Title: Lecture - Classical Physics, PHYS 612

Speakers: Aldo Riello

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Recap

config sp. $Q \ni q$

tangent bundle $TQ \ni (q, v)$

histories on config sp. $\Gamma_Q = C^\infty(\mathbb{R}, Q) \ni \xi$
($\mathbb{R}$: time)

Lagrangian f. $L: \mathbb{R} \times TQ \to \mathbb{R}$
$(t, (q, v)) \mapsto L(q, v, t)$

Action $S: \mathbb{R} \times \mathbb{R} \times \Gamma_Q \to \mathbb{R}$
$(t_0, t_1, \xi) \mapsto S(t_0, t_1, \xi) = \int_{t_0}^{t_1} L \circ \xi$

Hamiltonian Mech

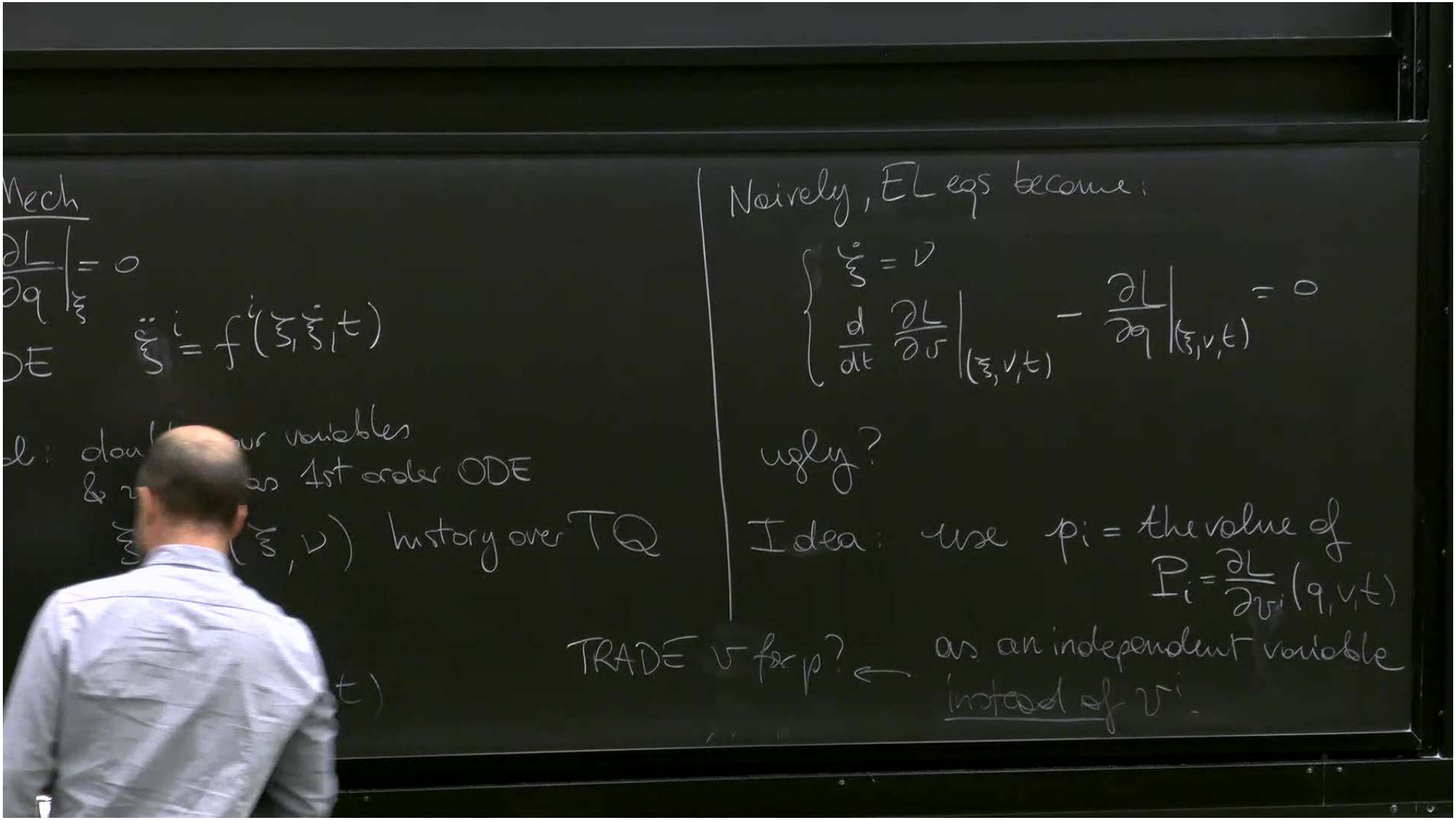
$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}\Big|_{\xi}\right] - \frac{\partial L}{\partial q}\Big|_{\xi} = 0$$

