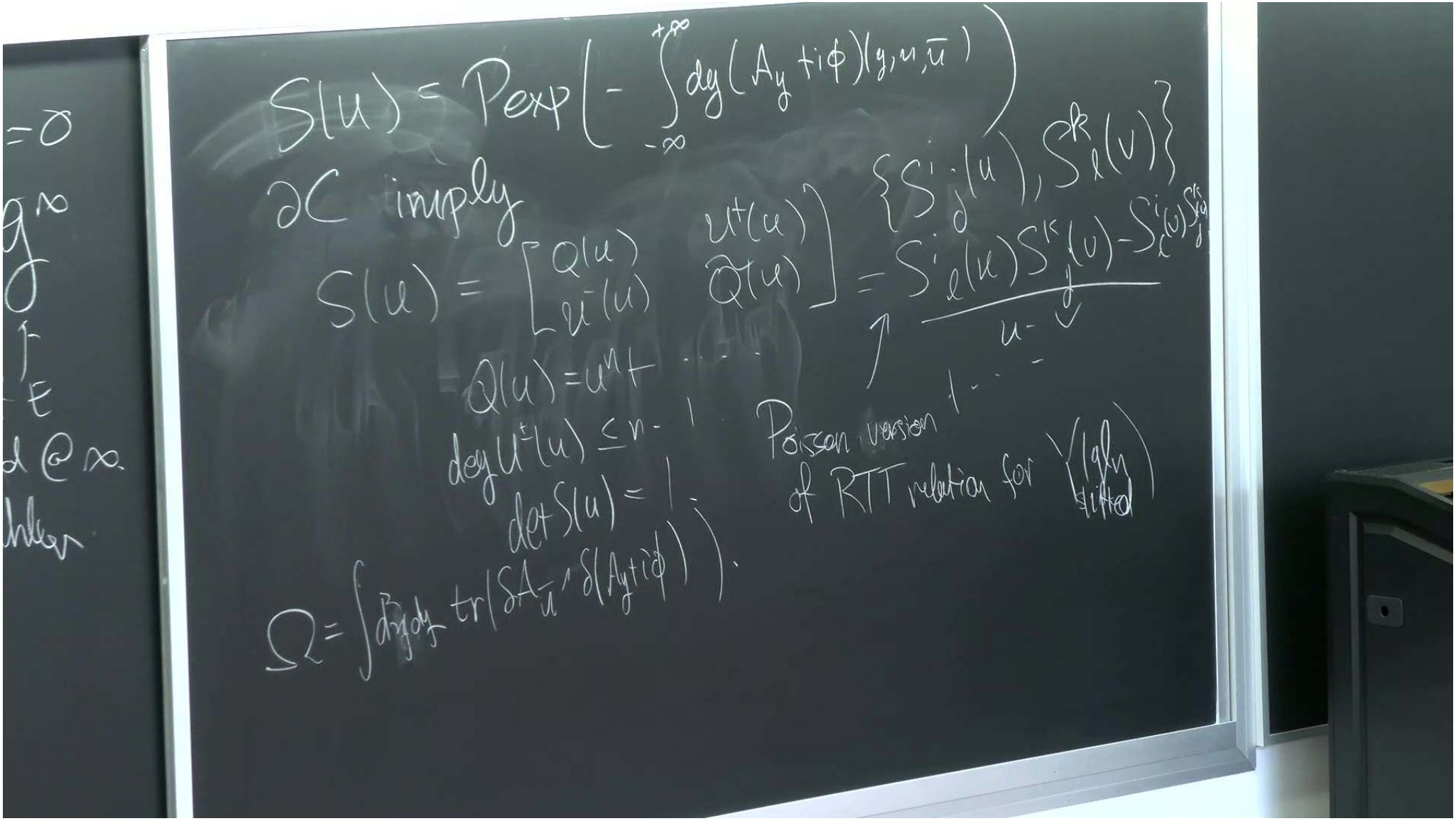
w/ asymptotic ac… following type:

$E|_{S^2_R} \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$

$\phi(\vec{x})$ —


$g = gl_2$.
monopoles are solns $F_A + *D_A\phi = 0$.
w/ asymptotic ∂_A of following type:
$E|_{S^2_R} \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ $n > 0$ charge.
$\phi(\vec{x}) \xrightarrow[r\to\infty]{} -i\,\mathrm{diag}(\phi_1, \phi_2) - \frac{i}{2r}\mathrm{diag}(n, -n) + \cdots$
$A \to$ Dirac connxn + smaller
$\mathcal{M}_n = \{(A, \phi) \mid F_A + *D$
$+ \partial$


$F_A + \star D_A\phi = 0 \Rightarrow$
$[D_u, D_y + i\phi] = 0$
$[D_u, D_x] + \frac{i}{2} D_y\phi = 0$
dividing by g_∞
$\mathcal{M}_n = \{(A,\phi) \mid F_A + \star D_A\phi = 0$ $+ \partial c$ $\}/g_\infty$
of following type:
$n > 0$ charge.
C^∞ aut's of E approach id @ ∞.
real dim $4n$, hyperkahler
$-i\,\mathrm{diag}(\phi_1, \phi_2) - \frac{i}{2r}\,\mathrm{diag}(n, -n) + \cdots$
connxn + smaller


$S(u) = P\exp\left(-\int_{-\infty}^{+\infty} dy\,(A_y + i\phi)(y,u,\bar{u})\right)$
∂C imply
$S(u) = \begin{bmatrix} Q(u) & u^+(u) \\ u^-(u) & \tilde{Q}(u) \end{bmatrix}$
$\{S^i_j(u), S^{kl}(v)\} = \dfrac{S^i_l(u)S^{kl}_j(v) - S^i_l(v)S^{kl}_j(u)}{u-v}$
$Q(u) = u^n + \ldots$
$\deg u^\pm(u) \le n-1$
$\det S(u) = 1$
Poisson version of RTT relation for Y(glN) shifted
$\Omega = \int_{\mathbb{R}} dy\,\mathrm{tr}(\delta A_{\bar{u}}^{-1}\delta(A_y + i\phi))$.

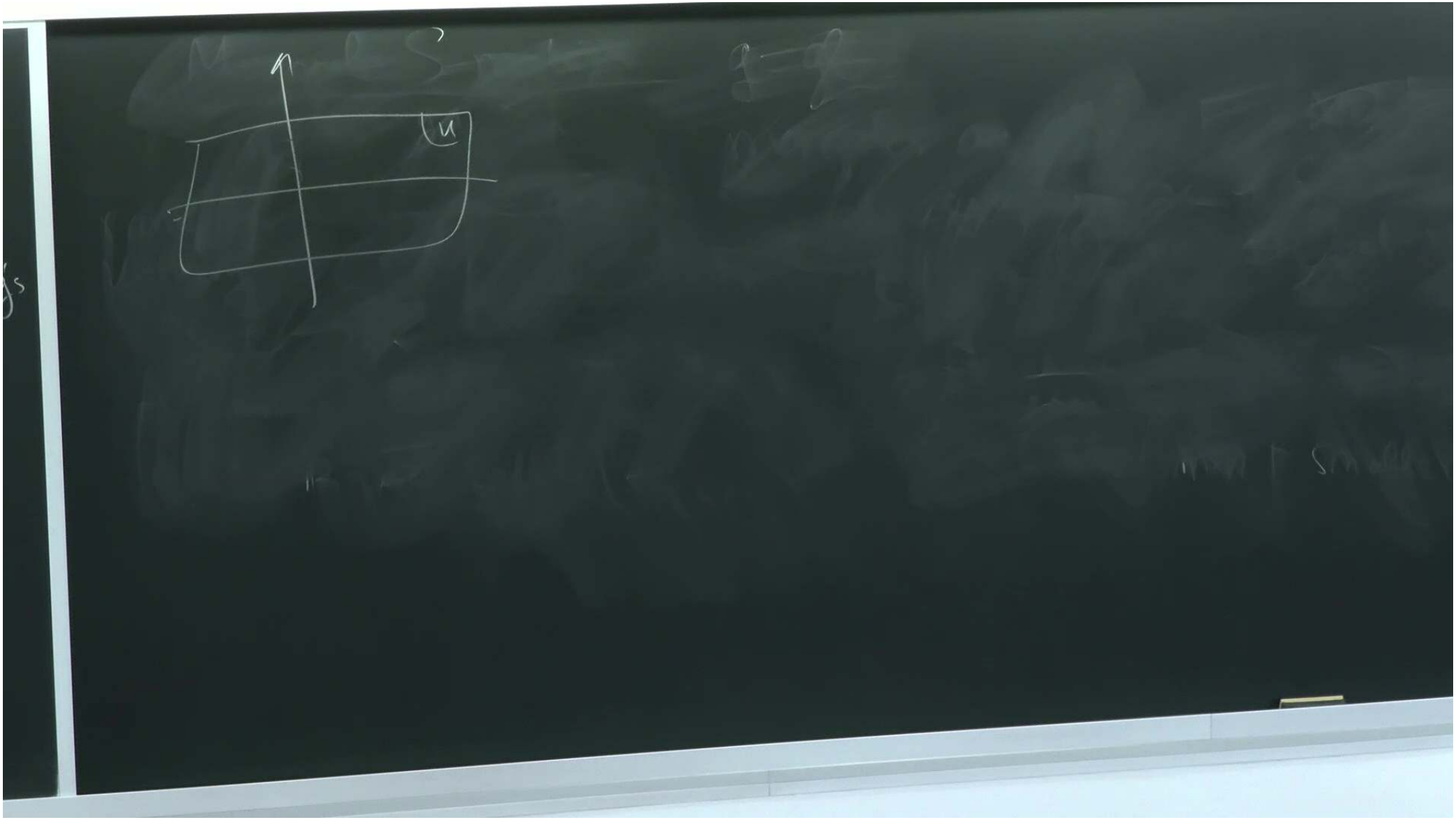
Instanton S-matrices

Motivation: Do an R-matrix formalism for Coulomb branches of general quiver gauge theories

These will be R-matrices for $Y(g_{\mathcal{Q}}, \mathcal{Q})$ (shifted)

- For $g_{\mathcal{Q}}$ = ADE type, corresponding R-matrices are scattering matrices for ADE monopoles/instanton defects of gen. affine Gr s/e.

