Title: Lecture - Quantum Theory (Core), PHYS 605

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Subject: Particle Physics, Quantum Fields and Strings, Quantum Information

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Transformation Procedures

Unitary: $\rho \rightarrow U \rho U^{+}$

- mixtures of U_i: $\rho \rightarrow \sum_i p_i U_i \rho U_i^{+}$
- nonselective mmts: $\rho \rightarrow \sum_i \Pi_i \rho \Pi_i = \sum_i p_i \frac{\Pi_i \rho \Pi_i}{p_i}$
- open systems $\rho_S \rightarrow \mathrm{Tr}_A \left[U_{SA} (\rho_S \otimes \tau_A) U_{SA}^{+} \right]$

$= \sum_i P_i \frac{\Pi_i \rho \Pi_i}{P_i}$

$\rho_S \otimes \tau_A)(U_{SA}^{+}]$

General Form (CPTP map)

- linear
- trace-preserving
- completely positive

operators $\mathcal{L}(\mathcal{H}) \equiv \mathcal{H} \to \mathcal{H}$

superoperators $\mathcal{L}(\mathcal{L}(\mathcal{H})) \equiv \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$

Kraus decomposition

Kraus decomposition: Any channel $T(\rho)$ can be written $T(\rho) = \sum_k A_k \rho A_k^\dagger$

for some A_k with $\sum_k A_k^\dagger A_k = \mathbb{I}$ (for trace-preservation)

ex) Amplitude damping channel

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$\mathcal{H}$

$\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$

$$\mathcal{E}(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$$

"energy dissipation" $= \begin{pmatrix} a + \gamma d & \sqrt{1-\gamma}\, b \\ \sqrt{1-\gamma}\, c & (1-\gamma) d \end{pmatrix}$

$A\left[{}_{SA}(U_S \otimes \mathbb{1}_A)U_{SA} \right]$

Adjoint of a channel

for $T: \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$ there is a unique $T^\dagger: \mathcal{L}(\mathcal{H}_{out}) \to \mathcal{L}(\mathcal{H}_{in})$

s.t. $\mathrm{Tr}\left(A\,T(B)\right) = \mathrm{Tr}\left(T^\dagger(A)\,B\right)$

$\mathrm{Tr}(A^2) = 0 \iff \langle A, T(B) \rangle$

$\mathrm{Tr}\left(E_k\,T(\rho)\right)$

$= \mathrm{Tr}\left(T^\dagger(E_k)\,\rho\right)$

$\begin{pmatrix} \sqrt{1-\gamma}\,c & (1-\gamma)d \end{pmatrix}$

If $T(\rho)$ has Krauss form $T(\rho) = \sum_i K_i \rho K_i^\dagger$

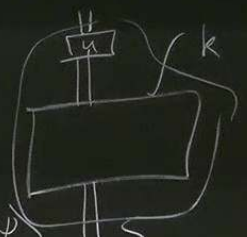
then $T^\dagger(\rho) = \sum_i K_i^\dagger \rho K_i$

note: $T^\dagger$ is always CP and also <u>unital</u> $\equiv T^\dagger(\mathbb{I}) = \mathbb{I}$

Ex) $U\rho U^\dagger \xrightarrow{\text{adjoint}} U^\dagger \rho U$ $\quad \rho \to |0\rangle\langle 0|$