2nd order ODE $\quad \ddot{\xi}^i = f^i(\xi, \dot{\xi}, t)$

Recipe for sol: double your variables & rewrite as 1st order ODE

$\xi \rightsquigarrow (\xi, v)$ history over TQ

$$\begin{cases} \dot{\xi}^i = v^i \\ \dot{v}^i = f^i(\xi, v, t) \end{cases}$$

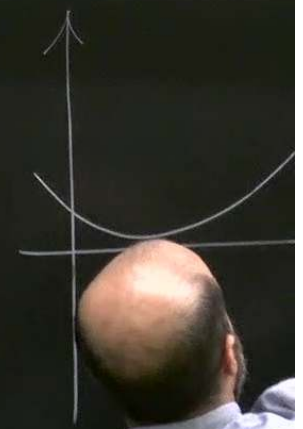

Mech
$\frac{\partial L}{\partial q}\Big|_{\xi} = 0$
$\ddot{\xi}^i = f^i(\xi, \dot{\xi}, t)$
DE
1st order ODE
history over TQ
Naively, EL eqs become:
$\begin{cases} \dot{\xi} = v \\ \frac{d}{dt}\frac{\partial L}{\partial v}\Big|_{(\xi,v,t)} - \frac{\partial L}{\partial q}\Big|_{(\xi,v,t)} = 0 \end{cases}$
ugly?
Idea: use p_i = the value of $P_i = \frac{\partial L}{\partial v^i}(q,v,t)$
TRADE v for p? ← as an independent variable instead of v.

$(t_0, t_1, \xi) \mapsto S(t_0, t_1, \xi) = \int_{t_0}^{t_1} L_\xi$

Aside: the Legendre transform.

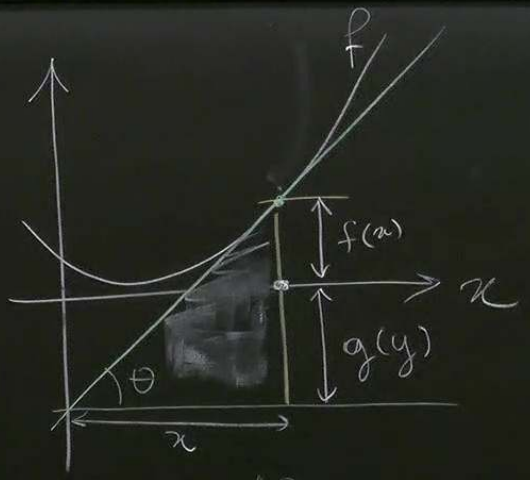
What I have: $x \longmapsto f(x)$

What I want: $y \longmapsto g(y)$ where $y = \frac{df}{dx}$ which contains "the same information of $x \mapsto f(x)$"



Rmk for this to work $x \mapsto y = \frac{df}{dx}$ must be invertible, i.e. monotonic, i.e. $\frac{d^2 f}{dx^2} > 0$ (or < 0)

i.e. f (or $-f$) is convex



$$\tan\theta = \frac{df}{dx} = y$$

$$y(x) := \frac{df}{dx} \overset{\text{inverse}}{\longleftrightarrow} x(y)$$

Define $g(y)$, the Legendre transform of $f(x)$

as

$$g(y) := y\,x(y) - f(x(y))$$

Reshuffling: $xy = f + g$

Rmk: this formula is "symmetric" under $(x, f) \leftrightarrow (y, g)$

Suggest: Legendre transform is an involution ($\mathcal{L}^2 = \mathrm{id}$) ↷ "same in"

invertible, i.e. monotonic, i.e. $\frac{d^2 f}{dx^2} > 0$ (or < 0)