Today: Describe analog for $\widehat{gl_n}$.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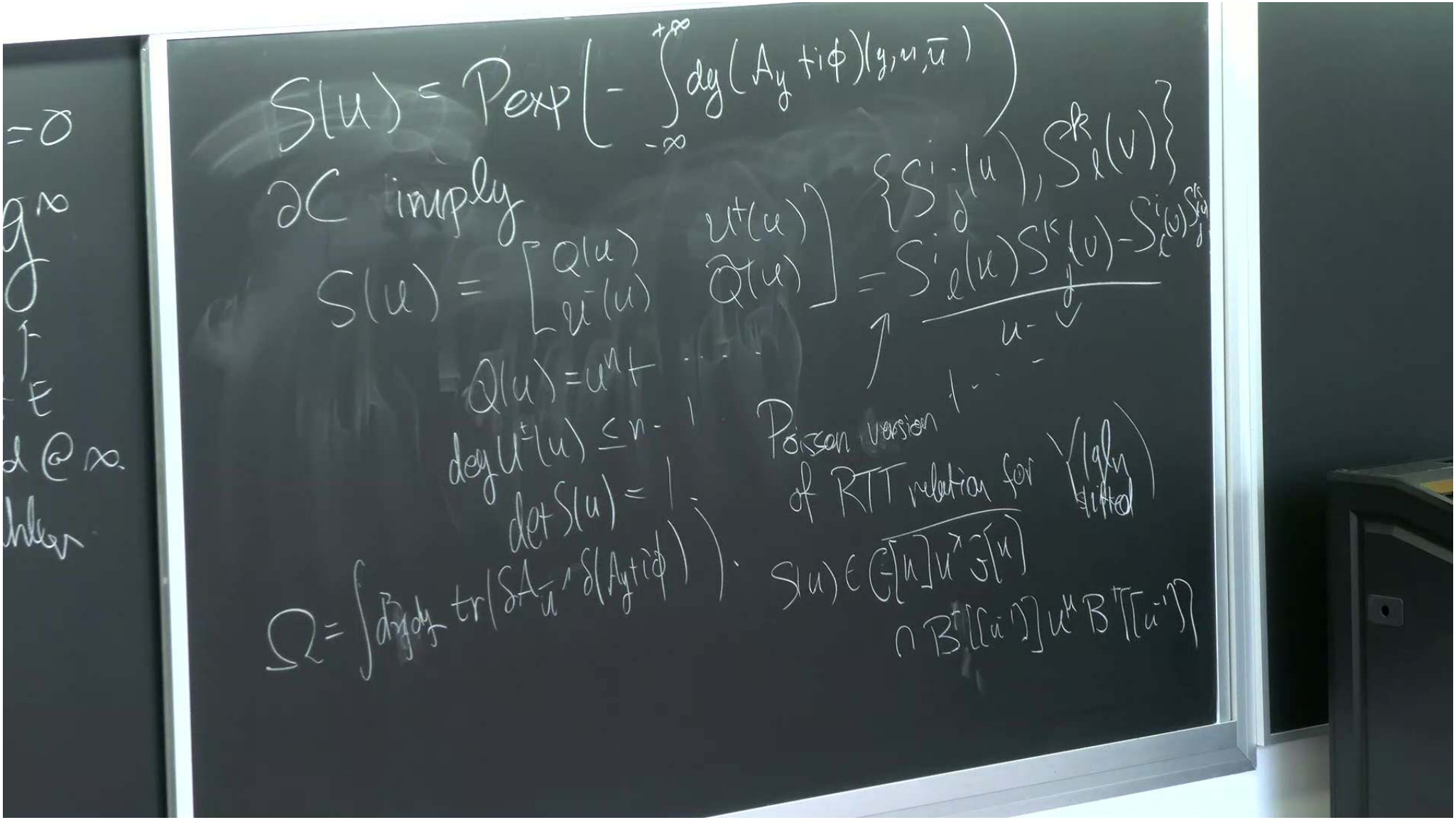
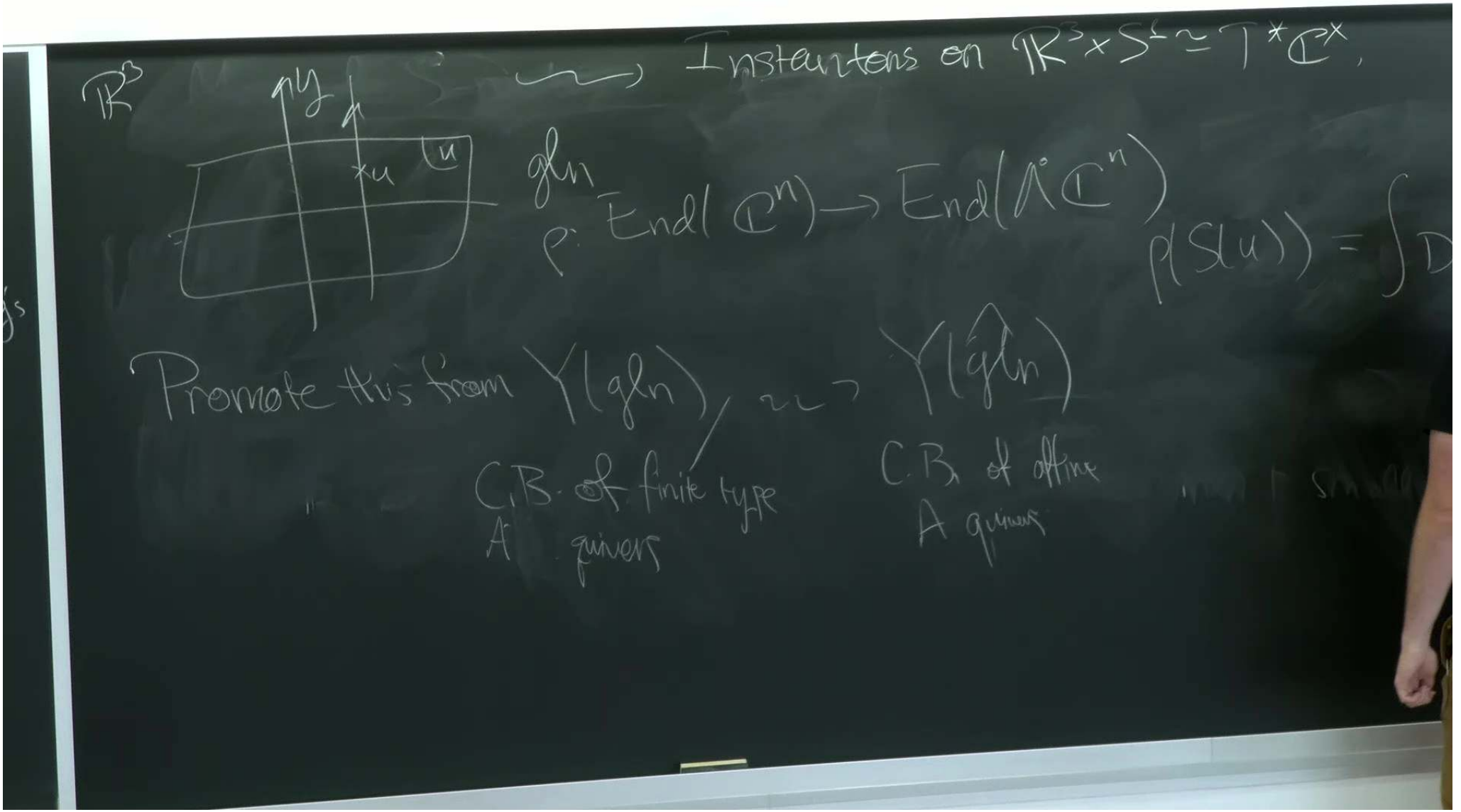
