Title: Lecture - Quantum Theory (Core), PHYS 605

Speakers: David Schmid

Collection/Series: Quantum Theory (Core), PHYS 605, September 9 - October 23, 2026

Subject: Particle Physics, Quantum Fields and Strings, Quantum Information

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David Schmid (office hours Mon 4PM
Fri 5PM)
Dan Wohns
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First tutorial: Fri 2PM

Hilbert Space: vector space with an inner product 2)

$\mathcal{H} := (\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ s.t.

1) closed under vector addition and scalar multiplication

so if $c_i \in \mathbb{C}$
$v_i \in \mathcal{H}$ $\Rightarrow \sum_i c_i v_i \in \mathcal{H}$

induced norm

$$\|v\| := \sqrt{\langle v, v \rangle}$$

2) $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ satisfying

- $\langle u, v \rangle = \langle v, u \rangle^*$ Conjugate Symmetry
- $\langle v, v \rangle \geq 0$ positive definiteness (equality iff $v = 0$)
- $\langle v, \alpha u + \beta w \rangle = \alpha \langle v, u \rangle + \beta \langle v, w \rangle$ linearity (in the second input)

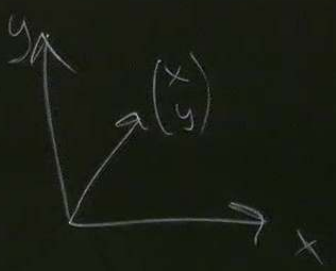
ex) $d=2$ $\mathcal{H}_2 \cong \mathbb{C}^2 = \{(\alpha,\beta)\}_{\alpha,\beta\in\mathbb{C}}$

inner product $\langle(\alpha_1,\alpha_2),(\beta_1,\beta_2)\rangle = \alpha_1^*\beta_1 + \alpha_2^*\beta_2$

$\Rightarrow$ norm $\|(\alpha,\beta)\| = \sqrt{|\alpha|^2+|\beta|^2}$

column vector $v = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

Intuition: Cartesian Vector space $\mathbb{R}^2$



two level atom

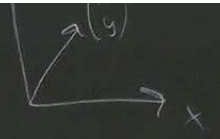
$-e_1 \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix} := |0\rangle$

$-e_0 \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix} := |1\rangle$

any state ψ in $\mathcal{H}_2$ can be written as $\alpha e_0 + \beta e_1 = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

but we could also use $e_\pm := \frac{e_0 \pm e_1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix} =: |\pm\rangle$

vector space $\mathbb{R}$



$A \in \mathcal{L}(\mathcal{H})$ $\quad A: \mathcal{H} \to \mathcal{H}$ s.t. $A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i |Av_i\rangle$

from linearity, can specify A fully by giving its action on a basis

$A|0\rangle = \alpha|0\rangle + \beta|1\rangle$
$A|1\rangle = \gamma|0\rangle + \delta|1\rangle$

then its action on any $|\psi\rangle = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \omega_1|0\rangle + \omega_2|1\rangle$ is fully determined

$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$

eg) $X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

$\Rightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Dual vectors and Inner products

↳ maps from $\mathcal{H} \to \mathbb{C}$

can represent as row vectors, since $(\gamma, \delta)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \gamma\alpha + \beta\delta \in \mathbb{C}$

or as "bra vectors" $\underset{\text{ket}}{|\psi\rangle} \overset{\text{dual}}{\longleftrightarrow} \underset{\text{bra}}{\langle\psi|}$

were the action of a bra on a ket is $\langle v|w\rangle := \langle v, w\rangle$

e.g. $\langle v|v\rangle = \langle(\alpha,\beta),(\alpha,\beta)\rangle = |\alpha|^2 + |\beta|^2 = (\alpha^*, \beta^*)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

y determined

$\mathcal{H}^*$ ≡ dual Hilbert space, of dual vectors

$|v\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longleftrightarrow \langle v| = (\alpha^* \; \beta^*)$

