Title: Lecture - Quantum Theory (Core), PHYS 605

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Collection/Series: Quantum Theory (Core), PHYS 605, September 9 - October 23, 2026

Subject: Particle Physics, Quantum Fields and Strings, Quantum Information

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Dynamics: Suppose S is in state $|i\rangle = |\psi(0)\rangle$. What is $|\psi(t)\rangle$?

$$|\psi(t)\rangle = U(t, t_0)|i\rangle \quad \text{or} \quad U(t)$$

Compositionality

Properties of U?

- Continuity: $\lim_{t \to t_0} |\psi(t)\rangle = |i\rangle \Rightarrow \lim_{t \to t_0} U(t) = \mathbb{I}$ (normality)
- Probability conservation (Unitarity)

$$\langle \psi(t) | \psi(t) \rangle = \langle i | i \rangle = 1$$

$$\langle i | U^\dagger(t) U(t) | i \rangle \quad \forall i$$

$$\Rightarrow U^\dagger(t) U(t) = \mathbb{I}$$

$|\psi(t)\rangle \quad t > t_0$

$t_2 > t_1 > t_0$

nality $\quad U(t_2, t_0) = U(t_2, t_1)\, U(t_1, t_0)$

units t^{-1} (frequency)

Claim: $U(t_0 + dt, t_0) = \mathbb{I} - i\Omega\, dt$ satisfies these constraints, for Ω Hermitian

$$U^\dagger U = (\mathbb{I} + i\Omega\, dt)(\mathbb{I} - i\Omega\, dt) = \mathbb{I} + \Omega^\dagger \Omega\, dt^2 \approx \mathbb{I}$$

$$U(t_0 + dt_1 + dt_2,\, t_0 + dt_1)\, U(t_0 + dt_1,\, t_0) = U(t_0 + dt_1 + dt_2,\, t_0)$$

so let's define

$$\hat{H} = \hbar\Omega$$

units Joule-seconds

Energy operator = Hamiltonian

II

So $U(t_0+dt, t_0) = \mathbb{I} - \frac{iH dt}{\hbar}$ for some $H = H^\dagger$

Now let's derive a diff eqtn for $U(t, t_0)$

$$U(t+dt, t_0) = U(t+dt, t)\, U(t, t_0) = \left(\mathbb{I} - \frac{iH dt}{\hbar}\right) U(t, t_0)$$

$$U(t+dt, t_0) - U(t, t_0) = \frac{-iH dt}{\hbar} U(t, t_0)$$

in differential form

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = H\, U(t, t_0)$$

Schrodinger equation for U

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Schrodinger equation for U

apply on both sides to the left

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

Solving this S.E.Q. = finding time evolution

$$e^{\frac{-iH(t-t_0)}{\hbar}} = \mathbb{I} + \frac{-iH(t-t_0)}{\hbar}\mathbb{I} + \frac{(-i)^2}{2}\left(\frac{H}{\hbar}\right)^2 2(t-t_0) + \left[\frac{H(t-t_0)}{\hbar}\right.$$

$$\frac{\partial}{\partial t}(\;) = -\frac{iH}{\hbar}\mathbb{I} + \frac{(-i)^2}{2}\left(\frac{H}{\hbar}\right)^2 2(t-t_0) +$$

$$= \frac{-iH}{\hbar}\left(\mathbb{I} + \frac{-iH(t-t_0)}{\hbar} + \cdots\right.$$

Case 1) H is independent of time (e.g., spin in a fixed $\vec{B}$ field)

$$\Rightarrow U(t,t_0) = e^{\frac{-iH(t-t_0)}{\hbar}}$$

Proof recall $\frac{d}{dt}(e^{at}) = ae^{at}$

$= H|\psi(t)\rangle$

$\frac{\partial}{\partial t}(\quad) = \frac{-iH}{\hbar}\mathbb{I} + \frac{(-i)^2}{2}\left(\frac{H}{\hbar}\right)^2 2(t-t_0) + \ldots$

$= \frac{-iH}{\hbar}\left(\mathbb{I} + \frac{-iH(t-t_0)}{\hbar} + \ldots\right) = \frac{-iH}{\hbar}e^{-\frac{iH}{\hbar}(t-t_0)}$

