Title: Lecture - Quantum Theory (Core), PHYS 605

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Free particles: find Hamiltonian $X_i \; P_i \rightarrow \hat{x} \; \hat{p}$

$$H = \frac{\vec{P}^2}{2m} = \frac{P_x^2 + P_y^2 + P_z^2}{2m}$$

$xp \rightarrow \hat{x}\hat{p} \quad \hat{p}\hat{x} \quad \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$

$$\left[x_i, F(\vec{p}) \right] = i\hbar \frac{\partial F}{\partial p_i}$$

$$\left[p_i, G(\vec{x}) \right] = -i\hbar \frac{\partial G}{\partial x_i}$$

Eom: $\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, H] = 0$

$$P_i^H(t) = P_i^H(0)$$

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, H] = \frac{1}{i\hbar} \frac{i\hbar}{2m} \frac{\partial}{\partial p_i} \left(\sum_{j=1}^{3} P_i^2 \right) = \frac{P_i}{m} = \frac{P_i(0)}{m}$$

$$X_i^H(t) = X_i^H(0) + \frac{P_i(0)}{m} t$$

Particle in a potential $V(\vec{x})$

$$H = \frac{\vec{P}^2}{2m} + V(\hat{x})$$

$$\frac{dP_i}{dt} = \frac{1}{i\hbar}\left[P_i, V(\hat{x})\right] = -\frac{\partial V(\vec{x})}{\partial x_i}$$

$$\frac{dx_i}{dt} = \frac{P_i}{m}$$

$$\frac{d^2 x_i}{dt} = \frac{d}{dt}\left(\frac{d}{dt} x_i\right) = \frac{1}{m}\frac{dP_i}{dt} = \frac{-1}{m}\frac{\partial V(\vec{x})}{\partial x_i}$$

$$m\frac{d^2\vec{x}}{dt^2} = -\nabla V(\vec{x})$$

$$\left.\right) + \frac{P_i(0)}{m} t$$

Ehrenfest's theorem

$$m\frac{d^2}{dt^2}\langle\vec{x}\rangle = \frac{d}{dt}\langle\vec{P}\rangle$$

$$= -\langle\nabla V(\vec{x})\rangle$$

wrt the (fixed!) Heisenberg picture state

Simple Harmonic Oscillator $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$

define $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ annihilation operator

$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ creation op.

$H = \hbar\omega($

So H and

$N|n\rangle$

$H|n\rangle$

$[x, p] = i\hbar$, so $[a, a^\dagger] = \frac{m\omega}{2\hbar}\left(\frac{-i}{m\omega}[x,p] + \frac{i}{m\omega}(p,x]\right) = \frac{2\hbar}{2\hbar} = 1$

define the Number operator $N = a^\dagger a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2}\right) + \frac{i}{2\hbar}[x,p]$

$$\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

nihilation operator

reation op.

$$H = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

So H and N are simultaneously diagonalize.

$$N|n\rangle = n|n\rangle$$

$$H|n\rangle = \underline{\left(n+\tfrac{1}{2}\right)\hbar\omega}|n\rangle$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$\left[x,p\right] + \frac{i}{m\omega}\left(p,x\right) = \frac{2\hbar}{2\hbar} = 1$$

$$N = a^\dagger a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2}\right) + \frac{i}{2\hbar}[x,p] = \frac{H}{\hbar\omega} - \frac{1}{2}$$

note

neously diagonalize.

$+\frac{1}{2})\hbar\omega$

note $[N, a] = [a^\dagger a, a] = a^\dagger \underbrace{[a, a]}_{0} + \underbrace{[a^\dagger, a]}_{-1} a = -a$

$$[N, a^\dagger] = a^\dagger$$

So $N\underbrace{a^\dagger|n\rangle} = (Na^\dagger - a^\dagger N + a^\dagger N)|n\rangle$

$$= ([N, a^\dagger] + a^\dagger N)|n\rangle$$

$$= \underbrace{(1+n)}\, a^\dagger |n\rangle$$

$N\underbrace{a|n\rangle} = ([N, a] + aN)|n\rangle = (n-1)\underbrace{a|n\rangle}$

So $a^\dagger|n\rangle$ and $a|n\rangle$ are e-states of $\hat{N}$ with e-values increased/decreased by 1. $a^\dagger$ "create" a quantum
a "annihilate" of energy

Since $N(a|n\rangle) = (n-1)\,a|n\rangle$, $a|n\rangle$ is an e-state of N with e-value $n-1$ ... but so is $|n-1\rangle$

So $a|n\rangle = c|n-1\rangle$ for $c \in \mathbb{C}$

to find c, note that $n = \langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = c^* c\langle n-1|n-1\rangle = |c|^2$

take c real, positive, $c = \sqrt{n}$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a^2|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle$

$a^3|n\rangle = \sqrt{n(n-1)(n-2)}\,|n-3\rangle$

n decreases without limit, making more energy e-states until we hit $a|0\rangle = 0$, which happens iff we started with n a positive integer.