e.g., define $|w\rangle\langle v|$ to be the linear operator from Z to W

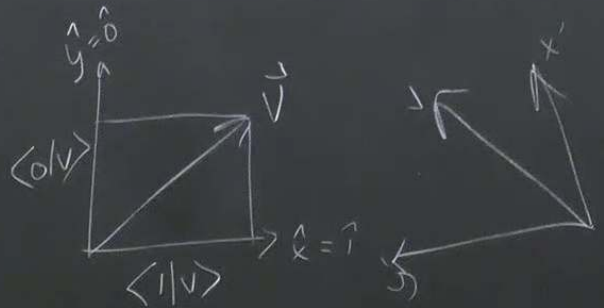
whose action is defined by $(|w\rangle\langle v|)(|z\rangle) := |w\rangle\langle v|z\rangle = \langle v|z\rangle |w\rangle$
$= |w\rangle\langle v|z\rangle$

formally, $|w\rangle\langle v|$ corresponds to the matrix outer product

e.g. $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} w_1 v_1 & w_1 v_2 \\ w_2 v_1 & w_2 v_2 \end{bmatrix}$

Identity operator $\mathbb{I}\,|v\rangle = |v\rangle \;\forall v$ as a Matrix $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

In bra-ket form $\sum_i |i\rangle\langle i|$ where $\{|i\rangle\}_i$ is any ON basis



$$\vec{v} = \langle 1|v\rangle\,|1\rangle + \langle 0|v\rangle\,|0\rangle$$

$$= |1\rangle\langle 1|v\rangle + |0\rangle\langle 0|v\rangle$$

$$= \left(|1\rangle\langle 1| + |0\rangle\langle 0|\right)|v\rangle$$

Linear operators on $\mathcal{H}$

$A \in \mathcal{L}(\mathcal{H})$ $\quad A: \mathcal{H} \to \mathcal{H}$ s.t. $A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i \left(A|v_i\rangle\right)$

from linearity, can specify A fully by giving its action on a basis

$A|0\rangle = \alpha|0\rangle + \beta|1\rangle$

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then its action on any $|\psi\rangle = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \omega_1|0\rangle + \omega_2$

$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ $\qquad \langle 1|X|0\rangle$

eg) $X|0\rangle = |1\rangle$

$X|1\rangle = |0\rangle$

$\Rightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Using this, we can represent operator A as

$$A = \mathbb{I} A \mathbb{I} = \left(\sum_i |i\rangle\langle i|\right) A \left(\sum_j |j\rangle\langle j|\right) = \sum_{ij} |i\rangle\langle i|$$

