Title: Lecture - Quantum Theory (Core), PHYS 605

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Collection/Series: Quantum Theory (Core), PHYS 605, September 9 - October 23, 2026

Subject: Particle Physics, Quantum Fields and Strings, Quantum Information

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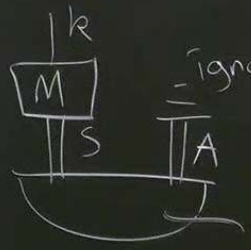
Example 2) access to only a subsystem



$|\Psi\rangle_{SA}$ = ignoring ~ doing a mmt with only one outcome

$\{\mathbb{I}\}$, $M = \{\Pi_k\}_k$

again, $\rho_S \geq 0$

$Tr[\rho_S] = 1$

$$P(k|\Psi_{SA}) = Tr_{SA}\left[|\Psi\rangle\langle\Psi|_{SA}\left(\mathbb{I}^A \otimes \Pi_k^S\right)\right] = Tr_S\left[\Pi_k^S \, Tr_A\left(|\Psi\rangle\langle\Psi|_{SA}\right)\right]$$

$$:= Tr_S\left[\Pi_k^S \rho^S\right]$$

ρ^S → probability catalogue

Postu

Postulate: Every preparation Procedure corresponds to an operator $\rho \in \mathcal{L}(\mathcal{H})$ satisfying - positive semidefiniteness $\rho \geq 0$

$\langle v|\rho|v\rangle \geq 0$

$\forall\ v \in \mathcal{H}$

↓

density matrix/operators

"state"

- unit trace

$\mathrm{Tr}[\rho] = 1$

- all parameters matter

$|\psi\rangle \sim e^{i\phi}|\psi\rangle$

$|\psi\rangle\langle\psi| = e^{i\phi}|\psi\rangle\langle\psi|e^{-i\phi}$

catalogue

Pure = convexly extremal states $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$

e.g. $|\pm\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \pm 1\end{bmatrix}$ $\xrightarrow{\text{density matrix}}$ $|\pm\rangle\langle\pm| = \frac{1}{2}\begin{bmatrix}1 & \pm 1\end{bmatrix}\begin{bmatrix}1\\ \pm 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}$

Mixed state := not extremal

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ for some nontrivi

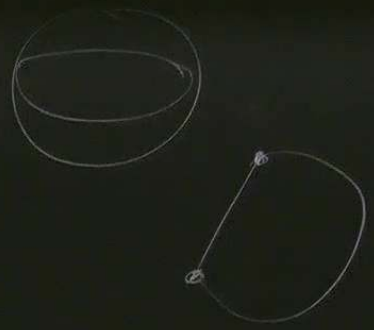
incoherent mixtures

ex) mix $|0\rangle$ and $|1\rangle$ $\quad p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| = p\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

Contrast with "coherent" superposition $|\psi\rangle = a|0\rangle + b|1\rangle$ $\rightarrow$

$|\psi\rangle \in \mathcal{H}$

$\frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$



ontrivial probabilistic ensemble

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + (1-p)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$

$\rightarrow |\psi\rangle\langle\psi| = (a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|) = \begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}$ $|a|^2 + |b|^2 = 1$

measurements in the $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ give the

Same probabilities for coherent/incoherent combinations

but measurements in any other basis will depend on the

"coherences"

$\{|+X+|, |-X-|\}$ coherences

$$\begin{bmatrix} |a|^2 & a b^* \\ a^* b & |b|^2 \end{bmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$\left(a^* \langle 0| + b^* \langle 1|\right)$$

Purification Principle: Every density operator can be obtained as the reduced/marginal state of some pure state (on two copies of the system)

given any ρ_A, $\exists\ |\psi\rangle_{AA'}$ s.t. $\rho_A = \mathrm{Tr}_{A'}\left[|\psi\rangle\langle\psi|_{AA'}\right]$

("Purifies ρ_A")

(also, $|\psi\rangle_{AA'}$ is unique up to reversible transformations on A')

$= (a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|) = \begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}$

m) Ambiguity of mixtures: Every ρ can be obtained by a mixture of pure states – and indeed, infinitely many such mixtures (if the state is not pure)

e.g. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\mathbb{I} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$

$= \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\psi^\perp\rangle\langle\psi^\perp|$