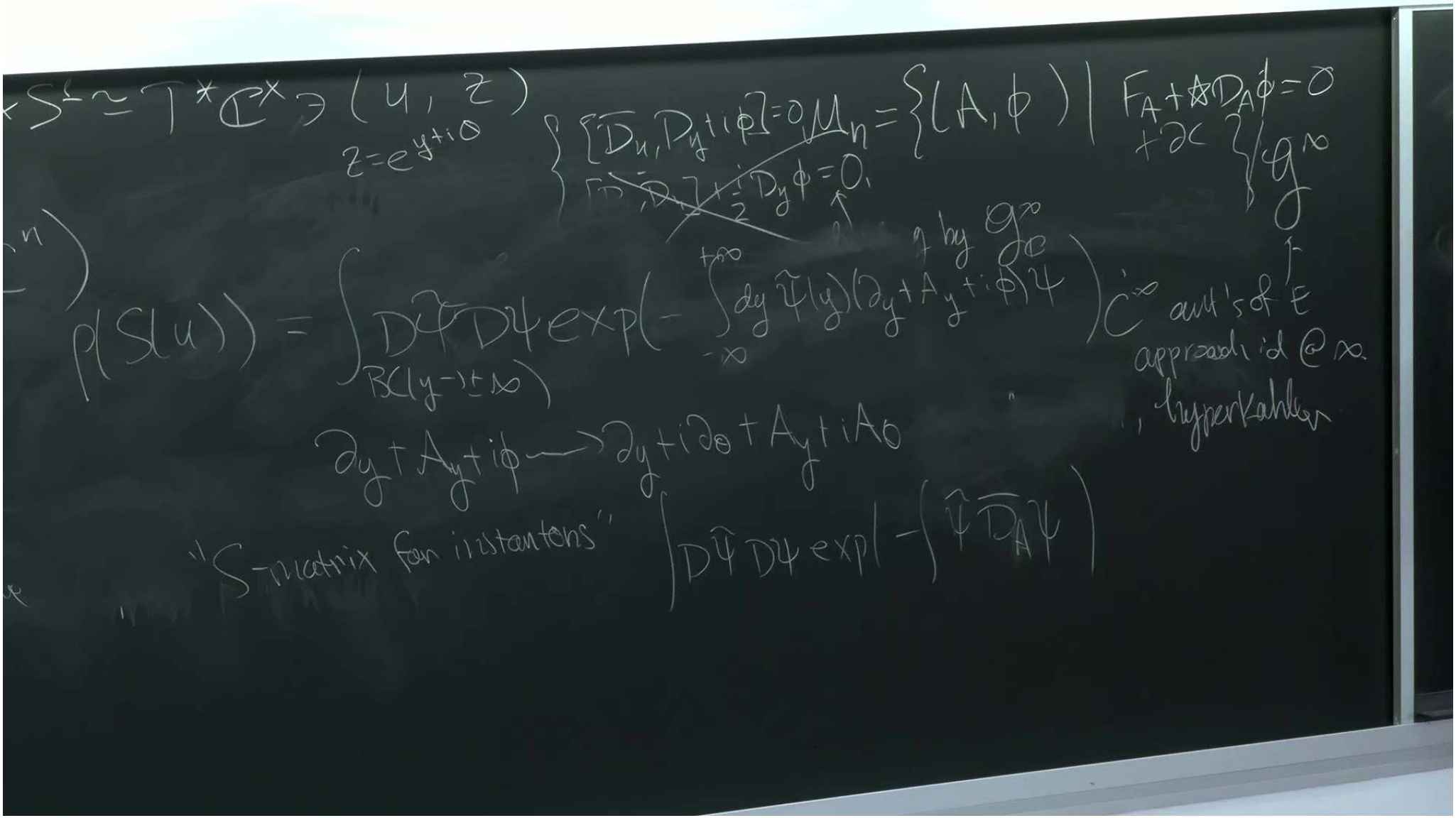
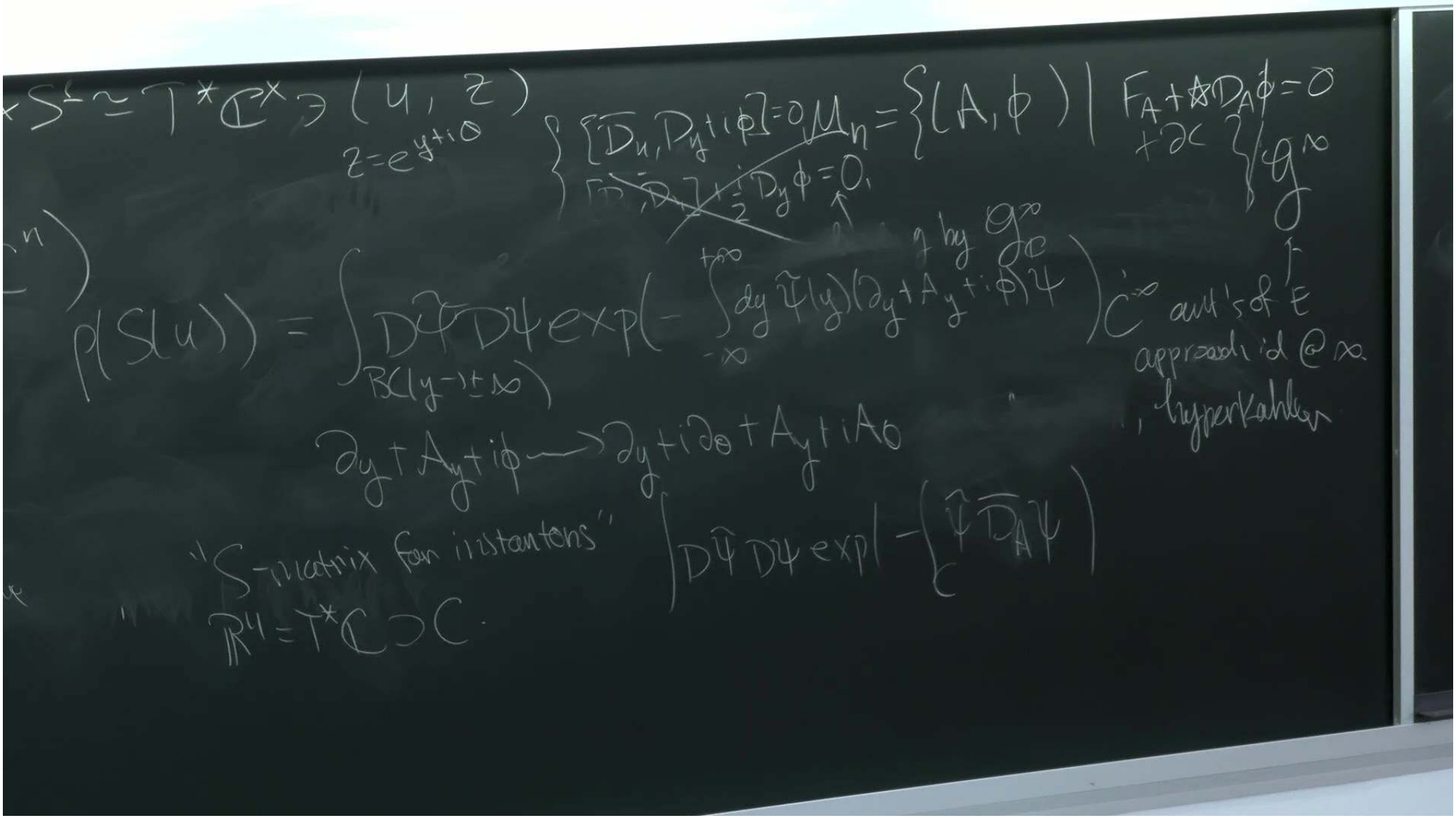
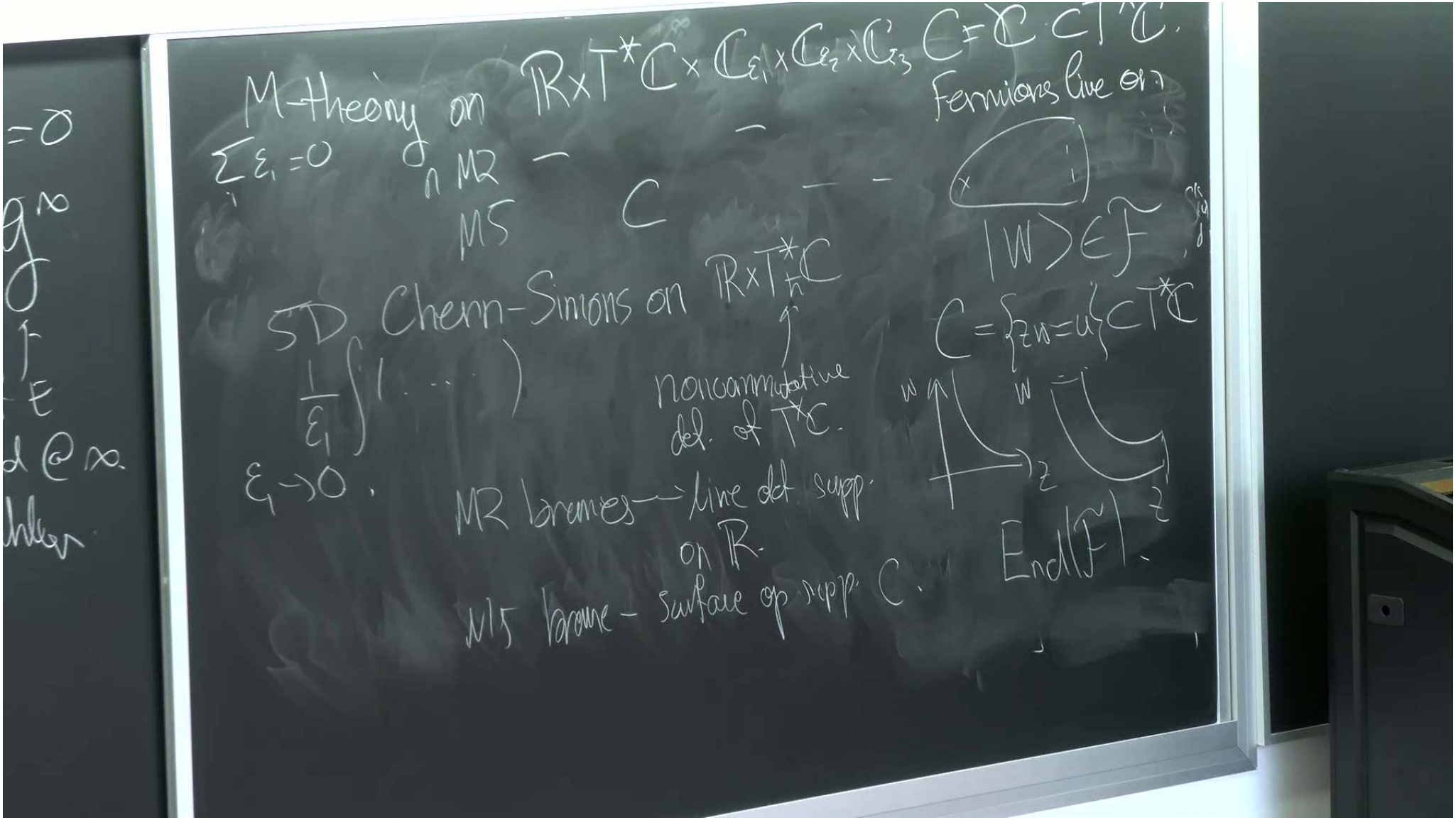
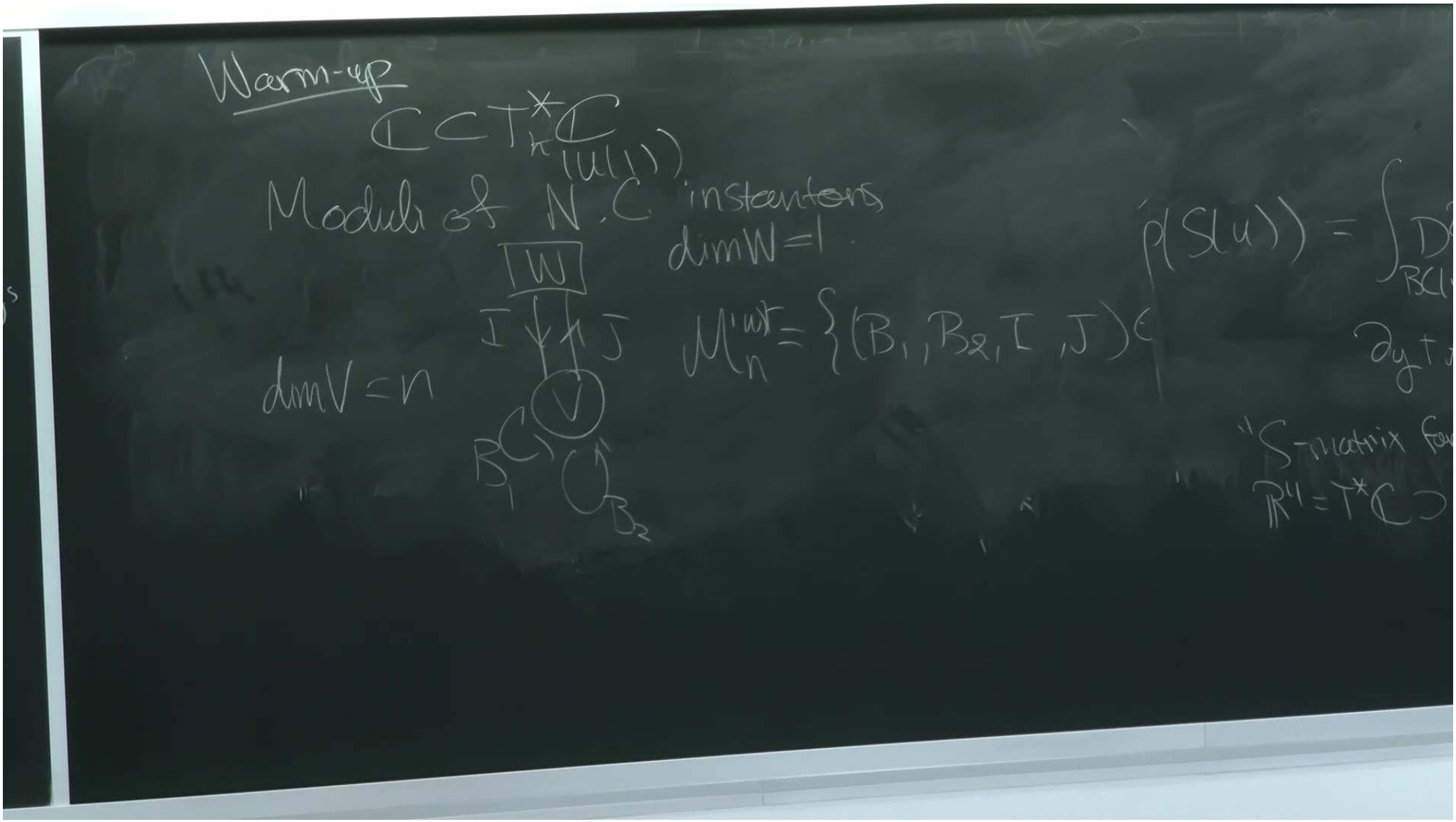