. E. Q. = finding time evolution

independent of time (e.g., spin in a fixed $\vec{B}$ field)

$e^{-\frac{iH(t-t_0)}{\hbar}}$

Proof recall $\frac{d}{dt}(e^{at}) = ae^{at}$

Schrödinger equation for U

Case 2) H depends on t, but $[H(t), H(t')] = 0 \quad \forall\, t, t_0 \geq$

$\Rightarrow$ soln is $U(t, t_0) = e^{\frac{-i}{\hbar}\int_{t_0}^{t} dt' H(t')}$

Proof: Same as above, but replace

$H(t - t_0)$ with $\int_{t_0}^{t} dt' H(t')$

Case 3] H a

e.g. spin in a $\vec{B}$ field with fixed direction, but with varying strength)

nt times varies and does not commute (e.g. varying $\vec{B}$ field)

$$) = \mathbb{I} + \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \; H(t_1) H(t_2) \cdots H(t_n)$$

Proof: see Sakurai, Chapter 5

Let's focus on case 1. How does $U(t,t_0)$ act on various states?

- easy answer for eigenstates of $\hat{H}$ (energy eigenstates) $\{|e\rangle\}$

$$U(t)|e\rangle = e^{\frac{-i}{\hbar}Ht}|e\rangle = e^{\frac{-i}{\hbar}et}|e\rangle$$

← same state up to phase

← "stationary state"