By positivity $n = \langle n|N|n\rangle$

$= (\langle n|a^\dagger)(a|n\rangle) \geq 0$

e-state with the lowest e-val for $\hat{N}$ $|0\rangle = |n=0\rangle$

take c real, positive, $c = \sqrt{n}$

now $a^+|0\rangle = \sqrt{1}\,|1\rangle$

$a^+|1\rangle = \sqrt{2}\,|2\rangle$

$a^+|2\rangle = \sqrt{3}\,|3\rangle$

$|1\rangle = a^+|0\rangle$

$|2\rangle = \frac{a^+}{\sqrt{2}}|1\rangle = \frac{a^+a^+}{\sqrt{2}}|0\rangle$

$|3\rangle = \frac{a^+}{\sqrt{3}}|2\rangle = \frac{(a^+)^3}{\sqrt{3}\sqrt{2}}|0\rangle$

$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$

these are the e-states of H

with $E_n = \left(n+\frac{1}{2}\right)\hbar\omega \quad n = 0, 1, 2, \ldots$

n a positive integer

Let's find their form in the position basis
- relate back to $\hat{x}$, $\hat{p}$,

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$

$$\hat{P} = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right)$$

Since $\langle n'|a|n\rangle = \sqrt{n}\,\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n',n-1}$

$\langle n'|a^\dagger|n\rangle = \sqrt{n+1}\,\langle n'|n+1\rangle = \sqrt{n+1}\,\delta_{n',n+1}$

$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$

$\langle n'|p|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$

not diagonal in energy e-basis

1, 2, ...

Want to find $\int dx\, |x\rangle\langle x|\psi\rangle = \int \psi(x)\,|x\rangle\, dx$ $\quad \langle x|x'\rangle = \delta(x-x')$

($\psi(x)$)

For the ground state $|0\rangle$, find the WF $\langle x|0\rangle$

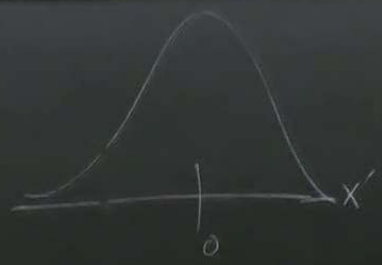
first, take $\langle x'|a|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x'|x + \frac{ip}{m\omega}|0\rangle = 0$ $\longrightarrow$ diff. eq. for $\langle x|0\rangle$

($a|0\rangle = 0$)

recall $\langle x'|p|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x'|\psi\rangle$

$|x\rangle\, dx \qquad \langle x|x'\rangle = \delta(x-x')$

$x_0 := \sqrt{\frac{\hbar}{m\omega}}$



e WF $\langle x|0\rangle$

$\frac{ip}{m\omega}|0\rangle = 0$ ⟶ diff. eq. for $\langle x'|0\rangle$

$\left(x' + x_0^2 \frac{d}{dx'}\right)\langle x'|0\rangle = 0$

$\langle x'|0\rangle = \psi_0(x') = \frac{1}{\pi^{1/4}\sqrt{x_0}}\, e^{-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2}$

$\rangle = -i\hbar \frac{\partial}{\partial x}\langle x'|\psi\rangle$

Similarly, $\langle x'|1\rangle = \langle x'|a^+|0\rangle = \frac{1}{\sqrt{2}\,x_0}\left(x' - x_0^2 \frac{d}{dx'}\right)\langle x'|0\rangle$

$\psi_1(x')$

$\langle x'|n\rangle \sim \left(x' - x_0^2 \frac{d}{dx'}\right)^n e^{-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2}$

Time evolution of SHO states. work in the Heis picture. write $x, p, a, a^\dagger$,