$S(u) = P\exp\left(-\int_{-\infty}^{+\infty} dy\,(A_y + i\phi)(y,u,\bar{u})\right)$
∂C imply
$S(u) = \begin{bmatrix} Q(u) & u^+(u) \\ u^-(u) & \bar{Q}(u) \end{bmatrix}$
$\{S^i_j(u), S^k_l(v)\} = S^i_l(u)S^k_j(v) - S^i_l(v)S^k_j(u)$
$\frac{}{u-v}$
$Q(u) = u^n + \ldots$
$\deg u^\pm(u) \le n-1$
$\det S(u) = 1$
Poisson version of RTT relation for Yangian
$\Omega = \int d^2x\,dy\,\mathrm{tr}\left(\delta A_{\bar{u}}\,\delta(A_y + i\phi)\right)$
$S(u) \in G[[u]]u^{\lambda}G[[u]]$
$\cap\, B^+[[u^{-1}]]u^{\mu}B^+[[u^{-1}]]$

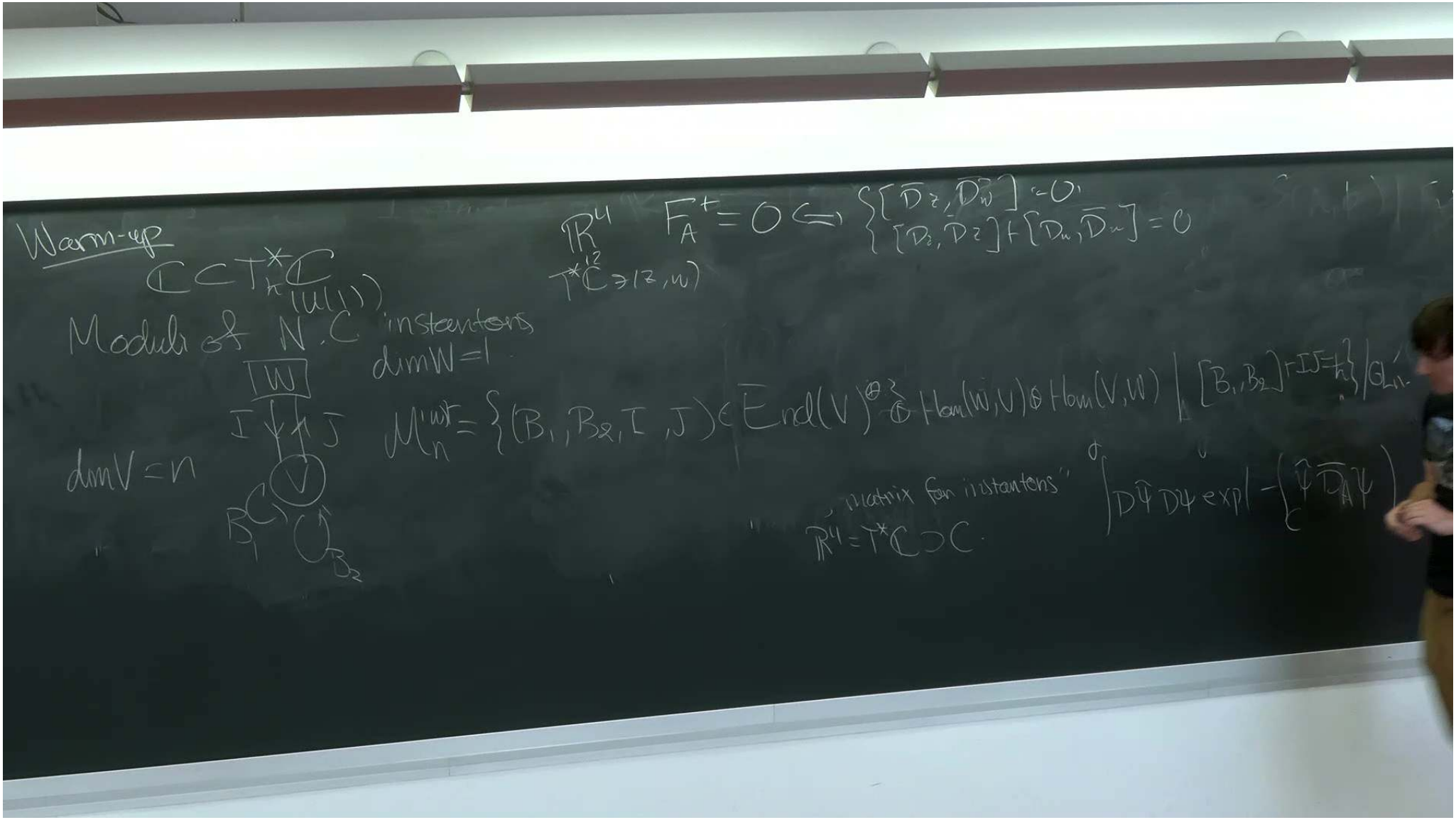

$\mathbb{R}^3$
xy
x_u
u
$\mathfrak{gl}_n$
$\rho: \mathrm{End}(\mathbb{C}^n) \to \mathrm{End}(\Lambda^\bullet \mathbb{C}^n)$
Instantons on $\mathbb{R}^3 \times S^1 \simeq T^*\mathbb{C}^\times$.
$\rho(S(u)) = \int_D$
Promote this from $Y(\mathfrak{gl}_n)$
$Y(\widehat{\mathfrak{gl}_n})$
C.B. of finite type A quivers
C.B. of affine A quivers


$T^*\mathbb{C}^\times \ni (u, z)$
$z = e^{y+i\theta}$
$\{[D_u, D_y + i\phi] = 0, \quad [D_u, D_{\bar{y}}] + \frac{i}{2} D_y \phi = 0\}$
$M_n = \{(A, \phi) \mid F_A + \star D_A \phi = 0$
$+ \partial c$
$\rho(S(u)) = \int_{B(y^{-1}(\pm\infty))} D\bar{\psi} D\psi \exp\left(-\int_{-\infty}^{+\infty} dy\, \bar{\psi}(y)(\partial_y + A_y + i\phi)\psi\right)$
$\partial_y + A_y + i\phi \longrightarrow \partial_y + i\partial_\theta + A_y + iA_\theta$
"S-matrix for instantons"
$\int D\bar{\psi} D\psi \exp\left(-\int \bar{\psi} \bar{D}_A \psi\right)$
C^∞ aut's of E
approach id @ ∞
hyperkähler


$T^*\mathbb{C}^\times \ni (u, z)$
$z = e^{y+i\theta}$
$\mathcal{M}_n = \{(A, \phi) \mid F_A + *D_A\phi = 0$
$[D_u, D_y + i\phi] = 0$
$[D_z, D_{\bar{z}}] + \frac{i}{2} D_y\phi = 0$
$+\partial c$
g by g_∞
g_∞
$\mathcal{P}(S(u)) = \int_{B(y^{-1}\pm\infty)} D\bar{\psi} D\psi \exp\left(-\int_{-\infty}^{+\infty} dy\, \bar{\psi}(y)(\partial_y + A_y + i\phi)\psi\right)$
C^∞ aut's of E
approach id @ ∞.
, hyperkahler
$\partial_y + A_y + i\phi \longrightarrow \partial_y + i\partial_\theta + A_y + iA_\theta$
"S-matrix for instantons"
$\int D\bar{\psi} D\psi \exp\left(-\int_C \bar{\psi} D_A \psi\right)$
$\mathbb{R}^4 = T^*\mathbb{C} \supset C$.


M-theory on $\mathbb{R}\times T^*\mathbb{C}\times\mathbb{C}_{\varepsilon_1}\times\mathbb{C}_{\varepsilon_2}\times\mathbb{C}_{\varepsilon_3}$ $C=\mathbb{C}\subset T^*\mathbb{C}$.
$\sum\varepsilon_i=0$
M2 —
M5 C — —
Fermions live on
$|W\rangle\in\mathcal{F}$
SD Chern-Simons on $\mathbb{R}\times T^*_{\hbar}\mathbb{C}$
$\frac{1}{\varepsilon_1}\int(\dots)$
$\varepsilon_1\to 0$.
noncommutative deformation of $T^*\mathbb{C}$.
$C=\{z_n=u\}\subset T^*\mathbb{C}$
M2 branes — line defect supp. on $\mathbb{R}$.
M5 brane — surface operator supp. C.
$\mathrm{End}(\mathcal{F})$.


Warm-up
$\mathbb{C} \subset T^*_{\mathbb{C}}\mathbb{C}$
$U(1)$
Moduli of N instantons
dim W = 1
W
I
J
dim V = n
V
B_1
B_2
$\mathcal{M}_n^{\text{inst}} = \{(B_1, B_2, I, J) \in$
$\rho(SU(4)) = \int_{BG}$
$\partial_y t$
"S-matrix for
$\mathbb{R}^4 = T^*\mathbb{C}$


Warm-up
$\mathbb{C} \subset T^*_{\mathbb{C}}\mathbb{C}$ (U(1))
Moduli of N.C instantons
dim W = 1
W
I
J
dim V = n
V
B_1
B_2
$\mathbb{R}^4$ $F_A^+ = 0 \hookleftarrow \{[\bar{D}_z, \bar{D}_w] = 0,\ [D_z, D_z] + [D_w, \bar{D}_w] = 0$
$T^*\mathbb{C}^2 \ni (z, w)$
$\mathcal{M}_n^{inst} = \{(B_1, B_2, I, J) \in \overline{End(V)}^{\oplus 3} \oplus Hom(W,V) \oplus Hom(V,W) \mid [B_1, B_2] + IJ = \hbar\}/GL_n$
"matrix fan instantons"
$\mathbb{R}^4 = T^*\mathbb{C} \supset C$
$\int D\bar{\psi}\, D\psi \exp\left(-\int_C \bar{\psi} \bar{D}_A \psi\right)$

$\mathbb{R}^4 \quad F_A^+ = 0 \Leftarrow \begin{cases} [\bar{D}_z, \bar{D}_w] = 0, \\ [D_z, D_z] + [D_w, \bar{D}_w] = 0 \end{cases}$