<u>NOT</u> all channels are mixtures of unitaries

Measurement upate rules

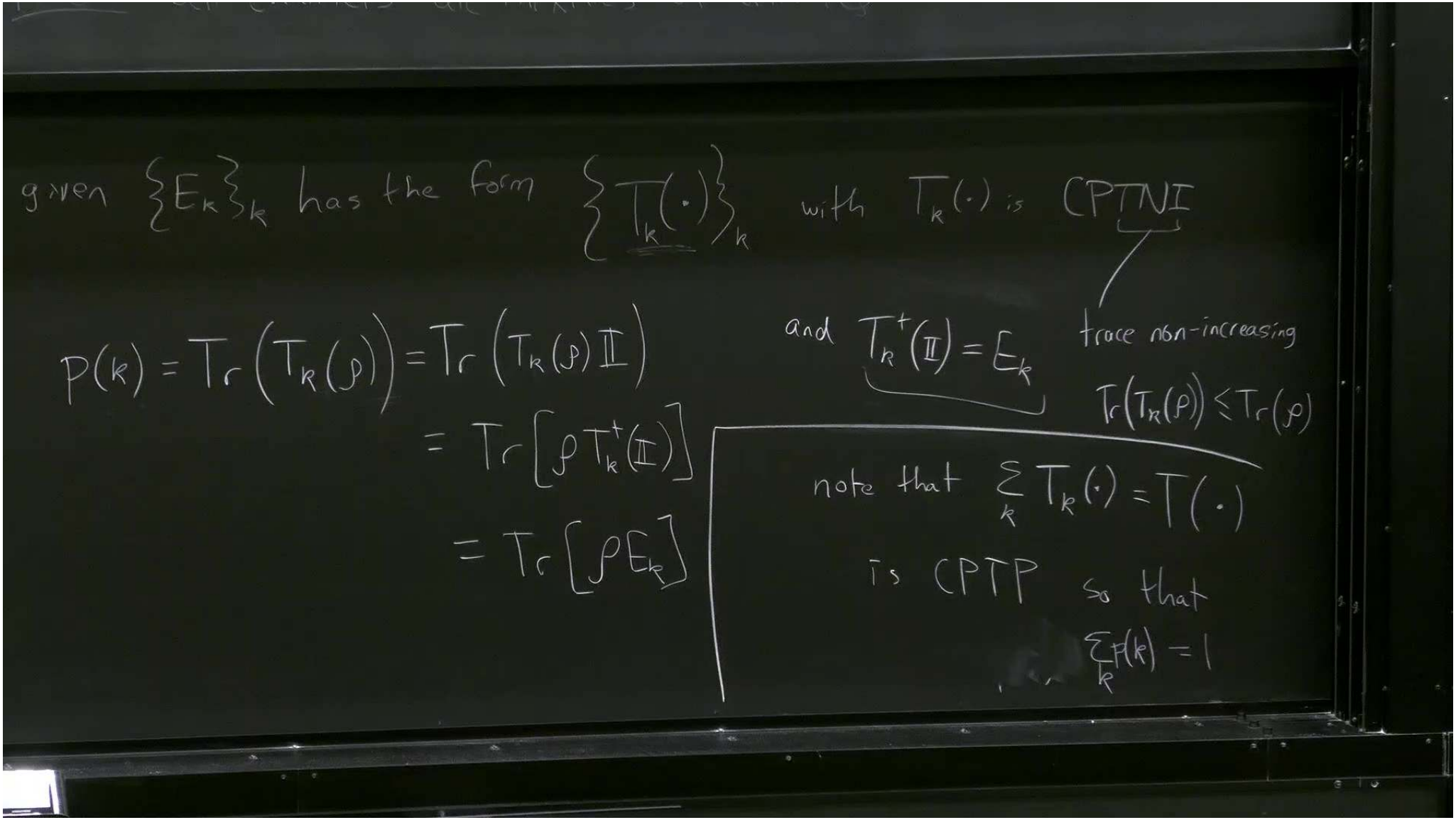


Lüders update $|\psi\rangle \xrightarrow{\Pi_k} |\psi_k\rangle = \frac{\Pi_k|\psi\rangle}{\sqrt{p_k}}$

"collapse" of the WF

$$|\psi\rangle \longrightarrow \frac{U\,\Pi_k|\psi\rangle}{\sqrt{p_k}}$$

Any update rule for a


given $\{E_k\}_k$ has the form $\{T_k(\cdot)\}_k$ with $T_k(\cdot)$ is CPTNI
trace non-increasing
$T_r(T_k(\rho)) \leq T_r(\rho)$
and $T_k^\dagger(\mathbb{I}) = E_k$
$p(k) = T_r(T_k(\rho)) = T_r(T_k(\rho)\mathbb{I})$
$= T_r[\rho T_k^\dagger(\mathbb{I})]$
$= T_r[\rho E_k]$
note that $\sum_k T_k(\cdot) = T(\cdot)$
is CPTP so that
$\sum_k p(k) = 1$