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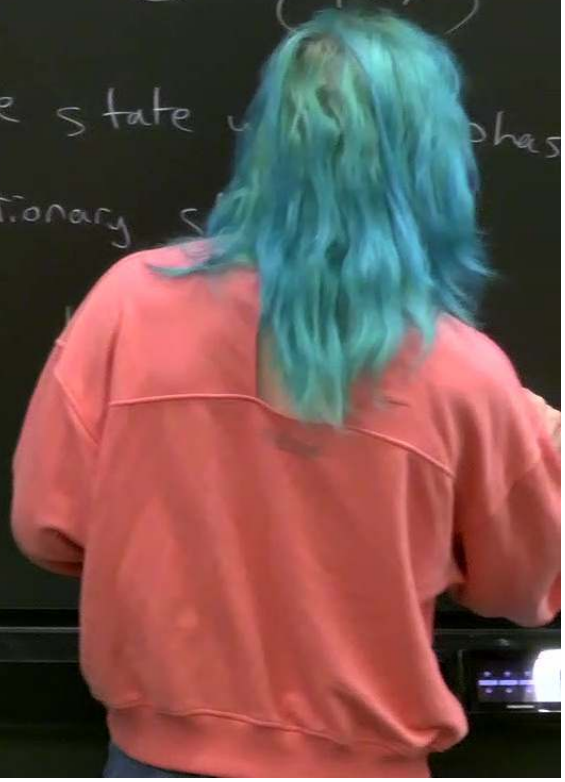
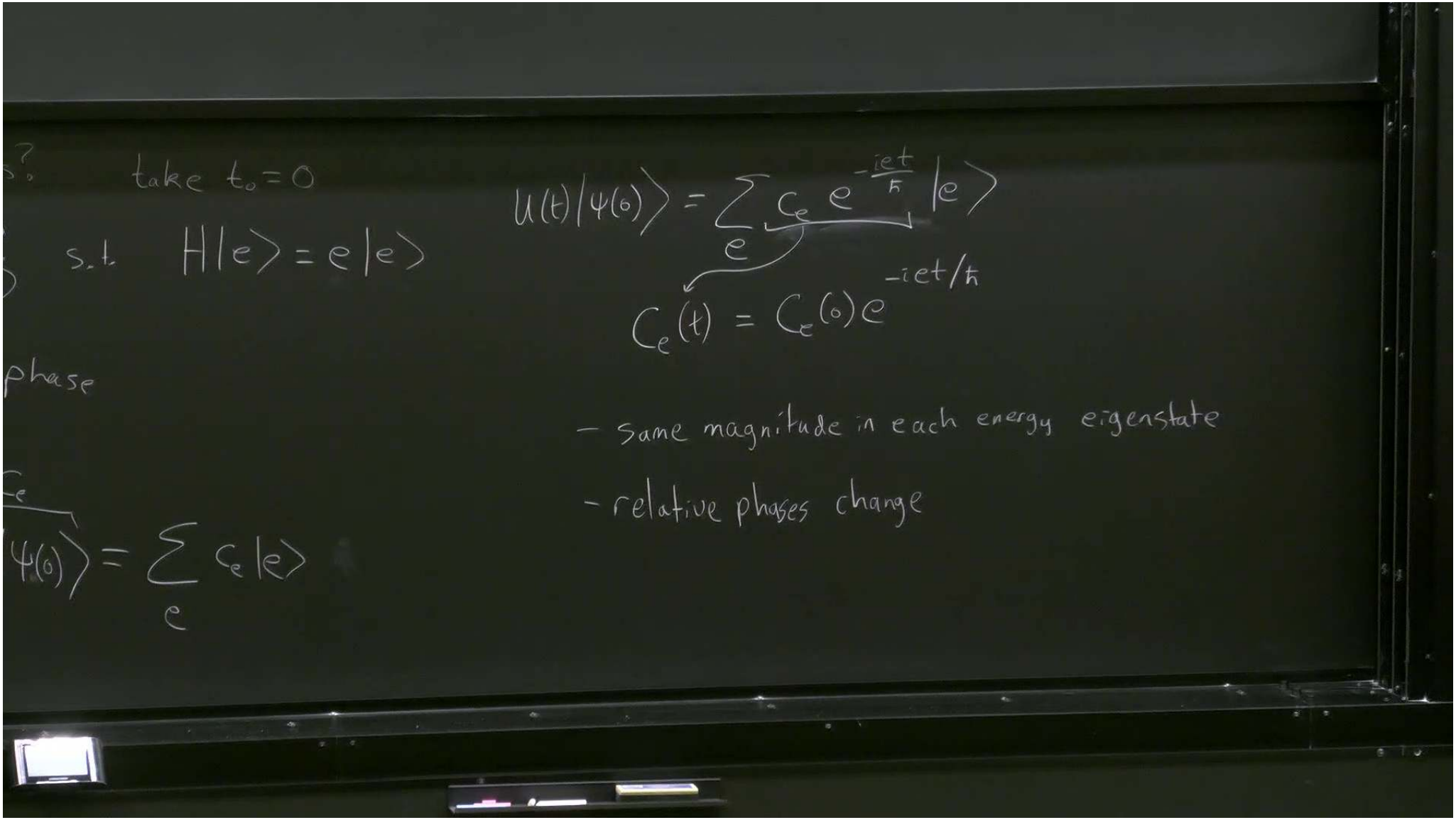
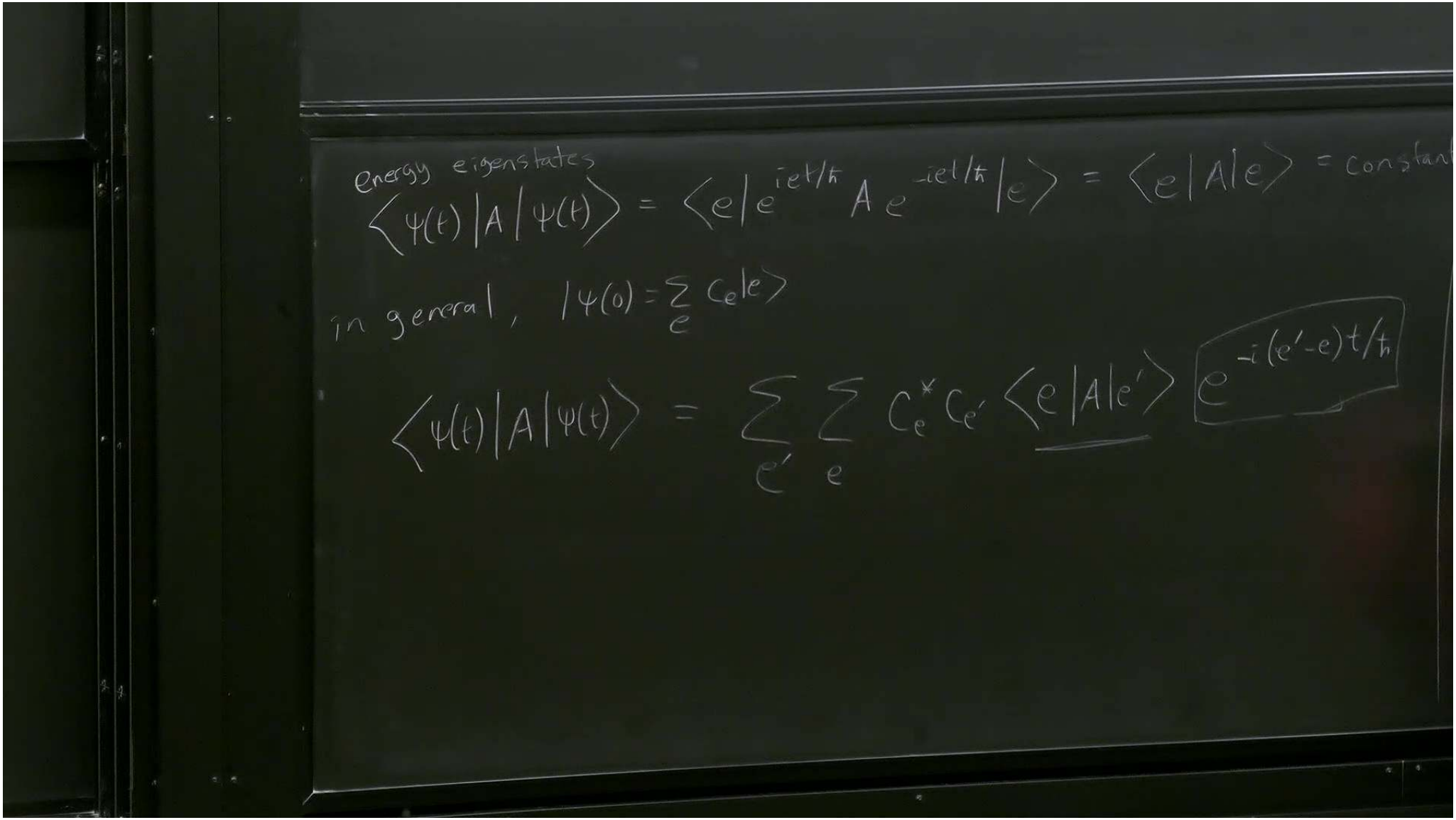
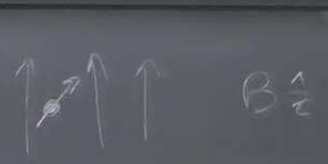
given any $|i\rangle = |\psi(0)\rangle$, expand it to $\{|e\rangle\}_e$,