H. EOM : $\frac{dp}{dt} = \frac{1}{i\hbar}[p, V(x)] = \frac{-\partial}{\partial x} V(x) = -m\omega^2 x$ ($V(x) = \frac{1}{2}m\omega^2 x^2$)

$\frac{dx}{dt} = \frac{p}{m}$ (Same as for free particle)

$\frac{da}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m} - i\omega x\right) = -i\omega a$

$\frac{da^\dagger}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m} + i\frac{m\omega^2}{\omega m} x\right) = i\omega a^\dagger$

soln) $a(t) = a(0)e^{-i\omega t}$

$a^\dagger(t) = a^\dagger(0)e^{i\omega t}$

Note that $a^\dagger a(t) = a^\dagger(0)a(0)$ ($\hat{N}$)

So $a(t) = \sqrt{\frac{m\omega}{2\hbar}}\left(x(t) + \frac{ip(t)}{m\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(x(0) + \frac{ip(0)}{m\omega}\right)e^{-i\omega t}$

$\Downarrow$

Hermitian part $\quad x(t) = x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin\omega t$

anti-Herm. part $\quad p(t) = -m\omega x(0)\sin\omega t + p(0)\cos\omega t$

looks like the classical EOMs

e.g. $\langle n|x|n\rangle \sim \langle n|a + a^\dagger|n\rangle = 0$

The Interaction Picture: say $H = H_0 + V(t)$

"free" "interaction" known

note

assume we have solved the $V(t)=0$ case, so we know $H_0|n\rangle = E_n|n\rangle$

Suppose at t_0 the state is $|\psi(t_0)\rangle$. at a later time, the Schr. pict. state is $|\psi$

Define $|\psi(t)\rangle^I = e^{iH_0 t/\hbar}|\psi(t)\rangle^S$ "removing the free evolution" chosen so $|\ \rangle^I = |$

For most operators,

Define $A^I(t) = e^{iH_0 t/\hbar} A^S e^{-iH_0 t/\hbar}$, $V^I(t) = e^{iH_0 t/\hbar} \underbrace{V^S(t)}_{V \text{ for short}} e^{-iH_0 t/\hbar}$