i.e. $f(\sigma - f)$ is convex

$y(x) := \frac{df}{dx}$

check

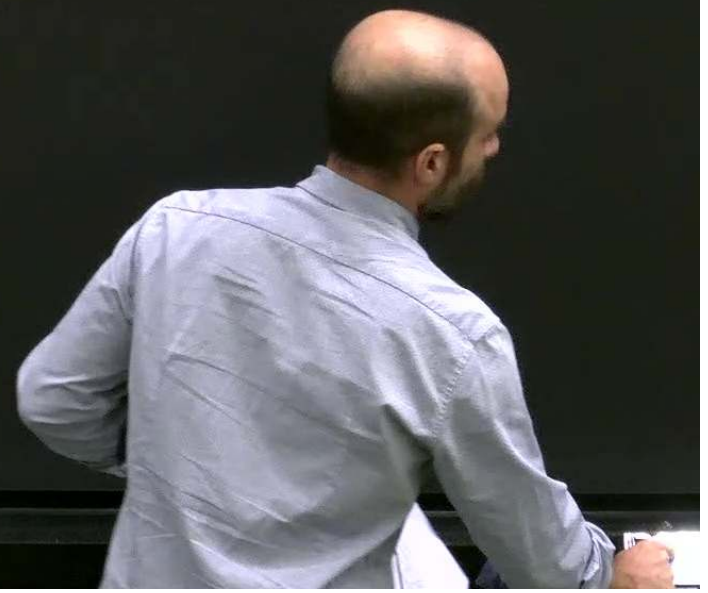
$$x \overset{?}{=} \frac{dg}{dy} = \frac{d}{dy}\left(x(y)\, y - f(x(y))\right)$$

$$= \frac{dx}{dy}\, y + x(y) - \underbrace{\left.\frac{df}{dx}\right|_{x(y)}}_{=y} \frac{dx}{dy}$$

$$= x(y) \qquad \blacksquare$$

Rmk $\mathcal{L}(f)(y) \neq f(x(y))$

the extra term is crucial for $\mathcal{L}^2 = id$.



i.e. $\frac{d^2 f}{dx^2} > 0$ (or < 0)

wex

$y(x) := \frac{df}{dx} \longleftrightarrow x(y)$

Suggest: Legendre tran
an involution

$\frac{dx}{dy}$

$\mathcal{L}^2 = 101.$

Back to physics

$(q, v, t) \longmapsto L(q, v, t)$

$P = \frac{\partial L}{\partial v}$

v ↑ x ; P ↑ y

(q&t : spectators)