Knowing the density matrix $\not\Rightarrow$ knowing how the system was prepared

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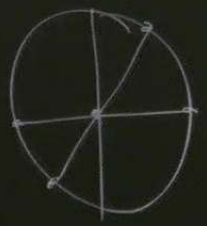
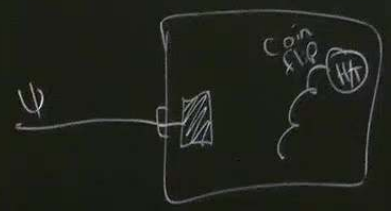


Knowing the density matrix $\not\Rightarrow$ knowing how the system was prepared

General Measurement Procedures

→ can be represented

ex)



$P(H|\psi) = 1/2 \quad \forall \psi$

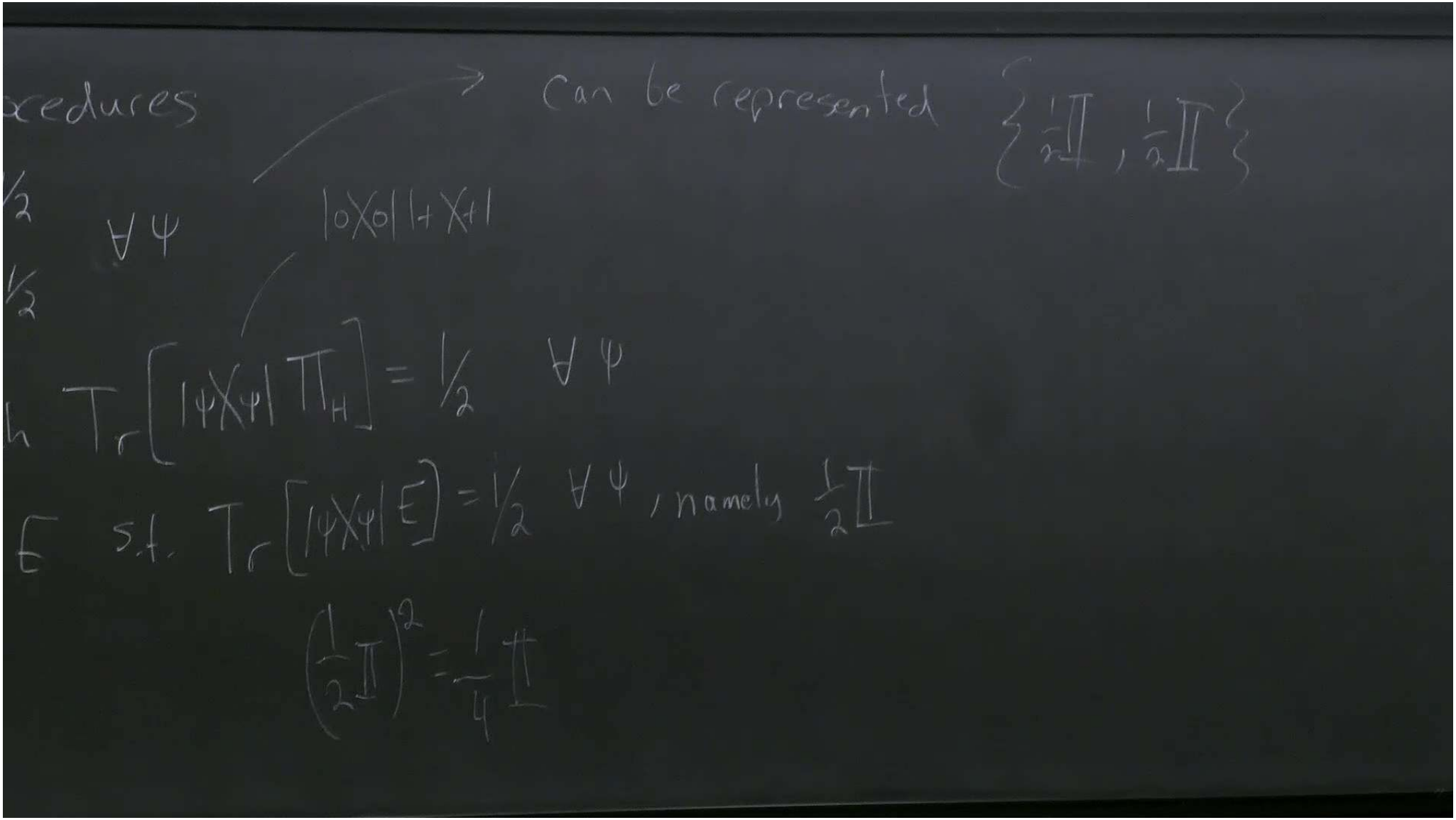
$P(T|\psi) = 1/2$

$|0\rangle\langle 0| \; |+\rangle\langle +|$

there is no projector for which $Tr[|\psi\rangle\langle\psi| \, \Pi_H] = 1/2 \quad \forall \psi$

however, there is an operator E s.t. $Tr[|\psi\rangle\langle\psi| E] = 1/2 \; \forall \psi$, namely $\frac{1}{2}\mathbb{I}$