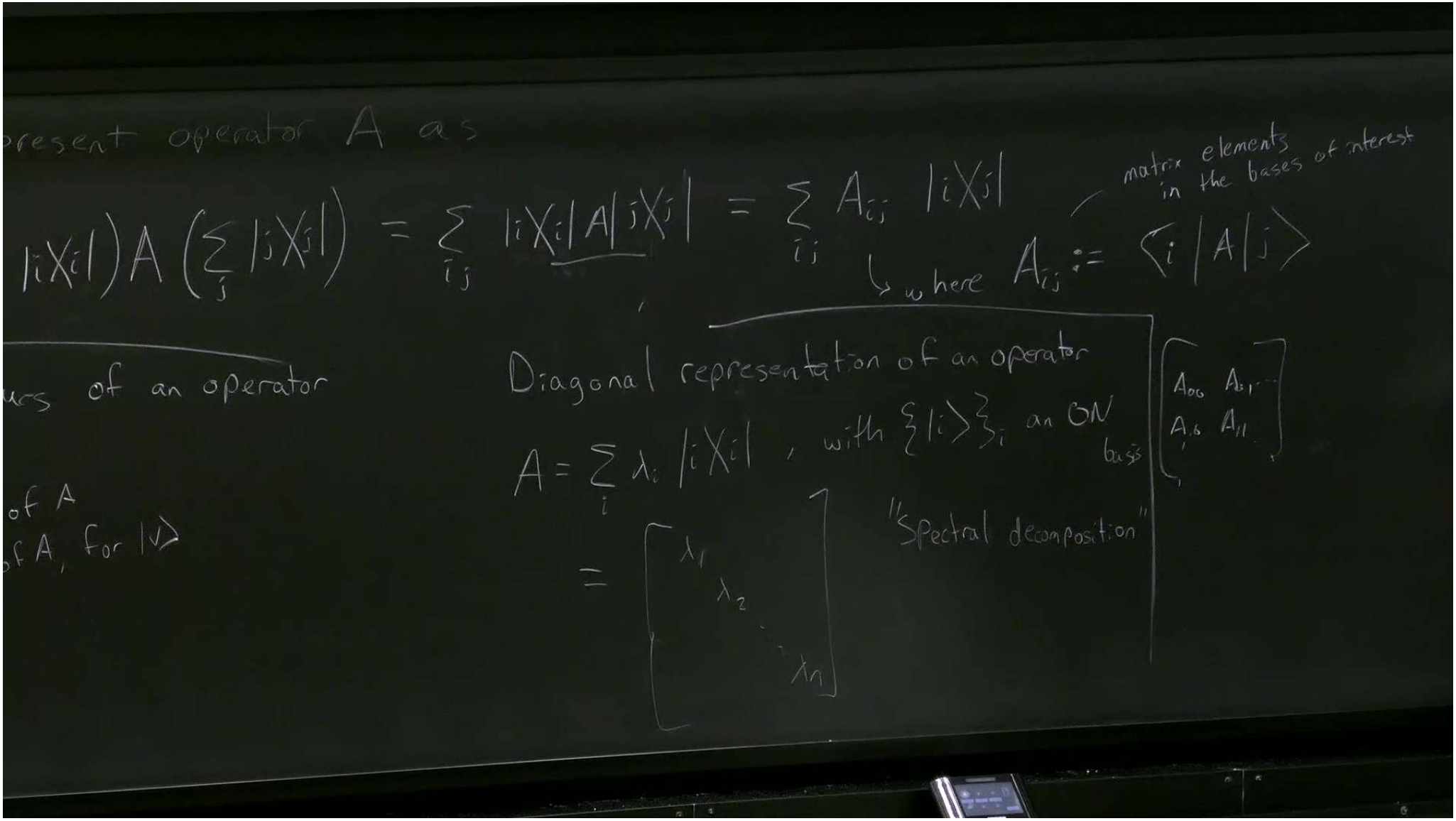
Eigenvectors and eigenvalues of an operator