take $t_0 = 0$
s.t. $H|e\rangle = e|e\rangle$
$U(t)|\psi(0)\rangle = \sum_e c_e e^{-iet/\hbar}|e\rangle$
$c_e(t) = c_e(0)e^{-iet/\hbar}$
phase
$|\psi(0)\rangle = \sum_e c_e|e\rangle$
- same magnitude in each energy eigenstate
- relative phases change


Energy eigenstates
$\langle \psi(t)|A|\psi(t)\rangle = \langle e|e^{iEt/\hbar} A e^{-iEt/\hbar}|e\rangle = \langle e|A|e\rangle$ = constant
in general, $|\psi(0)\rangle = \sum_e c_e|e\rangle$
$\langle \psi(t)|A|\psi(t)\rangle = \sum_{e'}\sum_{e} c_e^* c_{e'} \langle e|A|e'\rangle e^{-i(e'-e)t/\hbar}$

stant for $|\psi(0)\rangle = |e\rangle$ $\vec{B}$  $B\hat{z}$

Spin precession in static $\vec{B}$ field

$$\hat{H} = \frac{-e}{m_e c}\vec{S}\cdot\vec{B} = \frac{-eB}{m_e c}S_z = \omega S_z$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

eigenstates $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

eigenvalues $+\hbar/2$ $-\hbar/2$

eigenvalues are $e_\pm = \pm\frac{e\hbar B}{2m_e c} = \pm\frac{\hbar\omega}{2}$

eigenvectors $|e_\pm\rangle$

$$U(t,0) = e^{\frac{-i\omega t}{\hbar} S_z}$$

Say

Say $|\psi(0)\rangle = C_+ |+e\rangle + C_- |-e\rangle$

$$|\psi(t)\rangle = C_+ e^{\frac{-ie_+ t}{\hbar}} |+e\rangle + C_- e^{\frac{-ie_- t}{\hbar}} |-e\rangle$$