Define $A(t) = e^{\quad}$

↓ Contrast with

EOM: take time deriv. and multiply by $i\hbar$

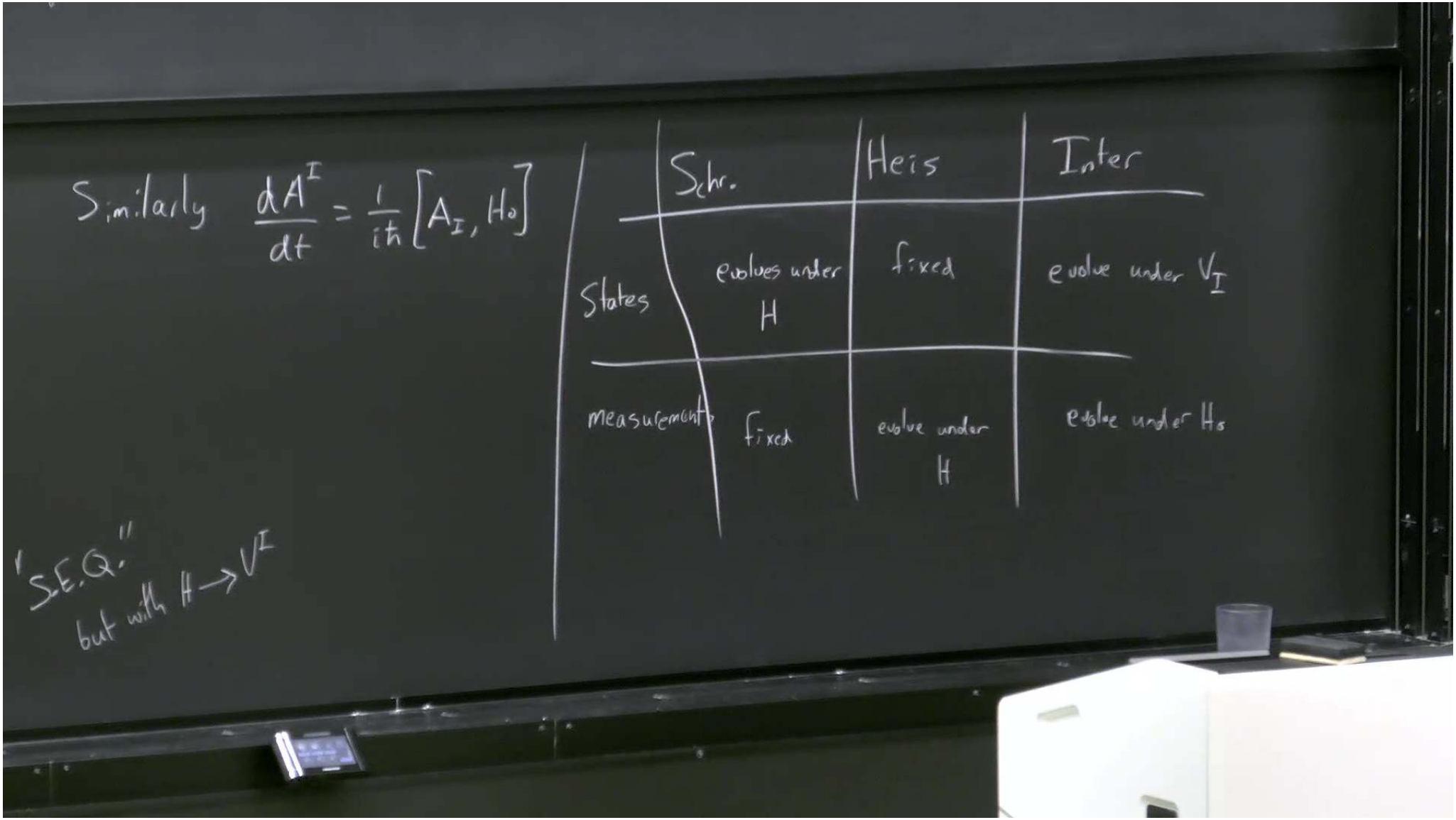
$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle^I = i\hbar \frac{iH_0}{\hbar} e^{iH_0 t/\hbar}|\psi(t)\rangle^S + i\hbar e^{iH_0 t/\hbar} \underbrace{\frac{d}{dt}|\psi(t)\rangle^S}_{\frac{(H_0+V)}{i\hbar}|\psi(t)\rangle^S}$$

$$e^{-iH_0 t/\hbar} \quad e^{+iH_0 t/\hbar}$$

$$= e^{iH_0 t/\hbar} V |\psi(t)\rangle^S = V^I(t)|\psi(t)\rangle^I$$

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle^I = V^I(t)|\psi(t)\rangle^I$$

"S.E.Q." but with $H \to V^I$


Similarly $\frac{dA^I}{dt} = \frac{1}{i\hbar}[A_I, H_0]$
Schr.
Heis
Inter
States
evolves under H
fixed
evolve under V_I
measurements
fixed
evolve under H
evolve under H_0
"S.E.Q."
but with $H \to V^I$

define $a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right)$ annihilation operator

$\frac{1}{2m}$ $\frac{1}{2}$ $H = \hbar\omega\left(N + \frac{1}{2}\right)$

$\subset$ H and N a

Say $|\psi(t_0)\rangle = \sum_n C_n |n\rangle$ ← eigenstates of H_0

$C_n = \langle n|\psi(t_0)\rangle$

then $\langle n | i\hbar \frac{d}{dt} | \psi(t)\rangle^I = \langle n | V^I(t) | \psi(t)\rangle^I$ (insert $\mathbb{I}$)