Legendre transform of L wrt v is Hamiltonian

$$H(q, p, t) = p\, V(q, p, t) - L(q, V(q, p, t), t)$$

where $v = V(q, p, t)$ iff $p = P(q, v, t) \equiv \frac{\partial L}{\partial v}$

Rmk

if $\partial_t L = 0$ then E

$H(q, p) = E(q, V(q$

The Hamiltonian is

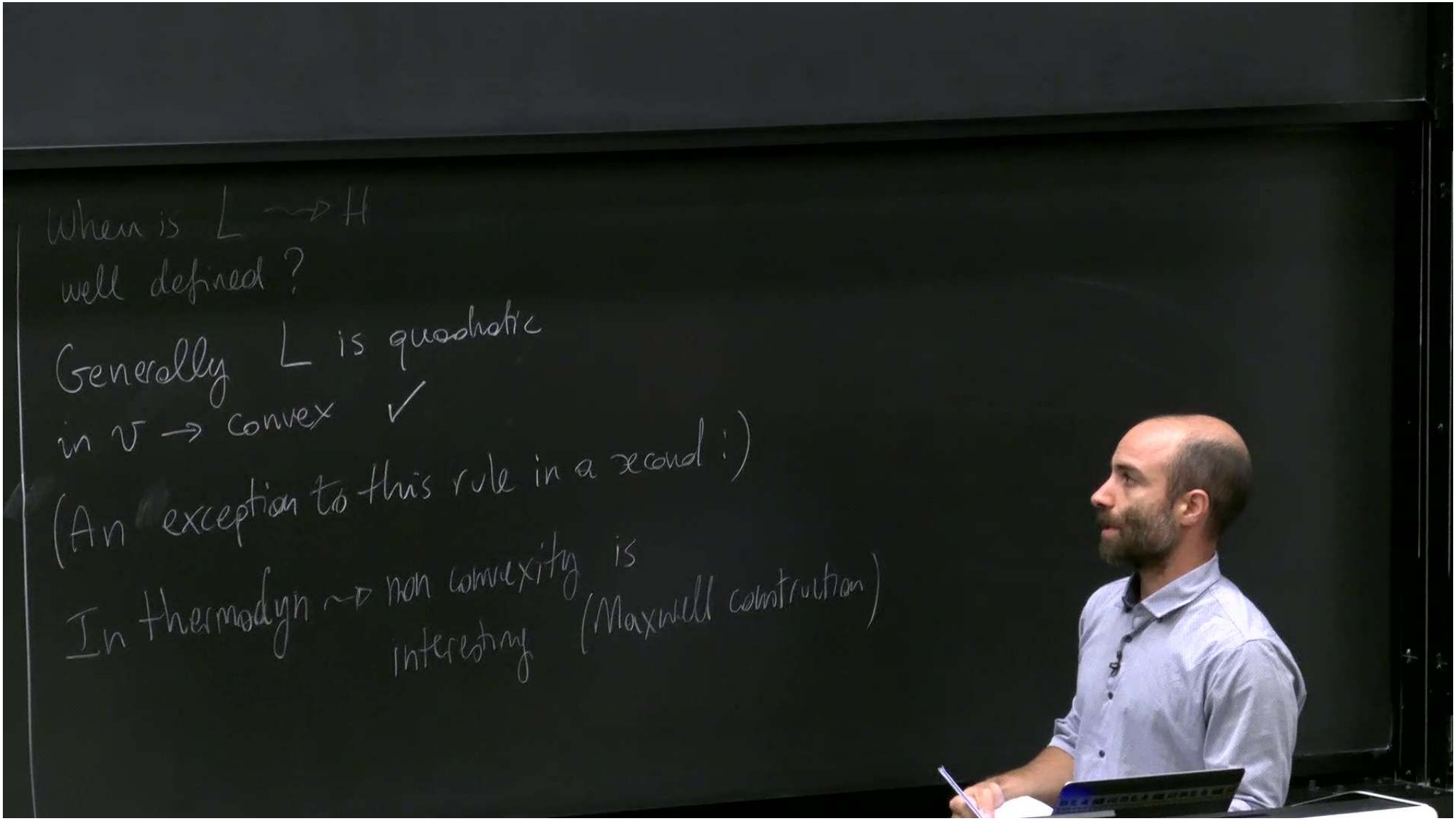
Ex 1

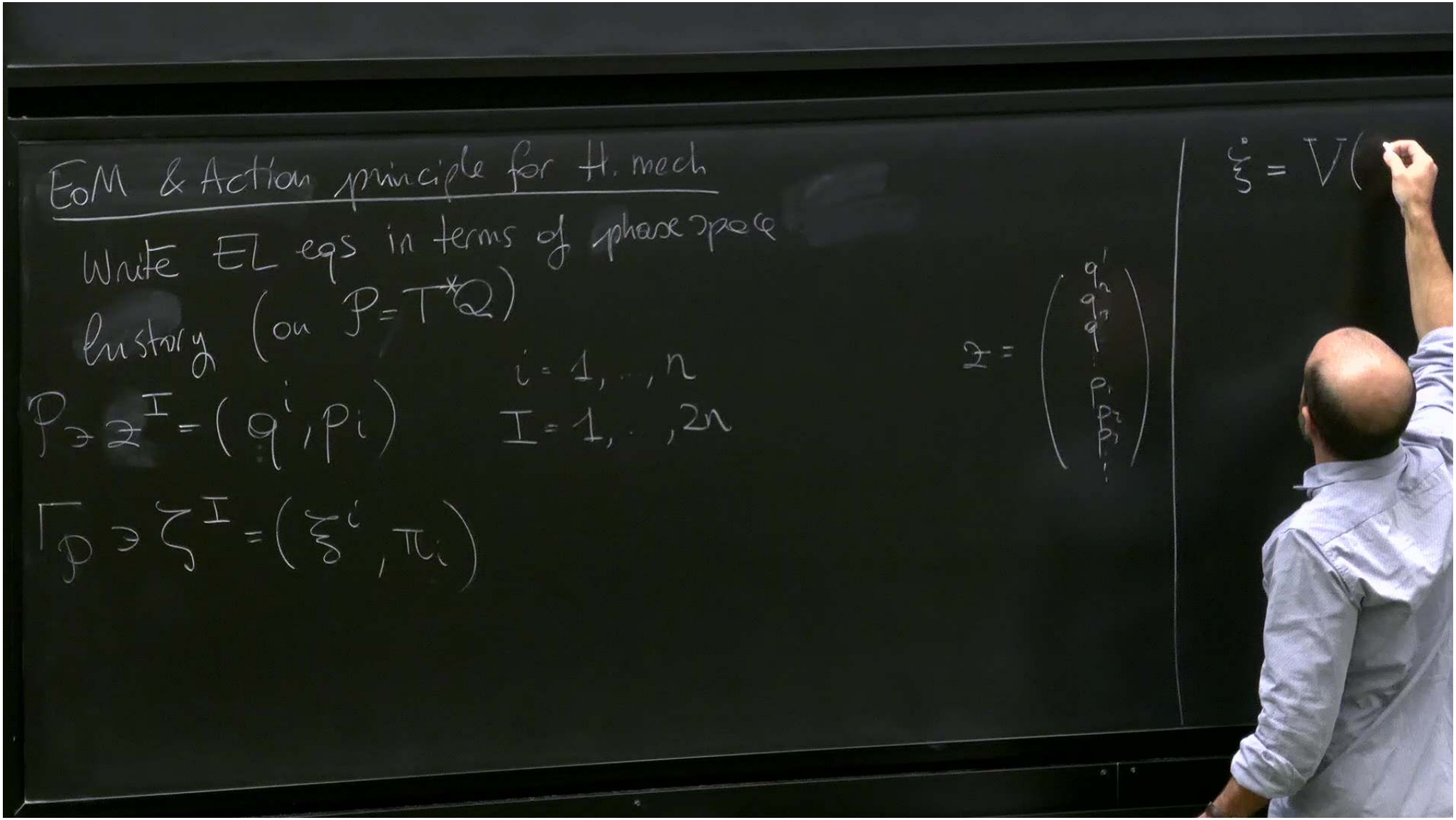
$$L = \frac{1}{2} m v^2 - \phi(q) - e A(q) v$$