$$= C_+ e^{\frac{-i\omega t}{2}} |+e\rangle + C_- e^{\frac{i\omega t}{2}} |-e\rangle$$

$$U(t,0) = e^{-i\frac{\omega t}{\hbar} S_z}$$

Say $|\psi(0)\rangle = C_+ |+e\rangle + C_- |-e\rangle$

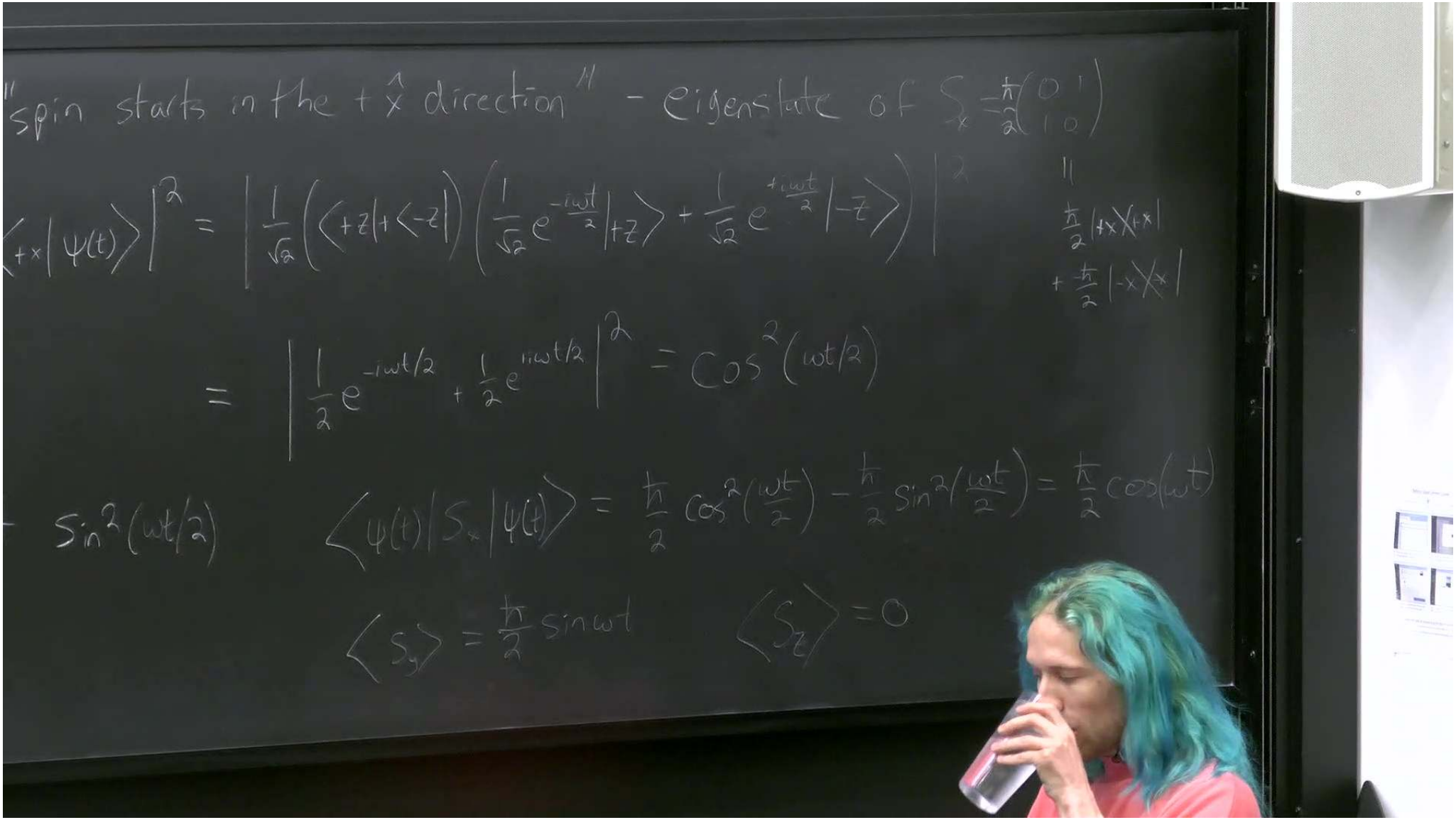
$$|\psi(t)\rangle = C_+ e^{-i\frac{e_+ t}{\hbar}} |+e\rangle + C_- e^{-i\frac{e_- t}{\hbar}} |-e\rangle$$

$$= C_+ e^{-i\frac{\omega t}{2}} |+e\rangle + C_- e^{i\frac{\omega t}{2}} |-e\rangle$$