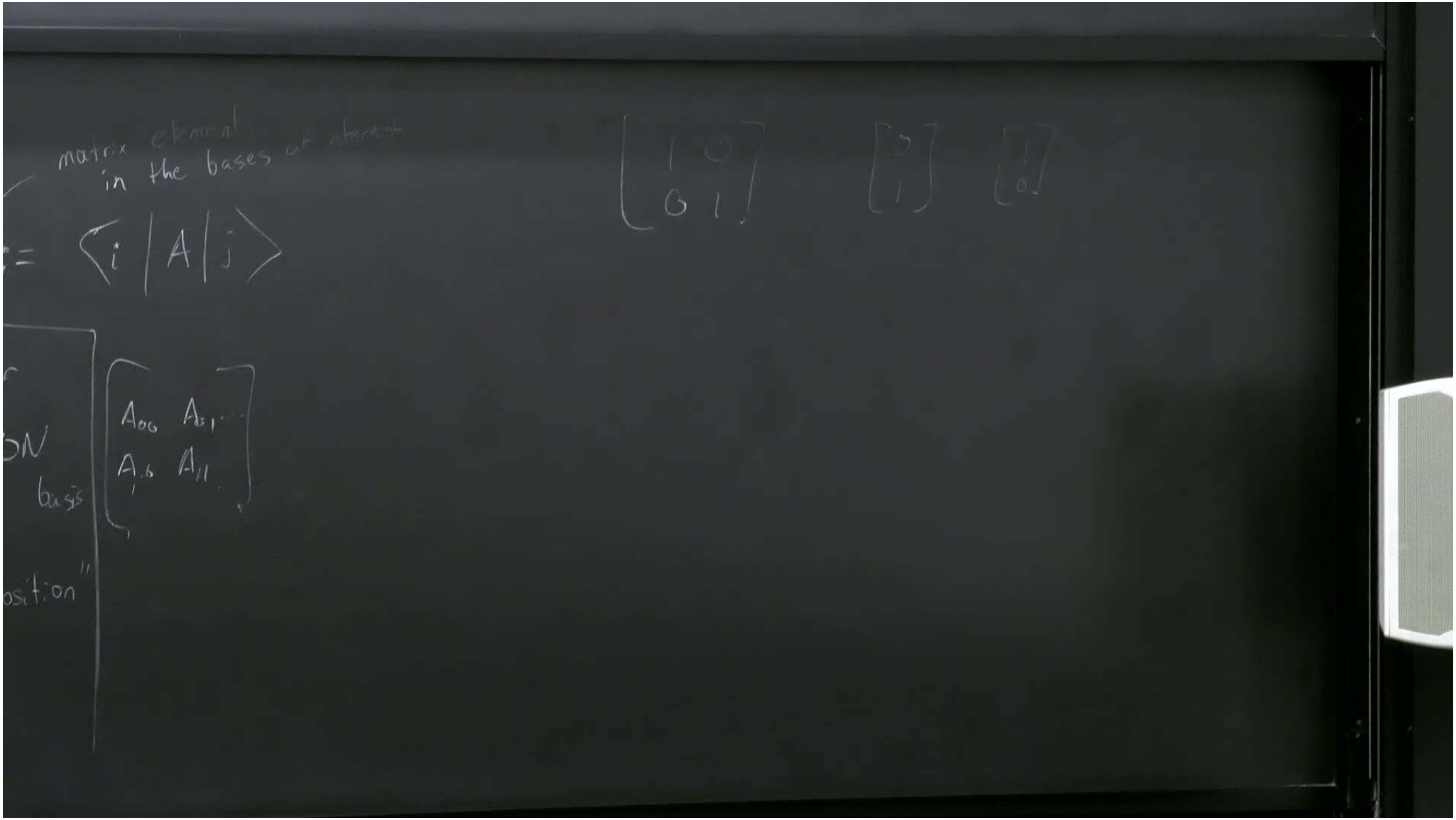
Diag

$$A|v\rangle = \lambda_v |v\rangle$$

($|v\rangle$: e-vector of A; λ_v: e-value of A, for $|v\rangle$)

$$A|v\rangle = v|v\rangle$$


resent operator A as
$\left(\sum_i |i\rangle\langle i|\right) A \left(\sum_j |j\rangle\langle j|\right) = \sum_{ij} |i\rangle\langle i|A|j\rangle\langle j| = \sum_{ij} A_{ij} |i\rangle\langle j|$
matrix elements in the bases of interest
where $A_{ij} := \langle i|A|j\rangle$
urs of an operator
of A
of A, for $|v\rangle$
Diagonal representation of an operator
$A = \sum_i \lambda_i |i\rangle\langle i|$, with $\{|i\rangle\}_i$ an ON basis
$= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
"Spectral decomposition"
$\begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$


matrix elements
in the bases of interest
= ⟨i|A|j⟩
basis
position"