$T^*\mathbb{C}_z(z,w)$

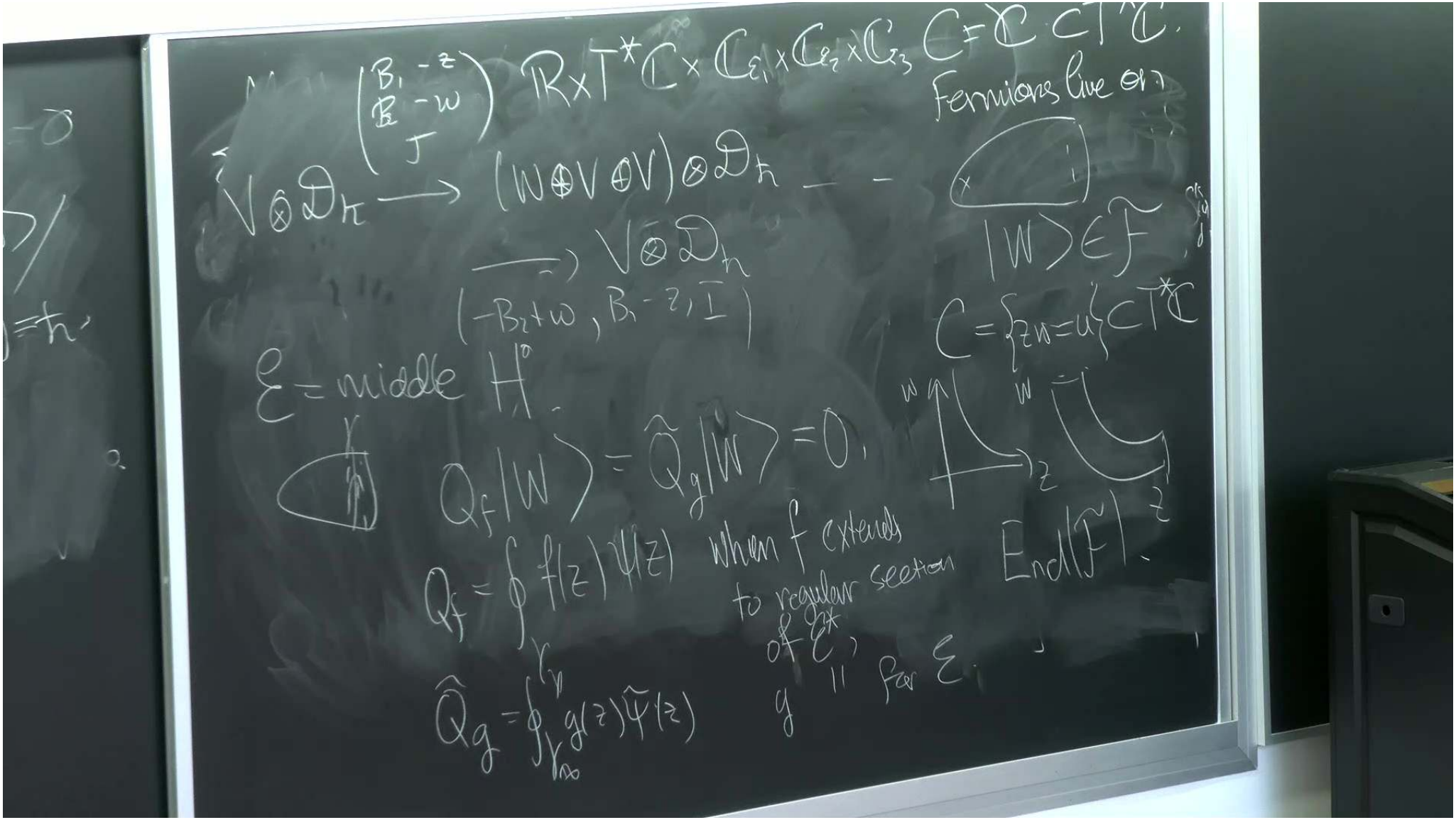
$\mathcal{E}$ = instanton bundle will be a right module for $\mathcal{O}_\hbar = \mathbb{C}\langle z, w\rangle / [w, z] = \hbar$.

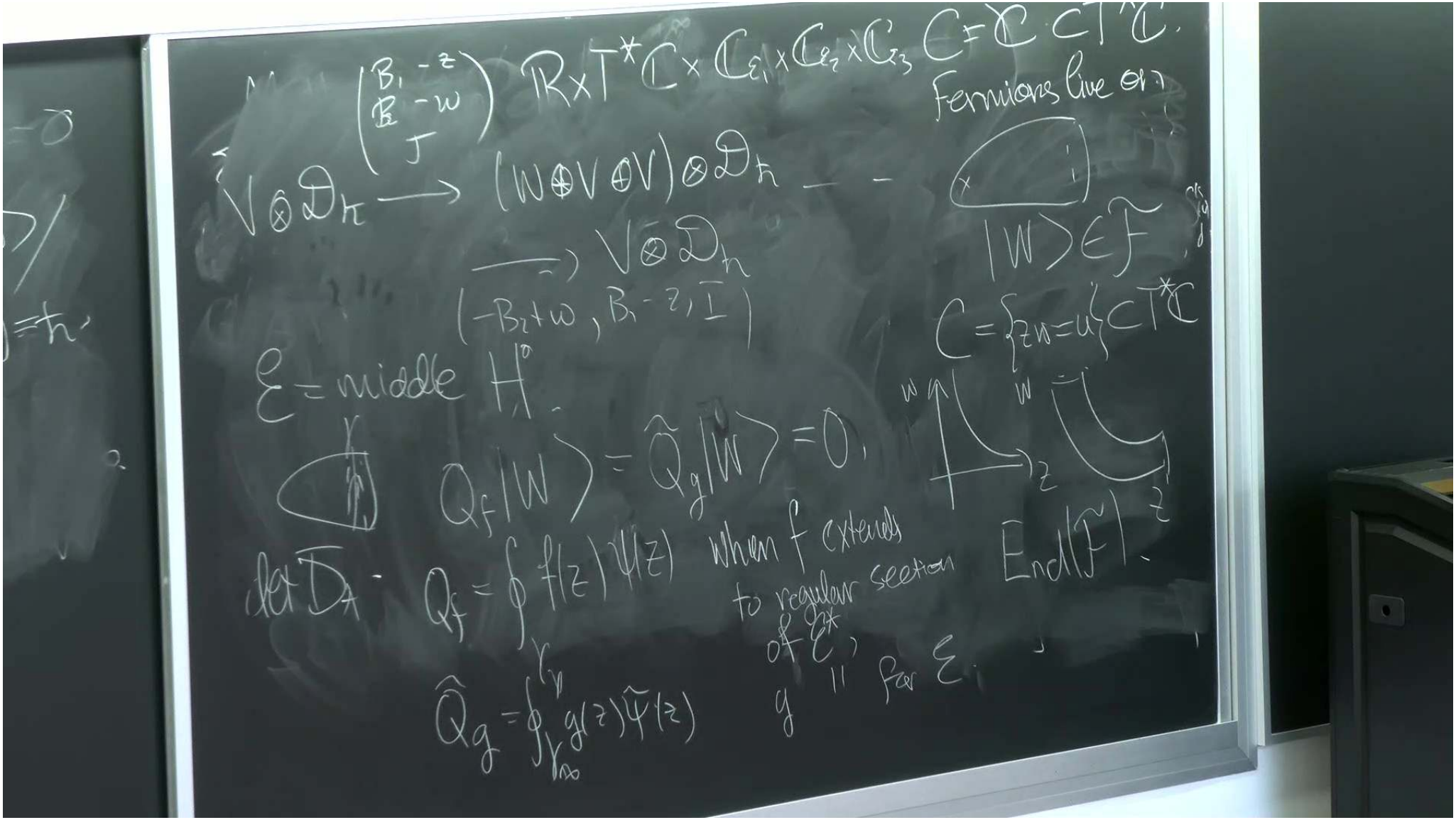
$B_2, I, J) \in \overline{\mathrm{End}(V)^{\oplus 2}} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W) \Big| [B_1, B_2] + IJ = \hbar \Big\} / GL_n.$

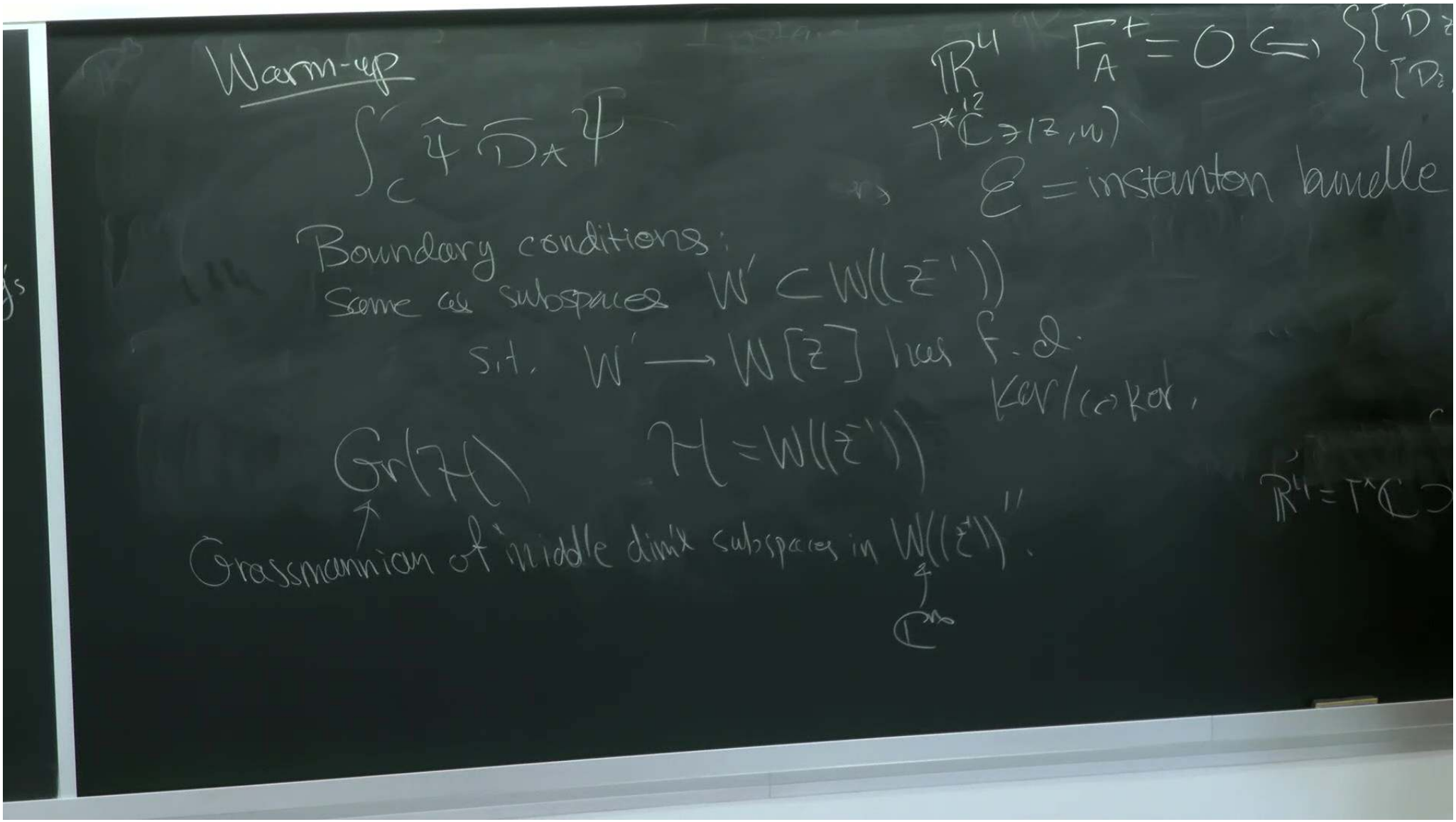
"matrix for instantons"