Any set of CPTNI maps that sums to a CPTP map.
$\Rightarrow$ this is a valid probabilistic transformation on the system] quantum instruments

$\rho_k' = T_k(\rho)$ is the unnormalized output state (conditioned on outcome k)

$$\mathrm{Tr}(\rho_k') = P(k)$$

$$= \mathrm{Tr}\left[T_k(\rho)\mathbb{I}\right] = \mathrm{Tr}\left[\rho \underbrace{T_k^\dagger(\mathbb{I})}_{E_k}\right]$$

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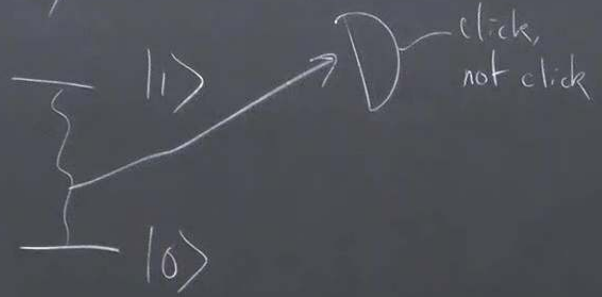
E_k effective POVM implemented by the instrument

ex) Monitored two-level atom



"measurement operator"

$$\{K_c, K_{no}\} = \left\{\sqrt{\gamma\Delta t}\,|0\rangle\langle 1|,\ |0\rangle\langle 0| + \sqrt{1-\gamma\Delta t}\,|1\rangle\langle 1|\right\}$$

click: $\rho \to K_c \rho K_c^\dagger = \gamma\Delta t\, |0\rangle\langle 1|\rho|1\rangle\langle 0|$, with $p(1) = \langle 1|\rho|1\rangle$

no click $\rho \to K_{no}\rho K_{no}^\dagger$

The associated POVM:

$$\{K_c^\dagger \mathbb{I} K_c,\ K_{no}^\dagger \mathbb{I} K_{no}^\dagger\}$$

$$= \left\{\gamma\Delta t\,|1\rangle\langle 1|,\ \mathbb{I} - \gamma\Delta t\,|1\rangle\langle 1|\right\}$$

click: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma d\Delta t & 0 \\ 0 & 0 \end{bmatrix}$ $P(\text{click}) = \gamma d\Delta t$

no click $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b\sqrt{1-\gamma\Delta t} \\ c\sqrt{1-\gamma\Delta t} & d(1-\gamma\Delta t) \end{bmatrix}$

If you ignore the outcome and take $\Delta t\gamma = 1$,

you get $\rho \longrightarrow K_c \rho K_c^\dagger + K_{no} \rho K_{no}^\dagger$

= amplitude damping.

Nonselective update map

$$T(\rho) = \sum_k P(k) \underbrace{\left(\frac{T_k(\rho)}{P(k)} \right)}_{\text{normalized state}} = \sum_k T_k(\rho)$$

Warning

$\{E_k\}$ tells us nothing about the update rule

$\{\Pi_R\}$ still tells us nothing

don't assume $\Pi_R \rho \Pi_R$ (collapse rule)

k P(k) k

Operational Postulates of QT

- Every Prepn procedure is associated with a density operator

  POVM

  mmt proc.

  CPTP map

  transf. proc

  CPTNI map

  measurement event

  (do a mmt; condition on one outcome)

Born Rule $P(k|P,T_i,M) = \mathrm{Tr}\left[E_k \mathcal{E}_{T_i}(\rho_P)\right]$

Bloch Representation (of a qubit)

Consider the space of Hermitian operators $\mathrm{Herm}(\mathcal{H})$

- this is a vector space on $\mathbb{R}$ (not on $\mathbb{C}$) $\quad i\begin{pmatrix}1&0\\0&1\end{pmatrix} + \left(i\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)^\dagger$

  $a\underline{M} + b\underline{N} \in \mathrm{Herm}(\mathcal{H})$ if $a, b \in \mathbb{R}$