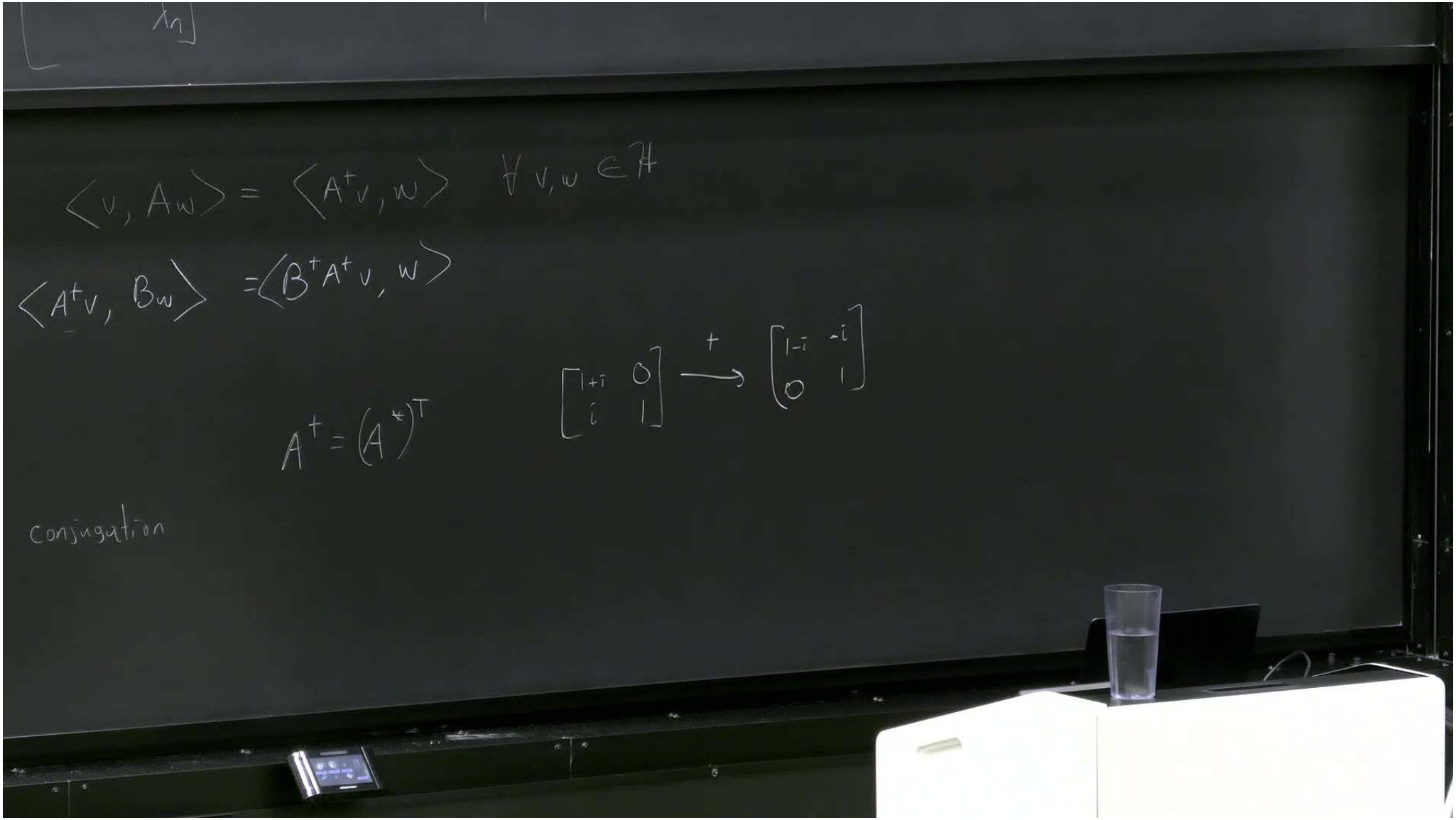
Adjoints and Hermitian Operators

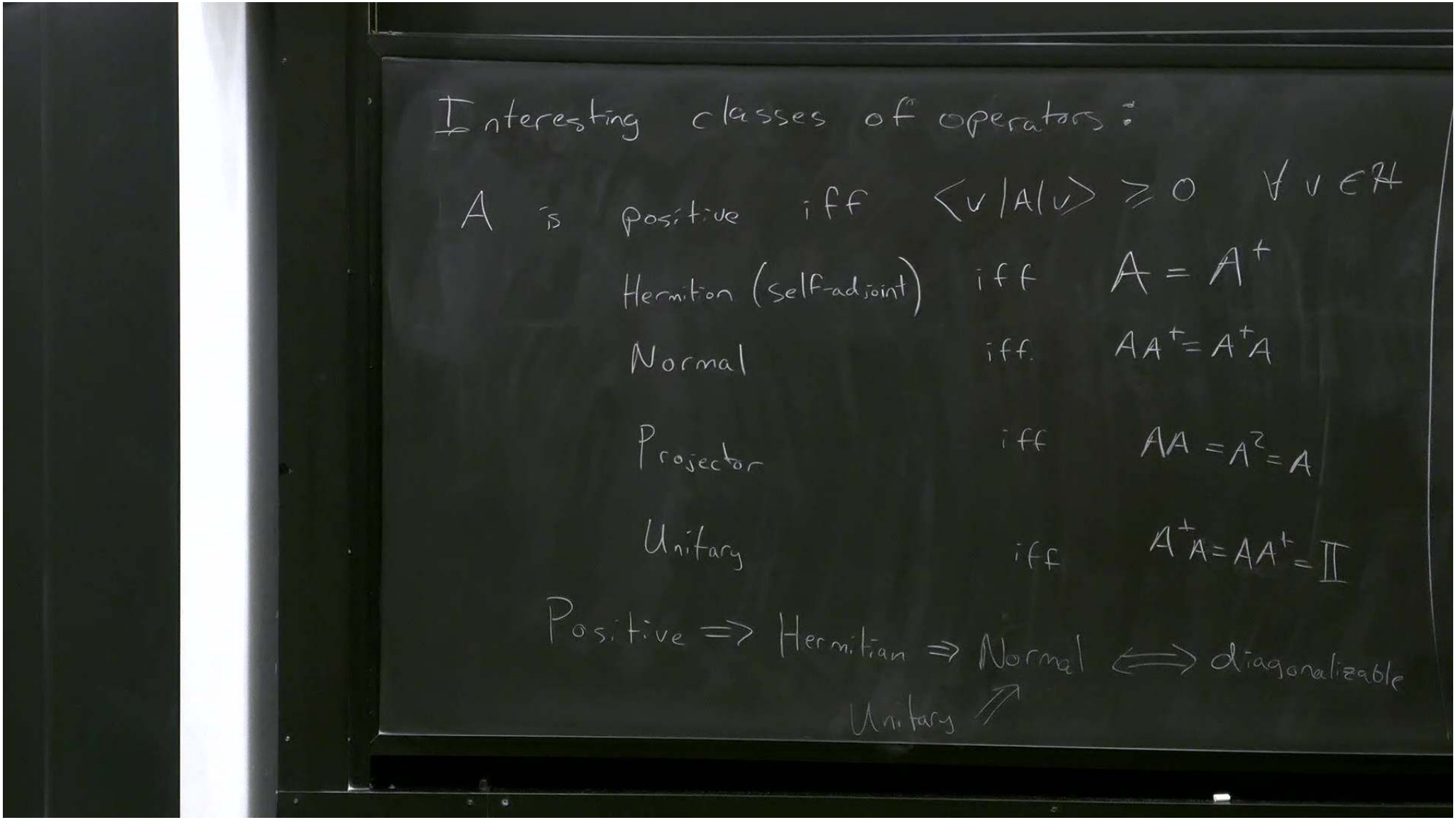
for $A \in \mathcal{L}(\mathcal{H})$, there is a unique linear operator $A^\dagger$ s.t. $\langle v, Aw \rangle = \langle A^\dagger v$

Clearly $(AB)^\dagger = B^\dagger A^\dagger$ since $\langle v, A(Bw) \rangle = \langle A^\dagger v, Bw \rangle = \langle B^\dagger A^\dagger v$

We take $|v\rangle^\dagger := \langle v|$, so that $(A|v\rangle)^\dagger = \langle v|A^\dagger$

In matrix representation, this is transpose + entry-wise Complex conjugation


$\langle v, Aw \rangle = \langle A^\dagger v, w \rangle \quad \forall v, w \in \mathcal{H}$
$\langle A^\dagger v, Bw \rangle = \langle B^\dagger A^\dagger v, w \rangle$
$A^\dagger = (A^*)^T$
$\begin{bmatrix} 1+i & 0 \\ i & 1 \end{bmatrix} \xrightarrow{\dagger} \begin{bmatrix} 1-i & -i \\ 0 & 1 \end{bmatrix}$
conjugation