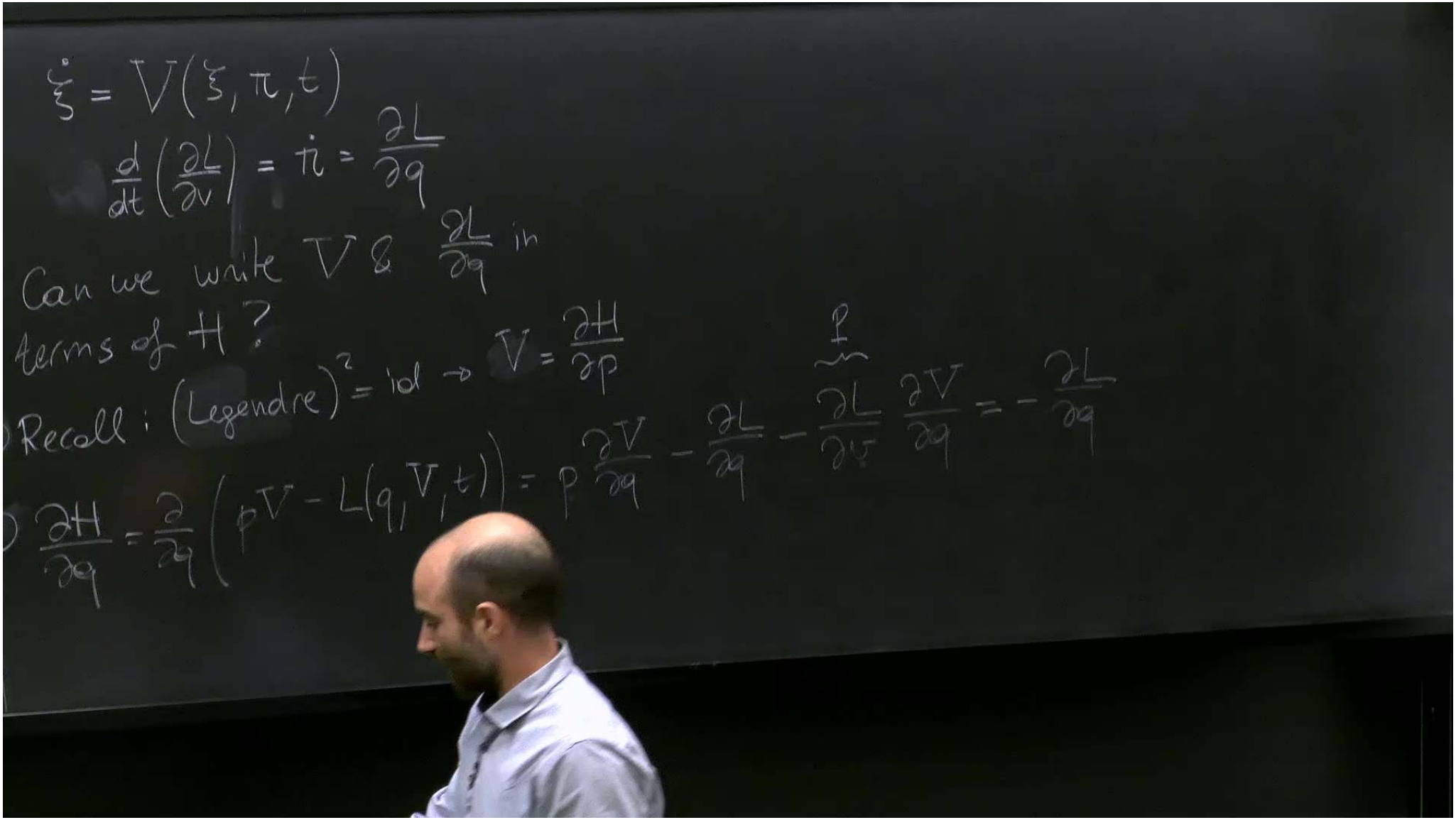
$$P(q,v) = \frac{\partial L}{\partial v} = mv - eA(q) \rightarrow v = V(q,p) = \frac{1}{m}\left(p + eA(q)\right)$$