$\left(\frac{1}{2}\mathbb{I}\right)^2 = \frac{1}{4}\mathbb{I}$


cedures
can be represented
$\{\frac{1}{2}\mathbb{I}, \frac{1}{2}\mathbb{I}\}$
$\frac{1}{2}$ $\forall\psi$ $|0\rangle\langle 0| + |+\rangle\langle +|$
$\frac{1}{2}$
$\mathrm{Tr}\left[|\psi\rangle\langle\psi| \Pi_H\right] = \frac{1}{2}$ $\forall\psi$
E s.t. $\mathrm{Tr}\left[|\psi\rangle\langle\psi| E\right] = \frac{1}{2}$ $\forall\psi$, namely $\frac{1}{2}\mathbb{I}$
$\left(\frac{1}{2}\mathbb{I}\right)^2 = \frac{1}{4}\mathbb{I}$

Ex 21

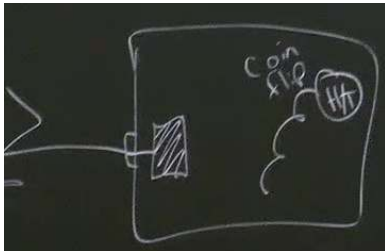
Imagine a noisy mmt $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ with $1-\epsilon$

$\{\mathbb{I}/2, \mathbb{I}/2\}$ with ϵ

$$P(k=0) = (1-\epsilon)\,P(\Pi_0) + \epsilon\,P(\mathbb{I}/2) = (1-\epsilon)\,P(\Pi_0) + \frac{\epsilon}{2}$$

$$P(k=1) = (1-\epsilon)\,P(\Pi_1) + \epsilon/2$$

Claim: effective mmt can be represented as $\left\{(1-\epsilon)|0\rangle\langle 0| + \frac{\epsilon\mathbb{I}}{2}, (1-\epsilon\right.$



$P(H|\psi) = 1/2 \quad \forall \psi$

$P(T|\psi) = 1/2$

$|0\rangle\langle 0| \; |+\rangle\langle +|$

$\mathrm{Tr}\left[|\psi\rangle\langle\psi| \, \Pi_H\right] = 1/2 \quad \forall \psi$