Interesting classes of operators:
A is positive iff $\langle v|A|v\rangle \geq 0$ $\forall v \in \mathcal{H}$
Hermitian (self-adjoint) iff $A = A^\dagger$
Normal iff $AA^\dagger = A^\dagger A$
Projector iff $AA = A^2 = A$
Unitary iff $A^\dagger A = AA^\dagger = \mathbb{I}$
Positive ⇒ Hermitian ⇒ Normal ⟺ diagonalizable
Unitary

Unitary operators preserve inner products

$$\langle Uv, Uw \rangle = \langle U^\dagger U v, w \rangle = \langle v, w \rangle$$

$$\langle Uv | Uw \rangle = \langle v | U^\dagger U | w \rangle = \langle v | w \rangle$$

now, take $\{ |w_i\rangle \}_i$ an ON basis.

then $\{ |v_i\rangle := U|w_i\rangle \}_i$

hence $U = U\mathbb{I} = U\left(\sum_i |w_i\rangle\langle w_i|\right) = \sum_i |v_i\rangle\langle w_i|$

Composite systems and tensor products

Consider $\mathcal{H}_m$ with basis $\{|i\rangle\}_{i=1}^{m}$

$\mathcal{H}_n$ with basis $\{|j\rangle\}_{j=1}^{n}$

notation

$|i\rangle|j\rangle = |i,j\rangle = |ij\rangle$

then $\mathcal{H}_{mn} = \mathcal{H}_m \otimes \mathcal{H}_n$ and a basis for $\mathcal{H}_{mn}$ is $\{|i\rangle \otimes |j\rangle\}_{i,j}$

Formally, $\otimes$ is a map from two Hilb. spaces to a larger Hilb space together with a bilinear map $\mathcal{H}_m \times \mathcal{H}_n \to \mathcal{H}_{mn}$, in particular, taking $(|i\rangle, |j\rangle)$

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Can define $A \otimes B$ via $(A \otimes B)(|v\rangle \otimes |w\rangle) := A|v\rangle \otimes B|v\rangle$

$$\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

- the inner product for $\mathcal{H}_{mn}$ is defined in the natural way from this

$$\left\langle \sum_i a_i |v_i\rangle \otimes |w_i\rangle, \sum_j b_j |v_j'\rangle \otimes |w_j'\rangle \right\rangle = \sum_{ij} a_i^* b_j \langle v_i | v_j' \rangle \langle w_i | w_j' \rangle$$

in matrix representation, the tensor product is $A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \\ \vdots & & \end{bmatrix}$

$|j\rangle)$ to the particular vector $|i\rangle \otimes |j\rangle \in \mathcal{H}_{mn}$

$$X \otimes Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & 1 \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ 1 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} \end{bmatrix} \end{bmatrix}$$

Operator functions

given $f: \mathbb{C} \to \mathbb{C}$ and an operator $A = \sum_a a\,|a\rangle\langle a| \in \mathcal{L}(\mathcal{H})$

define $f(A) := \sum_a f(a)\,|a\rangle\langle a|$

ex) $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad e^{\theta Z} = \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}$ $\quad \sqrt{Z} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Trace of A $\quad Tr[A] = \sum_i A_{ii} = \sum_i \langle a_i|A|a_i\rangle$ where $\{|a_i\rangle\}_i$ is any ON basis

this is linear and cyclic : $Tr(AB) = Tr(BA) \qquad Tr(ABC) = Tr(CAB)$

$Tr[UAU^\dagger] = Tr[U^\dagger UA] = Tr[A]$

$Tr[A|v\rangle\langle v|] = Tr[\langle v|A|v\rangle] = \langle v|A|v\rangle$

Commutator $[A,B] = AB - BA$ "A commutes with B" iff $[A,B] = 0$ iff both are simultaneously diagonalizable

Anticommutator $\{A,B\} = AB + BA$

So that $A = \sum_i a_i |i\rangle\langle i|$ and $B = \sum_i b_i |i\rangle\langle i|$