Say

$\langle\psi(t)|$

for


"spin starts in the $+\hat{x}$ direction" - eigenstate of $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$|\langle +x|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle +z| + \langle -z|\right)\left(\frac{1}{\sqrt{2}}e^{-i\omega t/2}|+z\rangle + \frac{1}{\sqrt{2}}e^{+i\omega t/2}|-z\rangle\right)\right|^2$
$= \left|\frac{1}{2}e^{-i\omega t/2} + \frac{1}{2}e^{i\omega t/2}\right|^2 = \cos^2(\omega t/2)$
$\sin^2(\omega t/2)$
$\frac{\hbar}{2}|+x\rangle\langle +x| + \frac{-\hbar}{2}|-x\rangle\langle -x|$
$\langle\psi(t)|S_x|\psi(t)\rangle = \frac{\hbar}{2}\cos^2\left(\frac{\omega t}{2}\right) - \frac{\hbar}{2}\sin^2\left(\frac{\omega t}{2}\right) = \frac{\hbar}{2}\cos(\omega t)$
$\langle S_y\rangle = \frac{\hbar}{2}\sin\omega t$
$\langle S_z\rangle = 0$

The Heisenberg Picture (vs Schrodinger)

$A^H(t) := U^\dagger(t) A U(t)$ $\quad A^H(0) = A$

associativity

recall $\left(\langle\psi(t_0)| U^\dagger\right) A \left(U|\psi(t_0)\rangle\right) = \langle\psi(t_0)| \left(U^\dagger_{(t)} A U_{(t)}\right) |\psi(t_0)\rangle$