- we can define an inner product on $\mathrm{Herm}(\mathcal{H})$ $\quad M, N \in \mathrm{Herm}(\mathcal{H})$

  via $\langle M, N \rangle = \mathrm{Tr}[MN]$ Hilbert-Schmidt inner product

$A \in \mathrm{Herm}(\mathcal{H})$ has the form $\begin{bmatrix} a & c+id \\ c-id & b \end{bmatrix}$ $a, b, c, d \in \mathbb{R}$

basis of operators has 4 elements

$\dim\left(\mathcal{H}\mathrm{erm}(\mathbb{C}^2)\right) = 4$

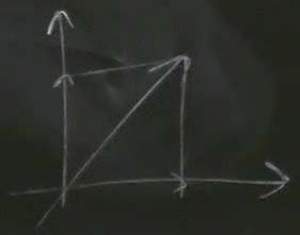
$A = \frac{a+b}{2}\mathbb{I} + \frac{a-b}{2}Z + cX + dY$

Pauli basis $\sigma_0 = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\sigma_1 = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\sigma_2 = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sigma_3 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



$\mathrm{Tr}[\sigma_i \sigma_j] = \begin{cases} 0 & i \neq j \\ 2 & i = j \end{cases}$

$\Rightarrow$ ON basis is $\left\{\frac{1}{\sqrt{2}}\sigma_i\right\}_i$

$$\rho = \sum_i \langle \rho, \tfrac{1}{\sqrt{2}}\sigma_i \rangle \tfrac{1}{\sqrt{2}}\sigma_i$$

$$= \frac{1}{2}\sum_i \underbrace{\mathrm{Tr}[\rho\sigma_i]}_{s_i}\sigma_i = \frac{1}{2}\sum_{i=0}^{3} s_i\sigma_i = \frac{1}{2}(s_0\mathbb{I} + s_1 X + s_2 Y + s_3 Z)$$

$$= \text{normalized } \frac{1}{2}(\mathbb{I} + \vec{s}\cdot\vec{\sigma}) \qquad \vec{\sigma} = \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+s_3 & s_1 - is_2 \\ s_1 + is_2 & 1-s_3 \end{bmatrix} \qquad \vec{s} = \begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix}$$