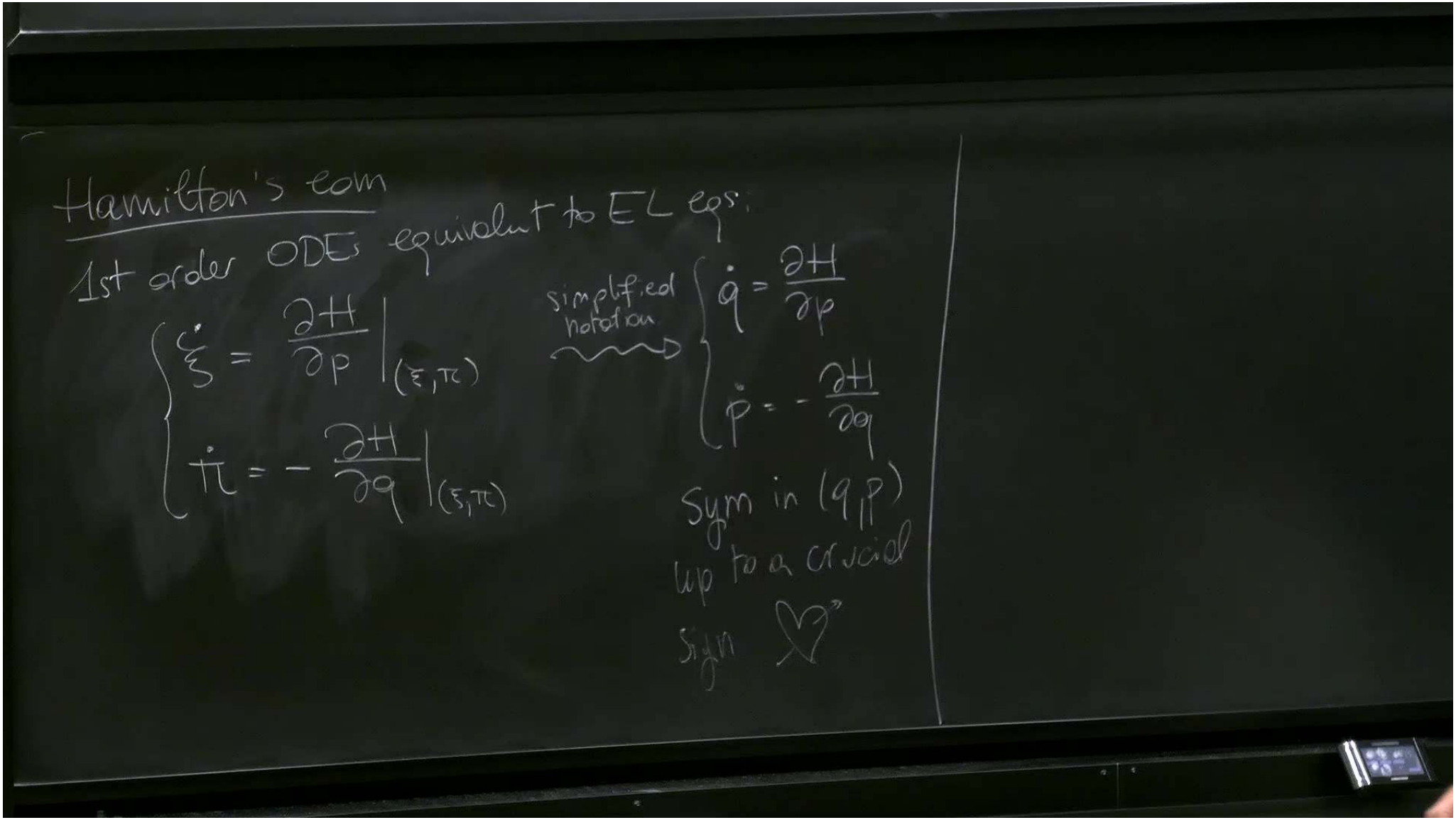
$$H = pV(q,p) - L(q,v) = p\frac{(p+eA)}{m} - \frac{1}{2} m \left(\frac{p+A}{m}\right)^2 + \phi(q) + eA\left(\frac{p+A}{m}\right)$$

$$= \frac{1}{2m}\left(p+eA\right)^2 + \phi(q)$$


When is L ⇝ H
well defined ?
Generally L is quasiconvex
in v → convex ✓
(An exception to this rule in a second :)
In thermodyn ⇝ non convexity is
interesting (Maxwell construction)