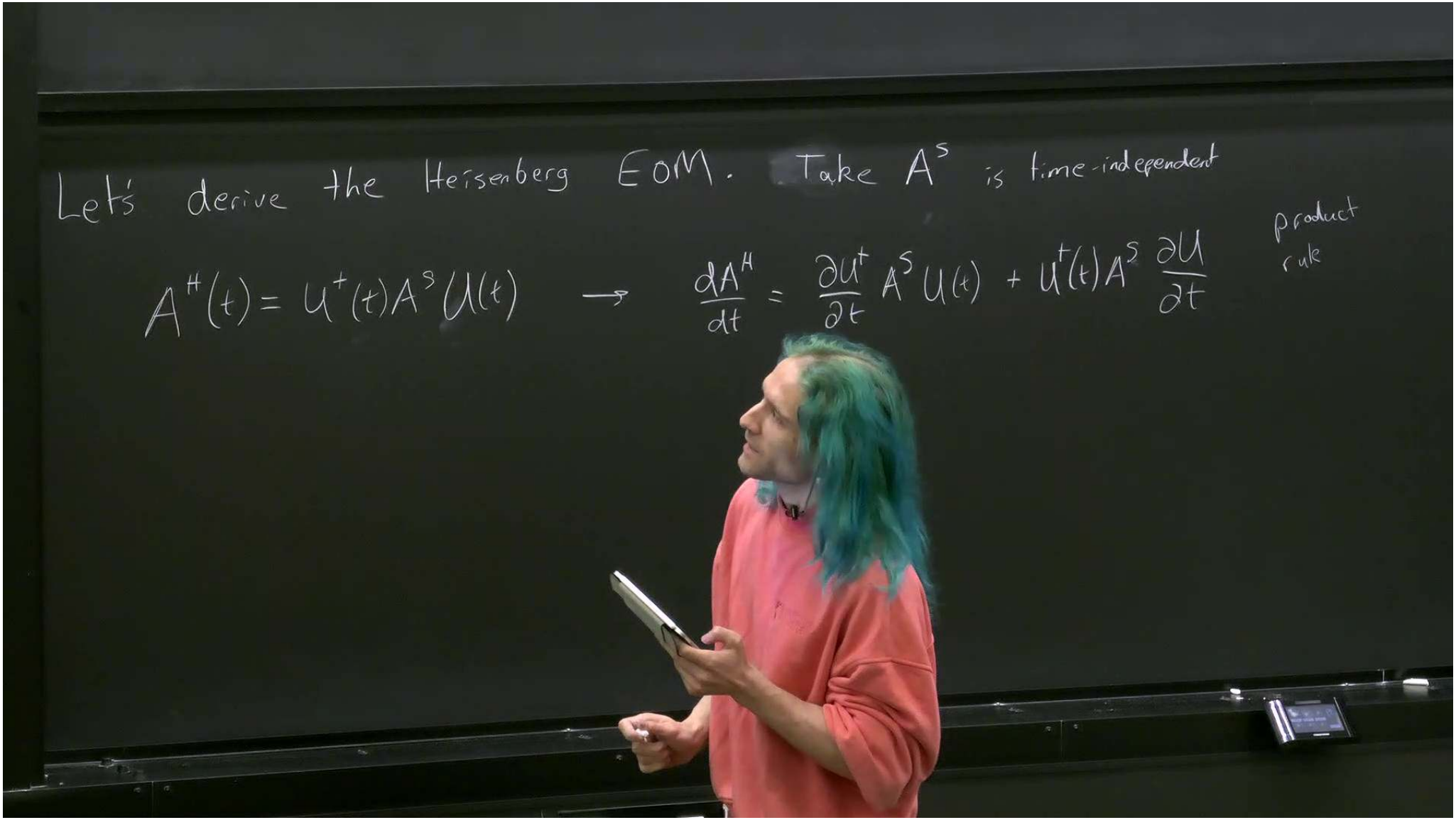
Schrodinger picture

Heisenberg picture
- evolution is applied to mmts
- state is fixed to $|\psi_0\rangle$

eigenvectors $|e_t\rangle$

$|\psi^H(t)\rangle = |\psi^S(0)\rangle$ constant

$$\langle \psi^H(t)| A^H(t) |\psi^H(t)\rangle = \langle \psi^S(0)| U^\dagger(t) A_S U(t) |\psi^S(0)\rangle = \langle \psi^S(t)| A_S |\psi^S(t)\rangle$$


Let's derive the Heisenberg EOM. Take A^S is time-independent
$A^H(t) = U^\dagger(t) A^S U(t) \rightarrow \frac{dA^H}{dt} = \frac{\partial U^\dagger}{\partial t} A^S U(t) + U^\dagger(t) A^S \frac{\partial U}{\partial t}$
product rule

$A^H(t) := U^\dagger(t) A U(t)$ $A^H(0) = A^S$

associativity

$|\psi^H(t)\rangle = |\psi^S(0)\rangle$ constant

$A U(t) |\psi(t_0)\rangle$

enberg picture

lution is applied to mmts

ate is fixed to $|\psi_0\rangle$

$\langle \psi^H(t)| A^H(t) |\psi^H(t)\rangle = \langle \psi^S(0)| U^\dagger(t) A_S U(t) |\psi^S(0)\rangle$

write out into $\{|e(t)\rangle, |f(t)\rangle\}$

$i\hbar$ A^H A^H

$U U^\dagger = \mathbb{1}$

$\frac{dA}{dt} = [A$

s Schrodinger)