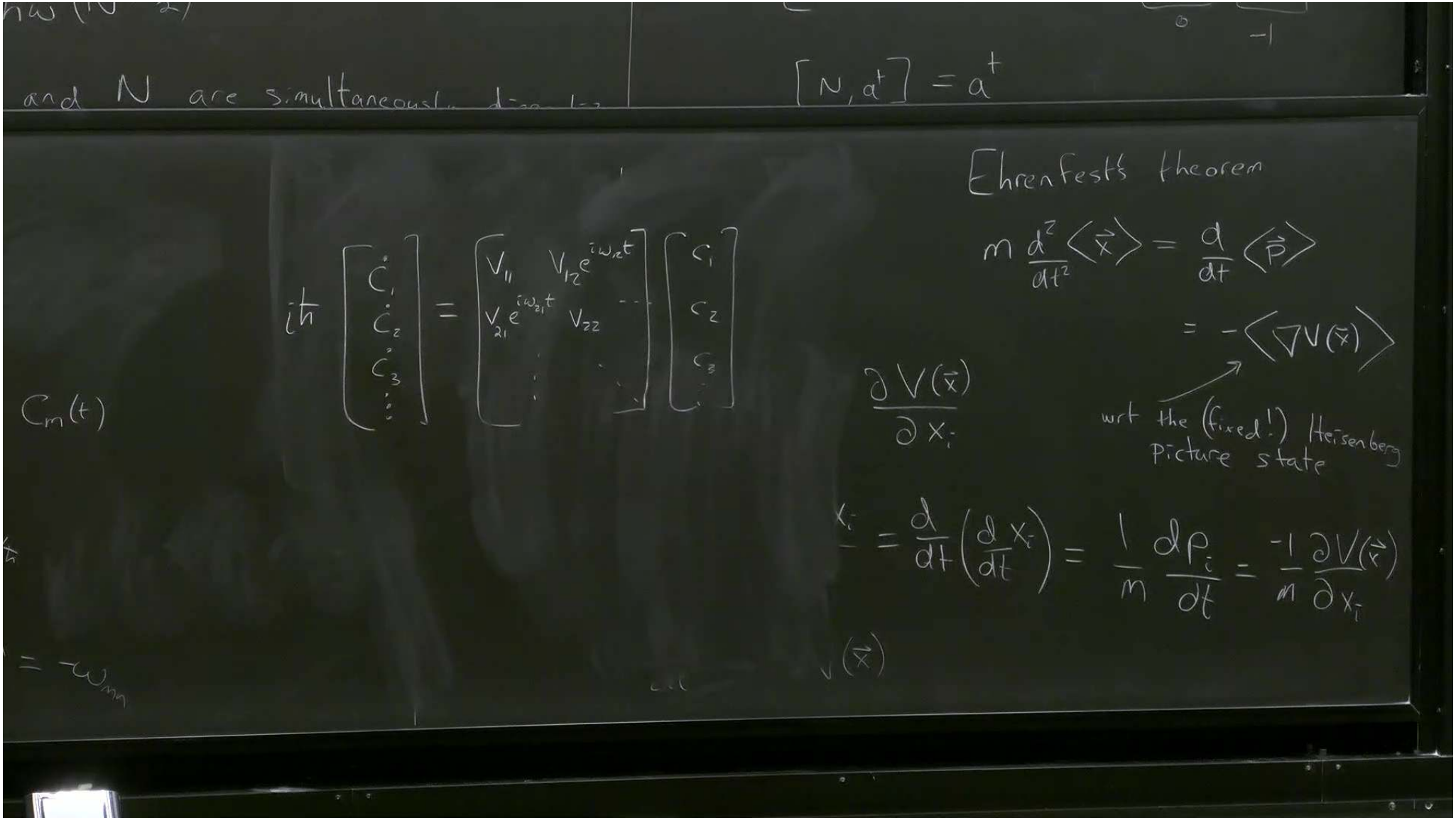
$= \sum_m \langle n|V^I(t)|m\rangle\langle m|\psi(t)\rangle^I$ ($V_{nm}(t)$, $C_m(t)$)

$= \sum_m \langle n | e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar} | m\rangle C_m(t)$

$V_{nm}(t)\, e^{i(E_n - E_m)t/\hbar}$

So $i\hbar \frac{d}{dt} C_n(t) = \sum_m V_{nm}(t)\, e^{i\omega_{nm}t} C_m(t)$

$\omega_{nm} = \frac{E_n - E_m}{\hbar} = -\omega_{mn}$


and N are simultaneous
$[N, a^\dagger] = a^\dagger$
Ehrenfest's theorem
$m \frac{d^2}{dt^2}\langle \vec{x} \rangle = \frac{d}{dt}\langle \vec{p} \rangle$
$= -\langle \nabla V(\vec{x}) \rangle$
wrt the (fixed!) Heisenberg picture state
$i\hbar \begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & \cdots \\ V_{21}e^{i\omega_{21}t} & V_{22} & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}$
$C_m(t)$
$\frac{\partial V(\vec{x})}{\partial x_i}$
$= \frac{d}{dt}\left(\frac{d}{dt} x_i\right) = \frac{1}{m}\frac{dp_i}{dt} = \frac{-1}{m}\frac{\partial V(\vec{x})}{\partial x_i}$