$\rho \geq 0 \Rightarrow s_1^2 + s_2^2 + s_3^2 \leq 1 \qquad |\vec{s}| \leq 1$

Bloch ball

Pure states = surface

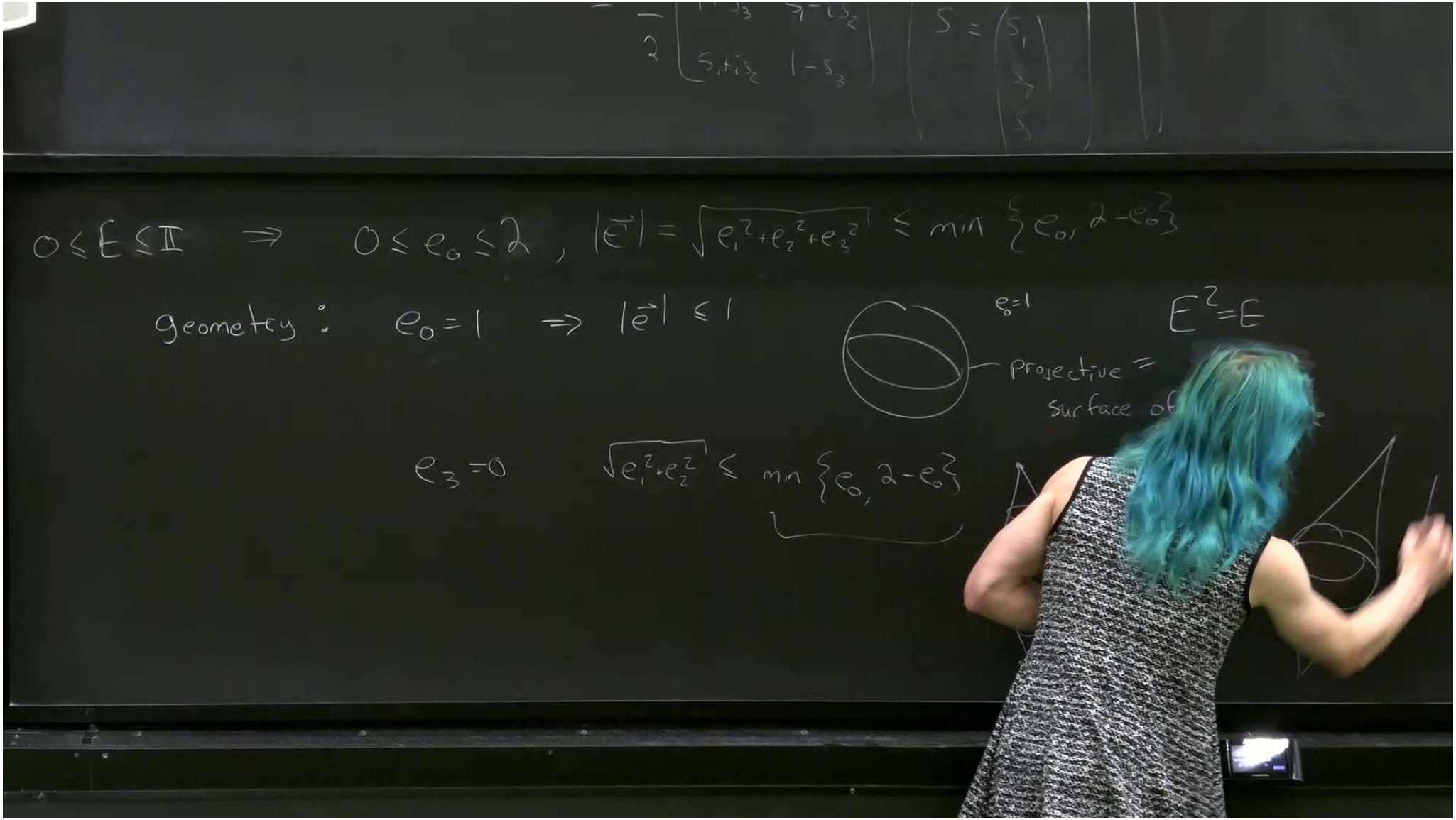
mixed states have $|\vec{s}| < 1$

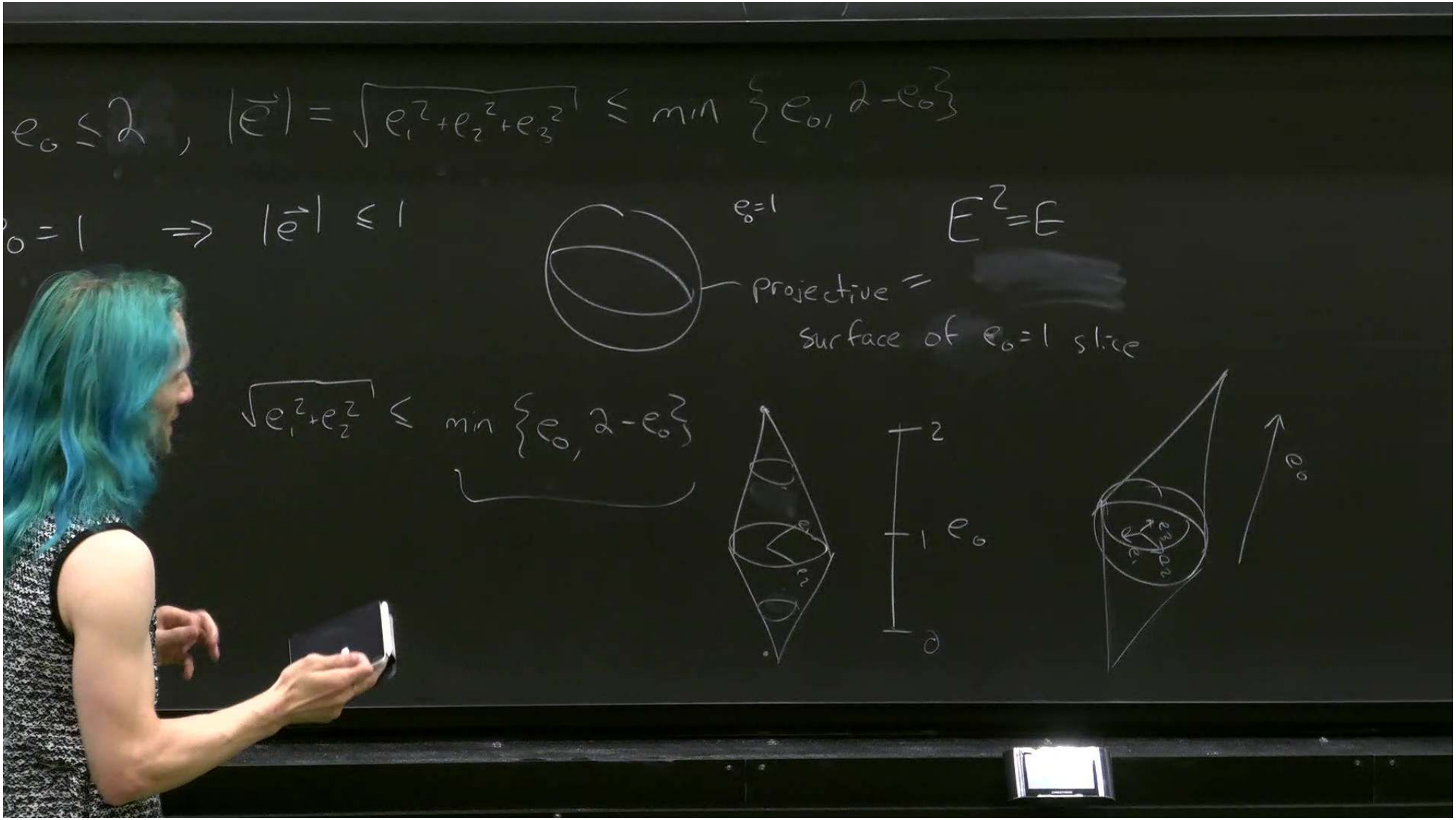
POVM elements

$$E = \sum_{i=0}^{3} \langle E, \tfrac{1}{\sqrt{2}}\sigma_i \rangle \sigma_i$$

$$= \frac{1}{2} \sum_{i=0}^{3} \underbrace{\mathrm{Tr}[E\sigma_i]}_{e_i} \sigma_i$$

$$= \frac{1}{2} \sum_i e_i \sigma_i = \frac{1}{2}\left(e_0 \mathbb{I} + e_1 X + e_2 Y + e_3 Z\right)$$


$\frac{1}{2}\begin{pmatrix} 1-s_3 & -s_1+is_2 \\ s_1+is_2 & 1-s_3 \end{pmatrix}$
$S=\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$
$0\leq E\leq II \Rightarrow 0\leq e_0\leq 2$, $|\vec{e}|=\sqrt{e_1^2+e_2^2+e_3^2}\leq \min\{e_0, 2-e_0\}$