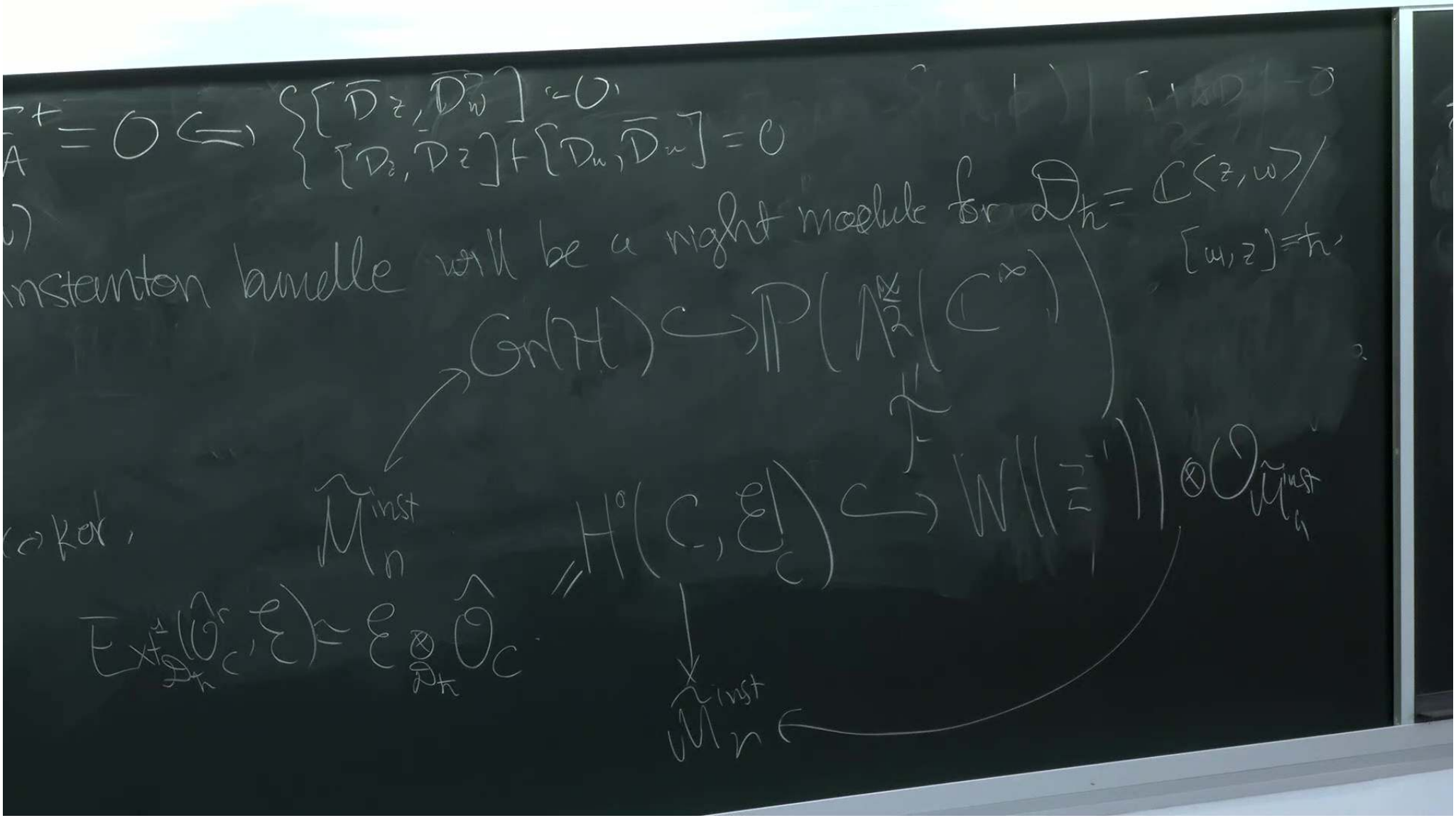
$\mathbb{R}^4 = T^*\mathbb{C} \supset C.$

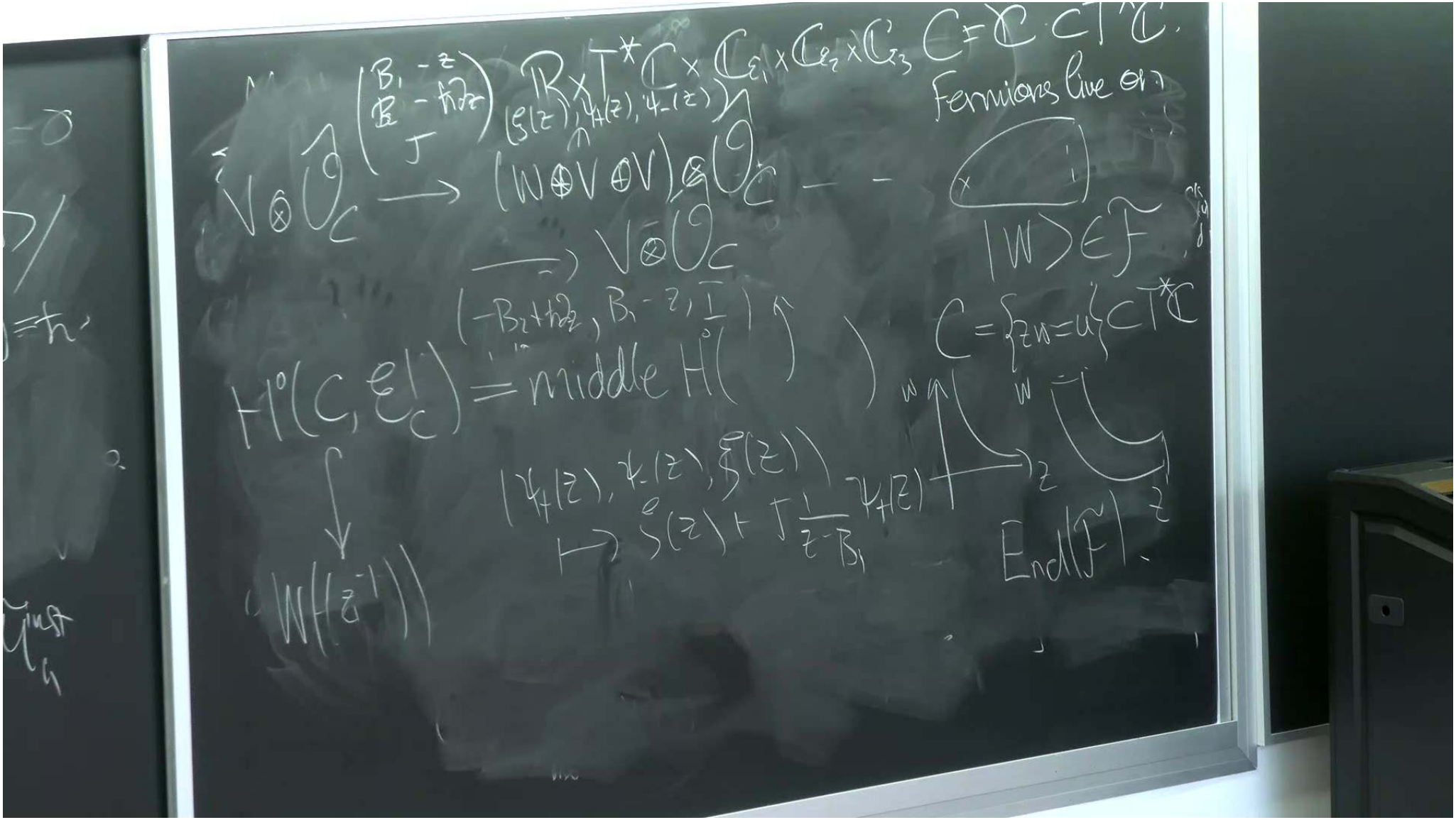
$\int D\bar{\psi}\, D\psi \exp\left(-\left\{ \bar{\psi} \bar{D}_A \psi \right.\right.$


$\begin{pmatrix} B_1 - z \\ B_2 - w \\ J \end{pmatrix}$ $\mathbb{R}\times T^*\mathbb{C}\times\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}\times\mathbb{C}_3$ $C=\mathbb{C}\subset T^*\mathbb{C}$.
Fermions live on
$V\otimes\mathcal{D}_R \longrightarrow (W\oplus V\oplus V)\otimes\mathcal{D}_R \longrightarrow V\otimes\mathcal{D}_R$
$(-B_2+w, B_1-z, \bar{I})$
$|W\rangle\in\mathcal{F}$
$\mathcal{E}$ = middle H^0.
$C=\{z=w=0\}\subset T^*\mathbb{C}$
$Q_f|W\rangle = \hat{Q}_g|W\rangle = 0$.
$Q_f = \oint_\gamma f(z)\psi(z)$ when f extends to regular section of $\mathcal{E}^*$,
$\hat{Q}_g = \oint_{\gamma_\infty} g(z)\hat{\psi}(z)$ g " for $\mathcal{E}$,
$\mathrm{End}(\mathcal{F})$.


$\begin{pmatrix} B_1 & -z \\ \bar{z} & -w \\ & J \end{pmatrix}$ $\mathbb{R}\times T^*\mathbb{C}\times\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}\times\mathbb{C}_{\epsilon_3}$ $C=\mathbb{C}\subset T^*\mathbb{C}$.
Fermions live on
$V\otimes\mathcal{D}_R \longrightarrow (W\oplus V\oplus V)\otimes\mathcal{D}_R$
$\longrightarrow V\otimes\mathcal{D}_R$
$(-B_2+w, B_1-z, \bar{I})$
$|W\rangle\in\tilde{\mathcal{F}}$
$C=\{z=w=0\}\subset T^*\mathbb{C}$
$\mathcal{E}=$ middle H^0.
$Q_f|W\rangle=\hat{Q}_g|W\rangle=0$.
$Q_f=\oint_\gamma f(z)\psi(z)$ when f extends to regular section of $\mathcal{E}^*$,
$\hat{Q}_g=\oint_{\gamma_\infty} g(z)\tilde{\psi}(z)$ g " for $\mathcal{E}$,
$\mathrm{End}(\tilde{\mathcal{F}})$.