$A^H(t) := U^\dagger(t) A U(t)$

associativity

$A^H(0) = A^S$

$\langle \psi(t_0) | (U_{(t)}^\dagger A U_{(t)}) | \psi(t_0) \rangle$

$|\psi^H(t)\rangle = |\psi^S(0)\rangle$ constant

Heis

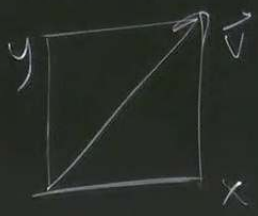
Heisenberg picture

- evolution is applied to mmts
- state is fixed to $|\psi_0\rangle$

write out ito

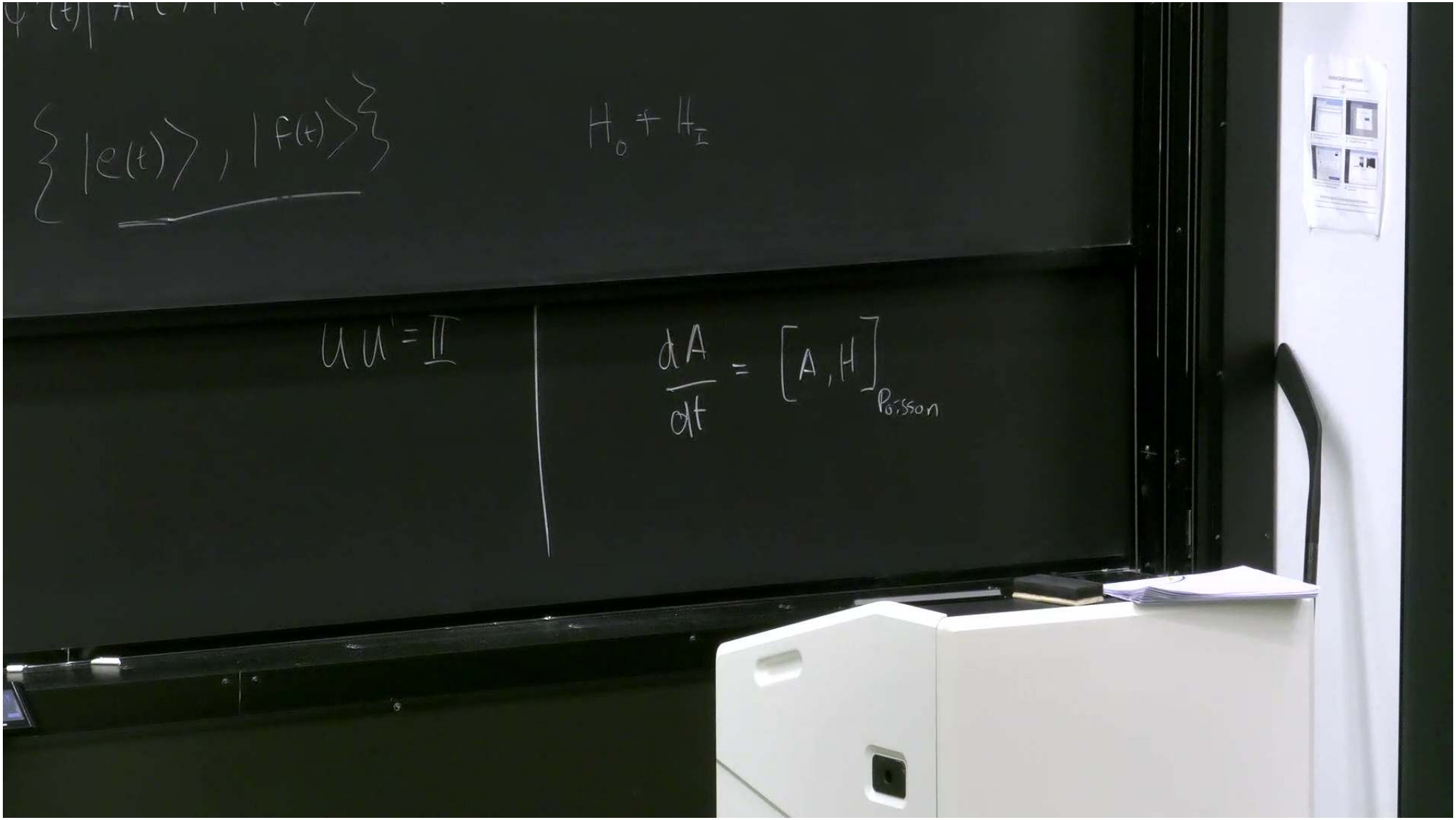
$\langle \psi^H(t) | A^H(t) | \psi^H(t) \rangle = \langle \psi^S(0) | U^\dagger(t)$

$\{ |e(t)\rangle, |f(t)\rangle \}$



$i\hbar$ A^H A^H

$UU^\dagger = \mathbb{I}$


$\{|e(t)\rangle, |f(t)\rangle\}$
$H_0 + H_I$
$UU^\dagger = \mathbb{I}$
$\frac{dA}{dt} = [A, H]_{Poisson}$