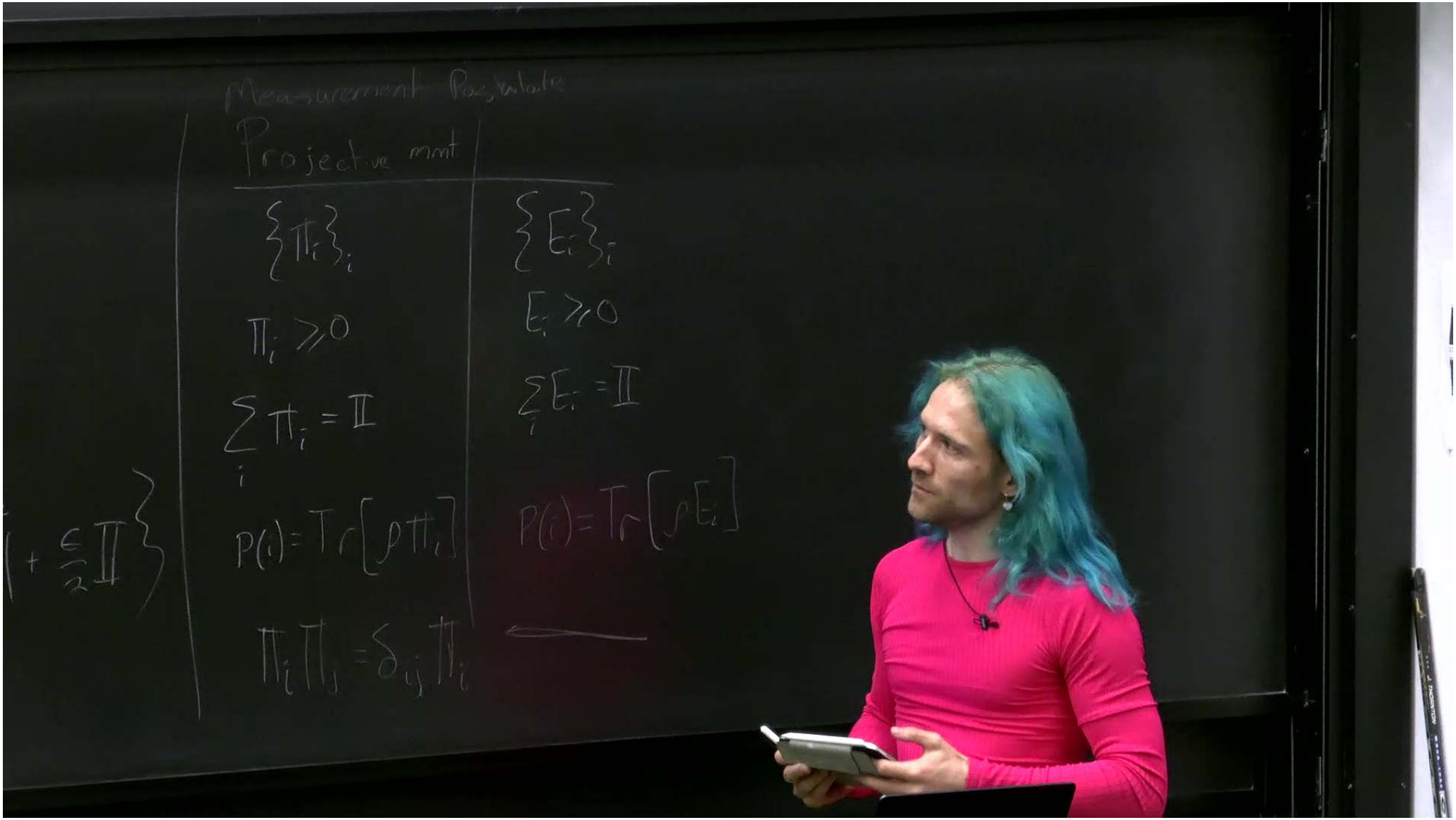
s no projector for which

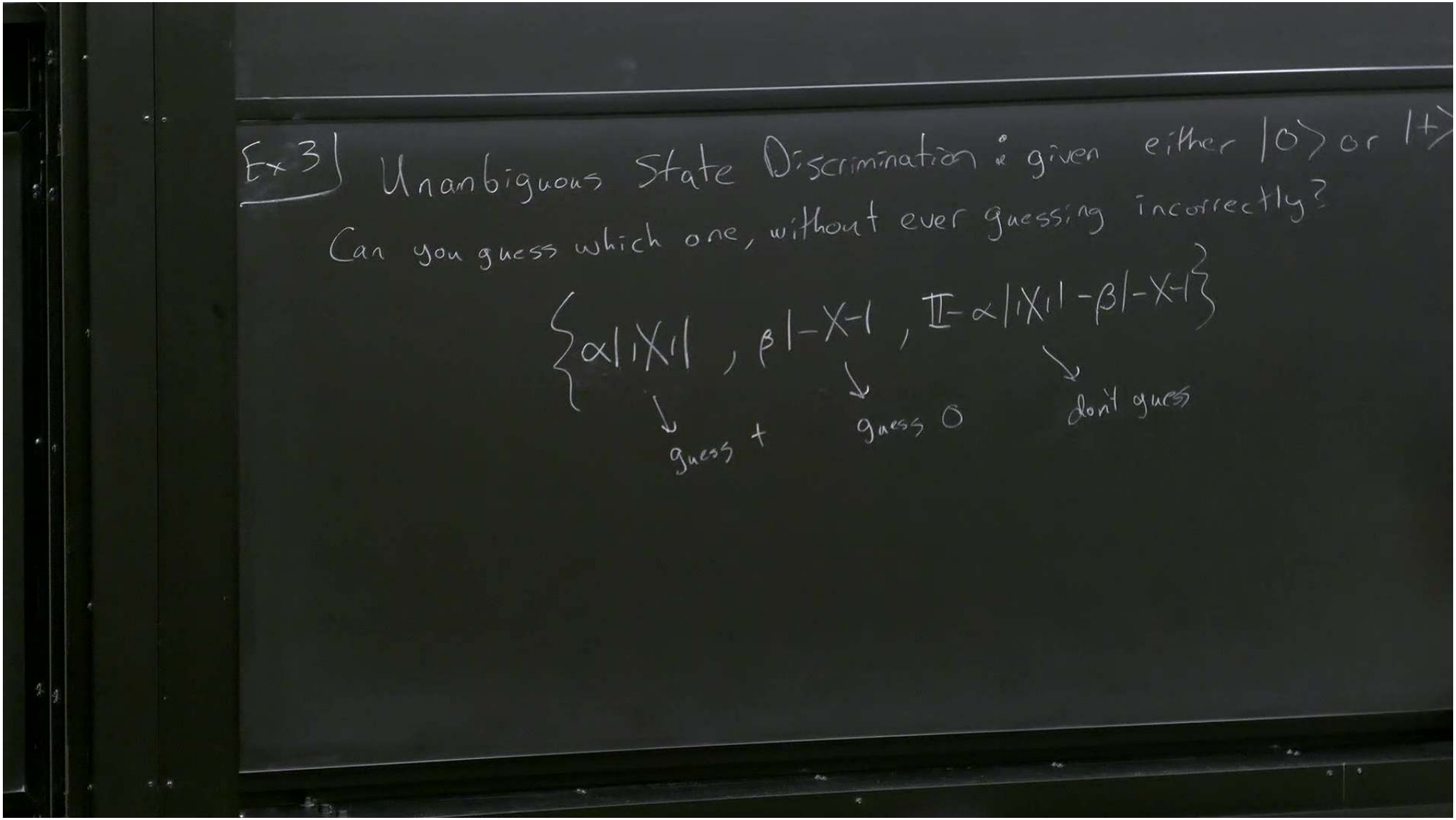
ver, there is an operator E s.t. $\mathrm{Tr}\left[|\psi\rangle\langle\psi| \, E\right] = 1/2 \quad \forall \psi$, namely $\frac{1}{2}\mathbb{I}$