Warm-up
$\int_C \bar{\psi} \bar{D}_A \psi$
Boundary conditions:
Same as subspaces $W' \subset W((z^{-1}))$
s.t. $W' \longrightarrow W[z]$ has f.d. ker/coker.
$Gr(\mathcal{H})$ $\mathcal{H} = W((z^{-1}))$
Grassmannian of "middle dimX subspaces in $W((z^{-1}))$".
$\mathbb{R}^4$ $F_A^+ = 0 \subset$
$T^*\mathbb{C}^2 \ni (z, w)$
$\mathcal{E}$ = instanton bundle
$\mathbb{R}^4 = T^*\mathbb{C}$


$\bar{D}_z, \bar{D}_w] = 0$
$[D_z, D_z] + [D_w, \bar{D}_w] = 0$
instanton bundle will be a right module for $\mathcal{O}_h = \mathbb{C}\langle z, w \rangle /$
$[w, z] = h$
$Gr(n) \hookrightarrow \mathbb{P}(\Lambda^{\frac{N}{2}}(\mathbb{C}^\infty))$
$\mathcal{M}_n^{inst}$
$H^0(\mathbb{C}, \mathcal{E}) \hookrightarrow W((z^{-1}))$
$\mathcal{O}_{\mathcal{M}_n^{inst}}$
$\mathcal{M}_n^{inst}$


Fermions live on
$|W\rangle \in \tilde{\mathcal{F}}$
$\mathrm{End}(\tilde{\mathcal{F}})$
middle H
$H^0(C, \mathcal{E}_C)$
$W(z^{-1})$

Instanton S-matrices

$$H^0(C, \mathcal{E}_\zeta) \hookrightarrow W((z^{-1})) \twoheadrightarrow W[z]$$

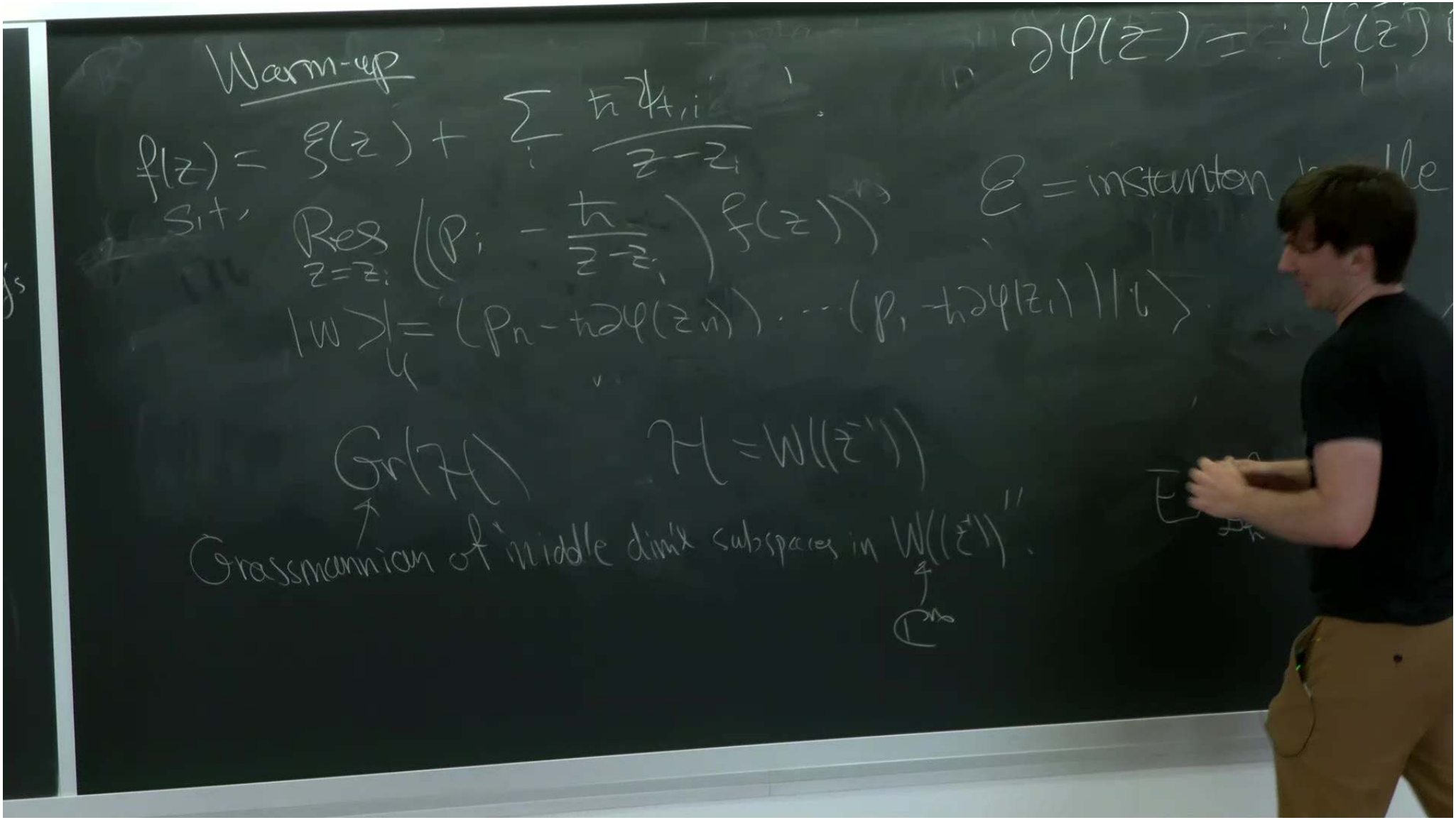
$$(\omega)\ker(\quad) \simeq (\omega)\ker(B_2) \qquad j_s$$