EoM & Action principle for H. mech
Write EL eqs in terms of phase space
history (on $\mathcal{P}=T^*Q$)
$i=1,\dots,n$
$\mathcal{P}\ni z^I=(q^i,p_i)$ $I=1,\dots,2n$
$\Gamma_{\mathcal{P}}\ni\zeta^I=(\xi^i,\pi_i)$
$z=$
$\dot{\xi}=V($


$\dot{\xi} = V(\xi, \pi, t)$
$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \dot{\pi} = \frac{\partial L}{\partial q}$
Can we write V & $\frac{\partial L}{\partial q}$ in terms of H?
Recall: (Legendre)^2 = id $\rightarrow$ $V = \frac{\partial H}{\partial p}$
$\frac{\partial H}{\partial q} = \frac{\partial}{\partial q}\left(pV - L(q, V, t)\right) = p\frac{\partial V}{\partial q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial v}\frac{\partial V}{\partial q} \overset{?}{=} -\frac{\partial L}{\partial q}$


Hamilton's eom
1st order ODEs equivalent to EL eqs:
$\begin{cases} \dot{\xi} = \frac{\partial H}{\partial p}\Big|_{(\xi,\pi)} \\ \dot{\pi} = -\frac{\partial H}{\partial q}\Big|_{(\xi,\pi)} \end{cases}$
simplified notation
$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$
Sym in (q,p)
up to a crucial
sign

Roughly: $\dot{z} = \text{sgrad}(H)$ — geometric; "symplectic"

Stay tuned!

$q \& p$ $\quad v \equiv q \& p$

$TP \ni (\mathcal{Z}^I, W^I)$ $\quad 4n$-dim

Define the first order Lagrangian over

$$\boxed{L_{1st}(z, w, t) = p v - H(p, q, t)}$$

⚠ we are treating $P = T^*Q$ as our config space in this Lagrangian

$$S_{1st}(t_0, t_1, \zeta) = \int_{t_0}^{t_1} dt\, L_{1st}(\zeta, \dot{\zeta}, t)$$

$\zeta^I = (\xi^i, \pi_i)$, $\dot{\zeta} = (\dot{\xi}^i, \dot{\pi}_i)$

$$\delta S_{1st} = \int \delta\pi\,\dot{\xi} + \pi\frac{d}{dt}\delta\xi - \frac{\partial H}{\partial q}\delta\xi - \frac{\partial H}{\partial p}\delta\pi$$

Rmk no velocities in $+l$!

$$= \int \delta\pi\left(\dot{\xi} - \frac{\partial H}{\partial p}\Big|_\xi\right) - \delta\xi\left(\dot{\pi} + \frac{\partial H}{\partial q}\Big|_\xi\right) + \Big[-\pi\delta\xi\Big]_{t_0}^{t_1}$$

EoM of S_{1st} $\rightarrow$ $\begin{cases} \dot{\xi} = \frac{\partial H}{\partial p} \\ \dot{\pi} = -\frac{\partial H}{\partial q} \end{cases}$ ✓

Boundary conds $\delta\xi(t_0) = \delta\xi(t_1) = 0$

only on ξ & not π ! $\rightarrow$ Not a bdry cond on entirety of $\zeta = (\xi, \pi)$

suggest: Legendre transform is an involution ($\mathcal{L}^2 = id$)

$p = \frac{\partial L}{\partial v}$

y

Rmk if $\partial_t L = 0$ then $E(q,v) := P(q,v)\,v - L(q,v)$ is the conserved generalized energy