geometry: $e_0=1 \Rightarrow |\vec{e}|\leq 1$
$e_0=1$
$E^2=E$
projective =
surface of
$e_3=0$
$\sqrt{e_1^2+e_2^2}\leq \min\{e_0, 2-e_0\}$


$e_0 \leq 2$, $|\vec{e}| = \sqrt{e_1^2+e_2^2+e_3^2} \leq \min\{e_0, 2-e_0\}$
$e_0 = 1 \Rightarrow |\vec{e}| \leq 1$
$e_0=1$
$E^2=E$
projective =
surface of $e_0=1$ slice
$\sqrt{e_1^2+e_2^2} \leq \min\{e_0, 2-e_0\}$
2
1
0
e_0

$\rho \geq 0 \Rightarrow s_1^2 + s_2^2 + s_3^2 \leq 1 \qquad |\vec{s}| \leq 1$

Bloch ball

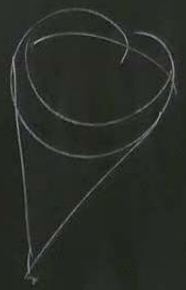
Pure states =

mixed states hav

POVM elements

$$E = \sum_{i=0}^{3} \langle E, \tfrac{1}{\sqrt{2}}\sigma_i \rangle \sigma_i$$

$$= \frac{1}{2} \sum_{i=0}^{3} \underbrace{\mathrm{Tr}[E\sigma_i]}_{e_i} \sigma_i$$