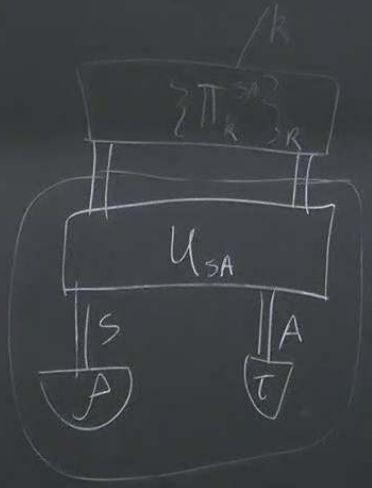
$\left(\frac{1}{2}\mathbb{I}\right)^2 = \frac{1}{4}\mathbb{I}$

$P(k=1) = (1-\epsilon) P(\Pi_1) + \epsilon/2$

be represented as $\left\{ (1-\epsilon)|0\rangle\langle 0| + \frac{\epsilon \mathbb{I}}{2}, \; (1-\epsilon)|1\rangle\langle 1| + \frac{\epsilon}{2}\mathbb{I} \right\}$


Measurement Postulate
Projective msmt
$\{\Pi_i\}_i$
$\Pi_i \geq 0$
$\sum_i \Pi_i = \mathbb{I}$
$P(i) = Tr[\rho \Pi_i]$
$\Pi_i \Pi_j = \delta_{ij} \Pi_i$
$\{E_i\}_i$
$E_i \geq 0$
$\sum_i E_i = \mathbb{I}$
$P(i) = Tr[\rho E_i]$
$+ \frac{\epsilon}{2} \mathbb{I} \}$


Ex 3] Unambiguous State Discrimination: given either |0⟩ or |+⟩
Can you guess which one, without ever guessing incorrectly?
{α|1⟩⟨1| , β|−⟩⟨−| , Π − α|1⟩⟨1| − β|−⟩⟨−|}
guess +
guess 0
don't guess



$$P(k) = Tr_{SA}\left[\Pi_k^{SA}\, U(\rho^S \otimes \tau^A) U^\dagger\right]$$

$$= Tr_{SA}\left[U^\dagger \Pi_k^{SA}\, U(\mathbb{I}^S \otimes \tau^A)(\rho^S \otimes \mathbb{I}^A)\right]$$

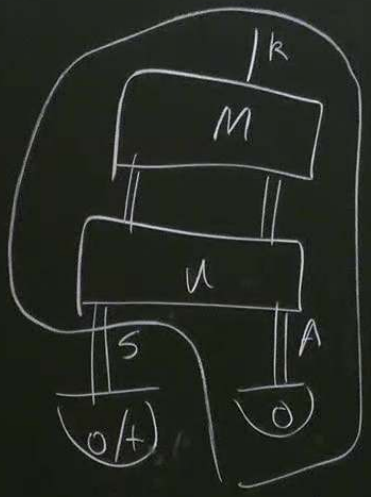
$$= Tr_S\left[Tr_A\left(U_{SA}\, \Pi_k^{SA}\, U_{SA}^\dagger\, \tau^A\right) \rho_S\right]$$

$$=: Tr_S\left[E_k^S\, \rho^S\right]$$

Naimark Dilation Theorem:

For every POVM, there is a way of implementing it by interacting with another system and then doing a projective mmt

dilation of U.S.D.



von Neumann dilation

$$U_{SA}\left(|0\rangle_S|0\rangle_A\right) = \sqrt{p}\,|0\rangle|0\rangle + \sqrt{1-p}\,|0\rangle|1\rangle \qquad \text{normalized if } p = 1$$

$$U_{SA}\left(|1\rangle_S|0\rangle_A\right) = -\sqrt{p}\,|0\rangle|0\rangle + \sqrt{2p}\,|1\rangle|0\rangle + \frac{p}{\sqrt{1-p}}|0\rangle|1\rangle$$

$$U_{SA}\left(|+\rangle_S|0\rangle_A\right) = \frac{1}{\sqrt{2}}\left(U|00\rangle + U|10\rangle\right) = \sqrt{p}\,|10\rangle + \frac{1}{\sqrt{2(1-p)}}|01\rangle$$

by linearity

$$M = \left\{ |00\rangle\langle 00|,\ |10\rangle\langle 10|,\ |11\rangle\langle 11| + |01\rangle\langle 01| \right\}$$

Π_0 Π_+ $\Pi_?$

if state is $|+\rangle$

$P(\Pi_0) = 0$

$$U_{SA}\left(|0\rangle_S|0\rangle_A\right) = \sqrt{p}\,|0\rangle|0\rangle + \sqrt{1-p}\,|0\rangle|1\rangle \qquad \text{normalized if } p = 1-\frac{1}{\sqrt{2}}$$

$$U_{SA}\left(|1\rangle_S|0\rangle_A\right) = -\sqrt{p}\,|0\rangle|0\rangle + \sqrt{2p}\,|1\rangle|0\rangle + \frac{p}{\sqrt{1-p}}|0\rangle|1\rangle$$

$$U_{SA}\left(|+\rangle_S|0\rangle_A\right) = \frac{1}{\sqrt{2}}\left(U|00\rangle + U|10\rangle\right) = \sqrt{p}\,|10\rangle + \frac{1}{\sqrt{2(1-p)}}|01\rangle$$

by linearity

$$M = \left\{ |00\rangle\langle 00|,\ |10\rangle\langle 10|,\ |11\rangle\langle 11| + |01\rangle\langle 01| \right\}$$

Π_0 Π_+ $\Pi_?$

if state is $|+\rangle$

$P(\Pi_0) = 0$ $P(\Pi_+) = p$ $P(\Pi_?) = 1-p$

General Transformation Procedures

"conjugation by U"

3)

$|\psi\rangle \rightarrow U|\psi\rangle$ $\qquad$ $|\psi\rangle\langle\psi| \longrightarrow U|\psi\rangle\langle\psi|U^\dagger$

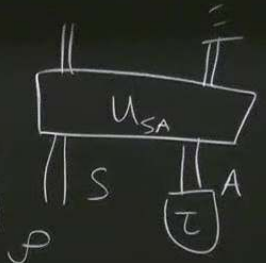
More generally: 1) mixtures of unitaries $\sum_i P(i)\, U_i \rho U_i^\dagger$

2) nondestructive and nonselective msmts $\sum_k P(k) \frac{\Pi_k \rho \Pi_k}{\sqrt{P(k)}\sqrt{P(k)}} = \sum_k \Pi_k \rho \Pi_k$

→ ignore outcome

3) interaction with extra systems

$$T(\rho) = \mathrm{Tr}_A\left[U_{SA}(\rho_S \otimes \tau_A)U_{SA}^{+}\right]$$



Π_R

Transformation Postulate

The general evolution of a density operator is given by a quantum channel / CPTP map $T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$

- linear $T\left(\sum_i a_i \rho_i\right) = \sum_i a_i T(\rho_i)$

- trace-preserving $\mathrm{Tr}(T(\rho)) = \mathrm{Tr}(\rho)$

- completely-positive $T^S \otimes \mathbb{I}^A(\rho^{SA})$ is positive

$T(\rho) \to$ positive