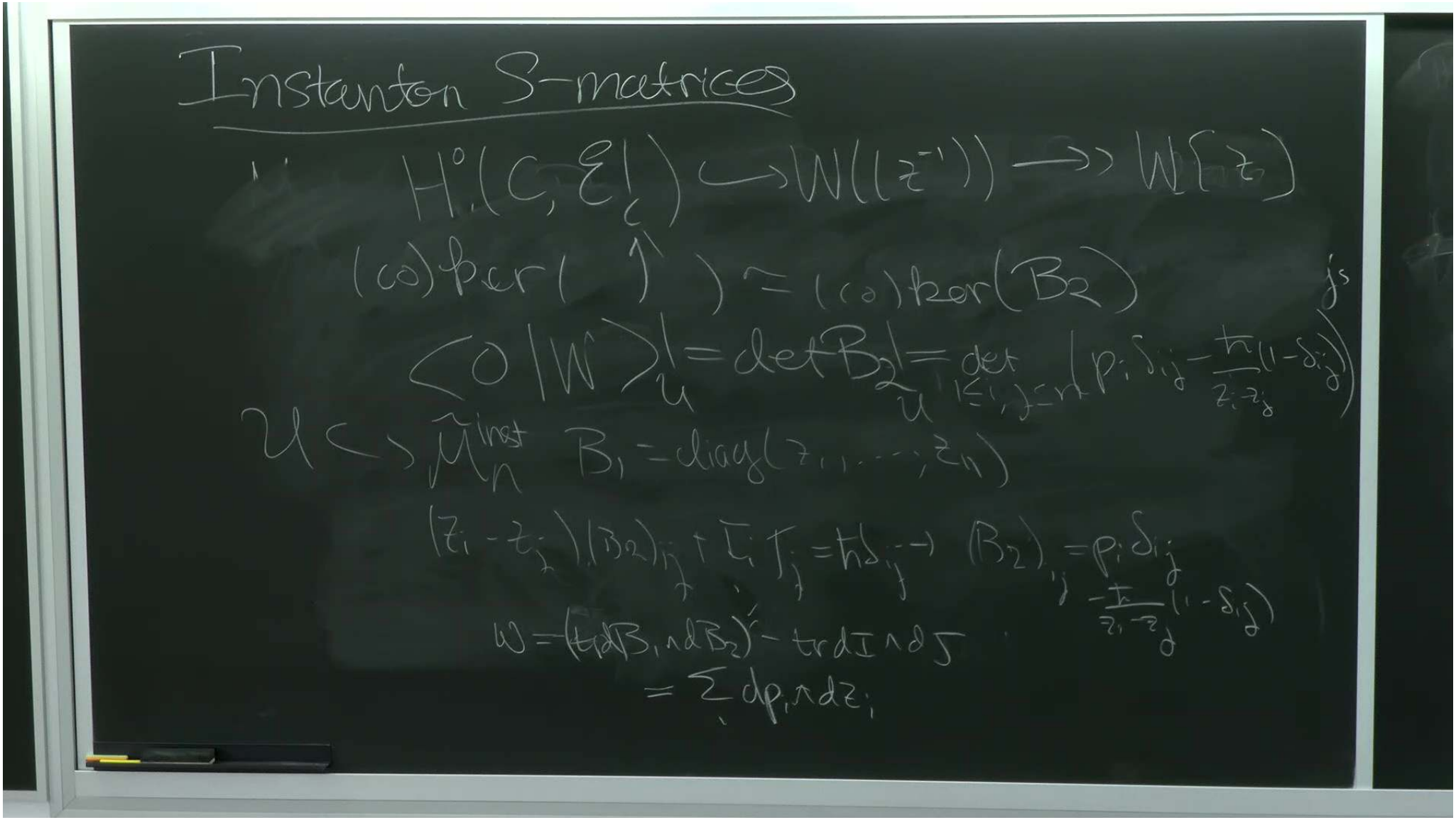
$$\langle 0|W\rangle = \det B_2 .$$

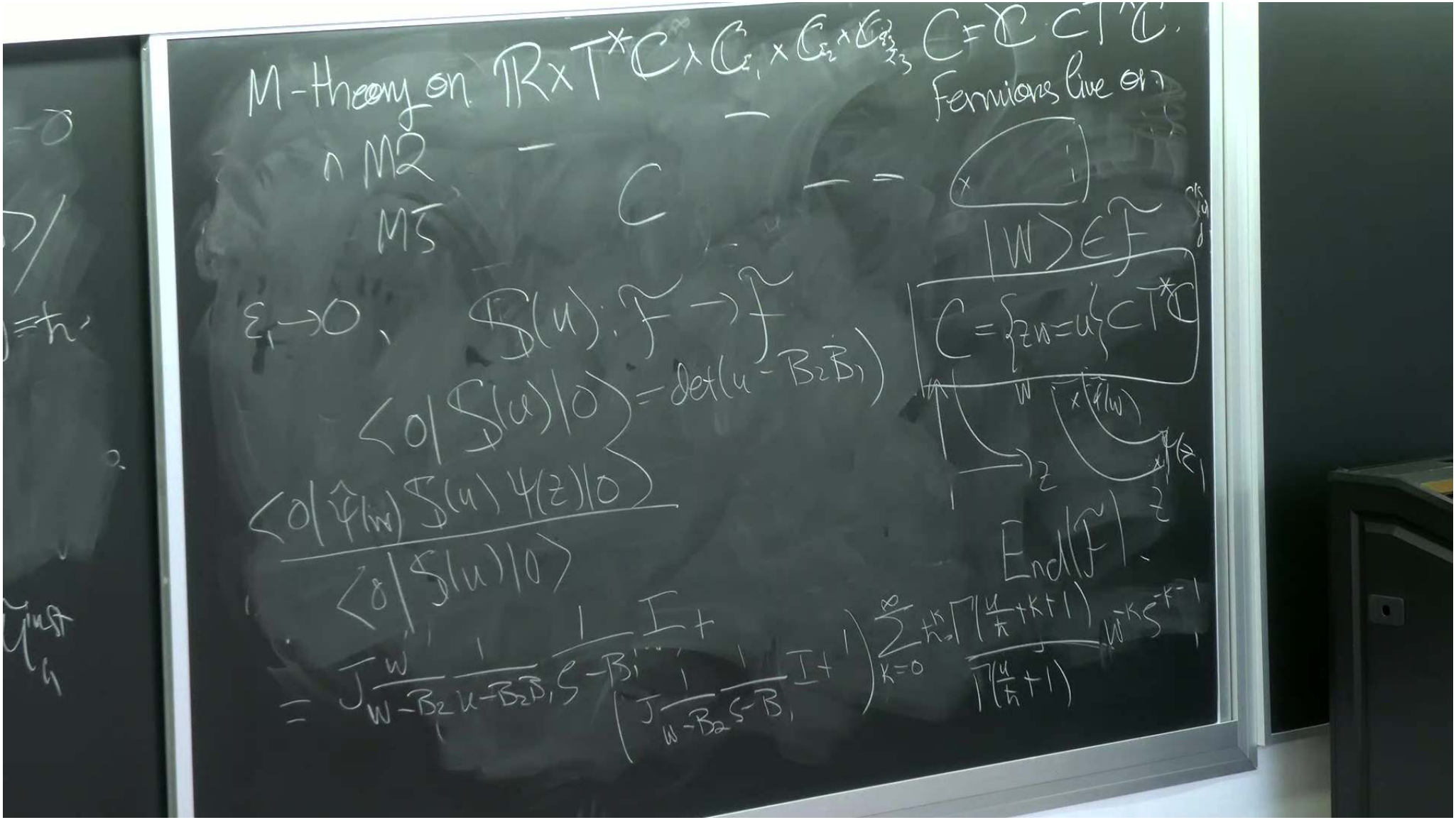
$$\mathcal{U} \leftrightarrow \tilde{\mathcal{M}}_n^{\text{inst}} \qquad B_1 = \operatorname{diag}(z_1, \ldots, z_n)$$

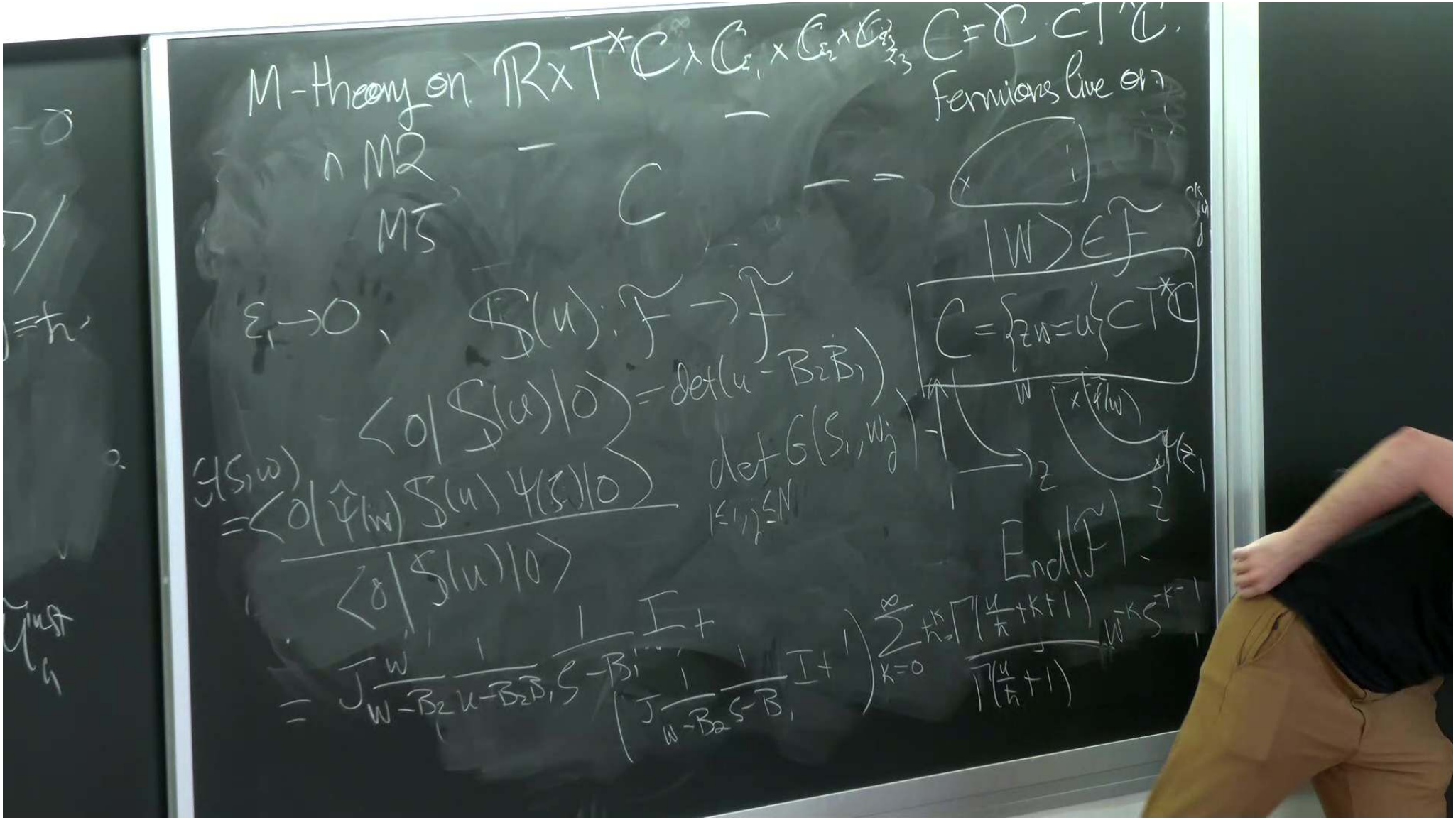
$$(z_i - z_j)(B_2)_{ij} + I_i J_j = \hbar\delta_{ij} \dashrightarrow (B_2)_{ij} = p_i\delta_{ij} - \frac{I_i J_j}{z_i - z_j}(1 - \delta_{ij})$$

$$\omega = (\operatorname{tr} dB_1 \wedge dB_2) - \operatorname{tr} dI \wedge dJ$$
$$= \sum dp_i \wedge dz_i$$


Warm-up
$f(z) = \xi(z) + \sum_i \frac{\hbar \psi_{+,i}}{z - z_i}$
$\partial\varphi(z) =: \psi(z)$
$\mathcal{E}$ = instanton
s.t.
$\operatorname{Res}_{z=z_i} \left(\left(p_i - \frac{\hbar}{z - z_i} \right) f(z) \right)$
$|w\rangle = (p_n - \hbar\partial\varphi(z_n)) \cdots (p_1 - \hbar\partial\varphi(z_1)) |1\rangle$
$Gr(\mathcal{H})$
$\mathcal{H} = W((z^{-1}))$
Grassmannian of "middle dim" subspaces in $W((z^{-1}))$


Instanton S-matrices
$H^0(C, \mathcal{E}|_C) \hookrightarrow W((z^{-1})) \twoheadrightarrow W[z]$
$(\omega)\ker(\quad) \simeq (\quad)\ker(B_2)$
$\langle 0|W\rangle_u = \det B_2|_u = \det_{1\le i,j\le n}\left(p_i\delta_{ij} - \frac{\hbar}{z_i - z_j}(1-\delta_{ij})\right)$
$u \leftrightarrow \tilde{\mathcal{M}}^{inst}_n \quad B_1 = \mathrm{diag}(z_1, \dots, z_n)$
$(z_i - z_j)(B_2)_{ij} + I_i J_j = \hbar\delta_{ij} \to (B_2)_{ij} = p_i\delta_{ij} - \frac{\hbar}{z_i - z_j}(1-\delta_{ij})$
$\omega = (\mathrm{tr}\, dB_1 \wedge dB_2) - \mathrm{tr}\, dI \wedge dJ$
$= \sum dp_i \wedge dz_i$


M-theory on $\mathbb{R}\times T^*\mathbb{C}\times\mathbb{C}_1\times\mathbb{C}_2\times\mathbb{C}_3$, $C=\mathbb{C}\subset T^*\mathbb{C}$.
Fermions live on
M2
M5
C
$\epsilon_1\to 0$, $S(u):\tilde{\mathcal{F}}\to\tilde{\mathcal{F}}$
$\langle 0|S(u)|0\rangle = \det(u-B_2B_1)$
$|W\rangle\in\tilde{\mathcal{F}}$
$C=\{z w=u\}\subset T^*\mathbb{C}$
$\mathrm{End}(\tilde{\mathcal{F}})$.
$\frac{\langle 0|\hat{\psi}(w)S(u)\psi(z)|0\rangle}{\langle 0|S(u)|0\rangle}$
$= J\frac{w}{w-B_2u-B_2B_1S-B_1}\frac{1}{\left(J\frac{1}{w-B_2S-B_1}\right)^{-1}+1}\sum_{k=0}^{\infty}\hbar^k\frac{\Gamma\left(\frac{u}{\hbar}+k+1\right)}{\Gamma\left(\frac{u}{\hbar}+1\right)}w^{-k}S^{-k-1}$


M-theory on $\mathbb{R}\times T^*\mathbb{C}\times\mathbb{C}_1\times\mathbb{C}_2\times\mathbb{C}_3/\mathbb{Z}_3$
Fermions live on
M2
M5
$\epsilon_1\to 0$,
$\mathbb{S}(u):\mathcal{F}\to\mathcal{F}$
$\langle 0|\mathbb{S}(u)|0\rangle = \det(u-B_2\bar{B}_1)$
$\det G(s_i,w_j)$
$|W\rangle\in\mathcal{F}$
$C=\{zw=u\}\subset T^*\mathbb{C}$
$\mathrm{End}(\mathcal{F})$