$$= \frac{1}{2} \sum_i e_i \sigma_i = \frac{1}{2}\left(e_0 \mathbb{I} + e_1 X + e_2 Y + e_3 Z\right)$$

$$e_0 = \mathrm{Tr}(\rho \mathbb{I}) \leq 1$$

$\rho \geq 0 \Rightarrow s_1^2 + s_2^2 + s_3^2 \leq 1 \qquad |\vec{s}| \leq 1$

Bloch ball

Pure states =

mixed states hav

POVM elements

$$E = \sum_{i=0}^{3} \langle E, \tfrac{1}{\sqrt{2}}\sigma_i \rangle \sigma_i$$

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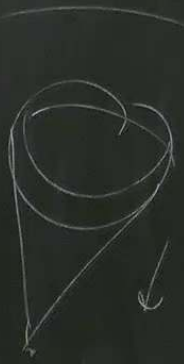


$s_0 = \mathrm{Tr}(\rho \mathbb{I}) \leq 1$

$$\begin{pmatrix} s_1+is_2 & 1-s_3 \end{pmatrix} \begin{pmatrix} s_2 \\ s_3 \end{pmatrix}$$

Born rule

$$\mathrm{Tr}(\rho E) = \frac{1}{4}\mathrm{Tr}\left[\sum_i \langle \rho, \sigma_i\rangle \sigma_i \sum_j \langle E, \sigma_j\rangle \sigma_j\right]$$

$$= \frac{1}{4}\sum_{ij} \langle \rho, \sigma_i\rangle \langle E, \sigma_j\rangle \mathrm{Tr}\left[\sigma_i \sigma_j\right] \quad (\mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij})$$

$$= \frac{1}{2}\sum_i \overbrace{\langle \rho, \sigma_i\rangle}^{s_i} \overbrace{\langle E, \sigma_j\rangle}^{e_j}$$

$$= \frac{1}{2}\left(s_0 e_0 + s_1 e_1 + s_2 e_2 + s_3 e_3\right) \overset{\text{normalized}}{=} \frac{1}{2}\left(e

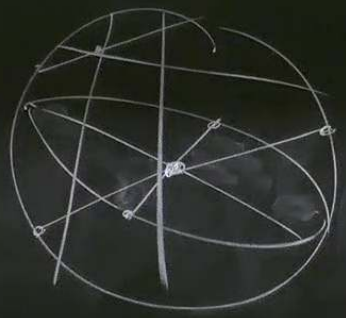
ule

$$\mathrm{Tr}(\rho E) = \frac{1}{4}\mathrm{Tr}\left[\sum_i \langle \rho, \sigma_i\rangle \sigma_i \sum_j \langle E, \sigma_j\rangle \sigma_j\right]$$

$$= \frac{1}{4}\sum_{ij} \langle \rho, \sigma_i\rangle \langle E, \sigma_j\rangle \mathrm{Tr}\left[\sigma_i \sigma_j\right] \quad (\mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij})$$

$$= \frac{1}{2}\sum_i \langle \rho, \sigma_i\rangle \langle E, \sigma_j\rangle \quad (\langle \rho, \sigma_i\rangle = s_i,\ \langle E, \sigma_j\rangle = e_j)$$

$$= \frac{1}{2}\left(s_0 e_0 + s_1 e_1 + s_2 e_2 + s_3 e_3\right) = \frac{1}{2}\left(e_0 + \vec{s}\cdot\vec{e}\right) \quad \text{normalized}$$



$$\langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle \leq 1$$

$$s_1^2 \qquad s_2^2 \qquad s_3^2$$

$T_k(\cdot)$ is CPTNI

and $\boxed{T_k^+(\mathbb{I}) = E_k}$ trace non-in

$\mathrm{Tr}\left(T_k(\rho)\right)$

note that $\sum_k T_k(\cdot) = T($

is CPTP so that

$\sum_k p(k) = 1$

$\left(e_0 + \vec{s}\cdot\vec{e}\right)$ Projective $\frac{1}{2}\left(1 + \vec{s}\cdot\vec{e}\right)$