$H(q,p) = E(q, V(q,p)) \iff E(q,v) = H(q, P(q,v))$

dual

The Hamiltonian is a f. of $(q,p) \in \mathcal{P} := T^*Q$

cotangent bundle of Q

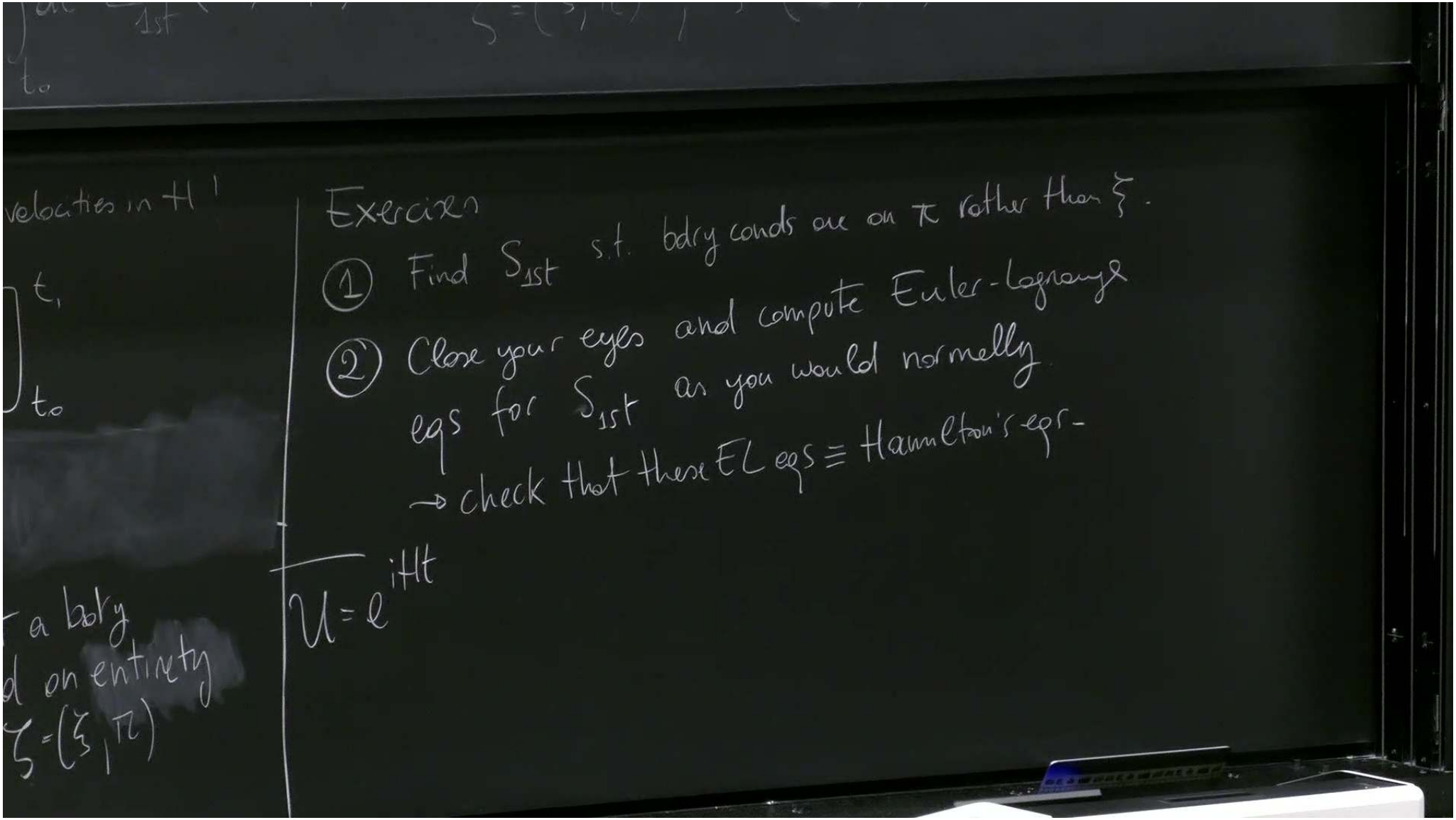
"dual"

wrt v is Hamiltonian

$-L(q, V(q,p,t), t)$

$p = P(q,v,t) \equiv \frac{\partial L}{\partial v}$

Q n-dim	$\mathcal{P}$ 2n-dim
$\xi \mapsto (q,v) = (\xi, \dot\xi)$	$\zeta = (\xi,\pi) \mapsto (z,w) = ((q,p),(v,r)) \mapsto ((\xi,\pi),(\dot\xi,\dot\pi))$
	$\pi = P(\xi,\dot\xi)$ is an eom


velocities in $\mathcal{H}'$
t_1
t_0
a bdry
d on entirety
$\zeta = (\xi, \pi)$
Exercises
① Find S_{1st} s.t. bdry conds are on π rather than ξ.
② Close your eyes and compute Euler-Lagrange eqs for S_{1st} as you would normally
→ check that these EL eqs ≡ Hamilton's eqs.
$U = e^{iHt}$