

Title: Analytic Functionals for Higher-Dimensional Conformal Bootstrap

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Collection/Series: Quantum Fields and Strings

Subject: Quantum Fields and Strings

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Abstract:

The conformal bootstrap has been extremely successful for simple critical systems such as the 3D Ising model and vector models, where it has led to very precise determinations of critical exponents.

However, obtaining similarly reliable nonperturbative data for more complicated theories, especially conformal gauge theories, remains difficult. One practical obstruction is that the standard derivative basis converges slowly when the external operators have high conformal dimension. In this talk, I will describe an analytic functional approach to this problem. I will start from the one-dimensional story, where analytic functionals give bases dual to generalized free spectra and turn crossing into discrete sum rules. I will then explain how the construction can be lifted to higher dimensions using the factorization of two-dimensional global blocks together with dimensional reduction formulas for higher-dimensional conformal blocks. The result is a class of product analytic functionals acting directly on higher-dimensional crossing equations. I will discuss the convergence and positivity properties of these functionals and show that, unlike the standard derivative basis, they maintain fast convergence even when the external operators have high conformal dimension. I will then present numerical applications, including bounds and data extraction in the 3D Ising model, as well as extensions to bootstrap problems with global symmetry. I will end with possible applications to mixed-correlator bootstrap and strongly coupled conformal gauge theories.

Analytic Functionals for Higher-Dimensional Conformal Bootstrap

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Based on 2605.xxxxx with Zechuan Zheng

Perimeter Institute
05/05/2026



List of critical exponents [\[edit\]](#)

For symmetries, the group listed gives the symmetry of the order parameter. The group S_n is the n -element [symmetric group](#), $O(n)$ is the [orthogonal group](#) in n dimensions, \mathbb{Z}_2 is the [cyclic group](#) of order 2 (parity, or Ising symmetry), and **1** is the [trivial group](#). [Mean-field theory](#) result is indicated with (MF). In the two-dimensional Ashkin-Teller model, the exponents in general depend continuously on a parameter along the critical line.^[2] The spherical model can be viewed as the $n \rightarrow \infty$ limit of the $O(n)$ symmetry.^[3]

Appearance

Text

Small

Standard

Incomplete List

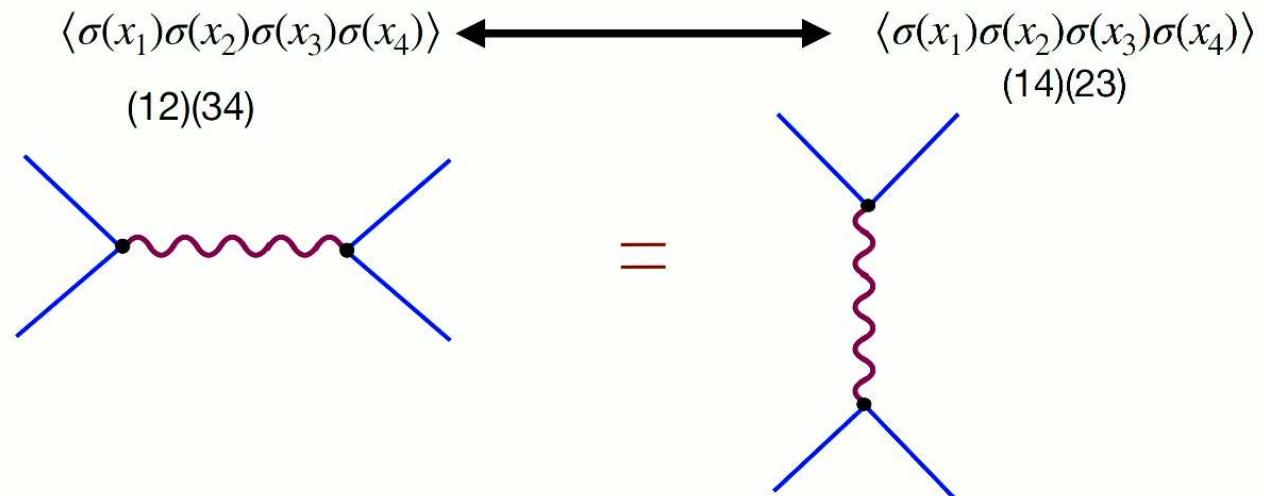
Class	Dimension	Symmetry	α	β	γ	δ	ν	η
3-state Potts	2	S_3	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{13}{9}$	14	$\frac{5}{6}$	$\frac{4}{15}$
Ashkin–Teller (at 4-state Potts point)	2	S_4	$\frac{2}{3}$	$\frac{1}{12}$	$\frac{7}{6}$	15	$\frac{2}{3}$	$\frac{1}{4}$
Ordinary percolation	1	1	1	0	1	∞	1	1
	2	1	$-\frac{2}{3}$	$\frac{5}{36}$	$\frac{43}{18}$	$\frac{91}{5}$	$\frac{4}{3}$	$\frac{5}{24}$
	3	1	−0.625(3)	0.4181(8)	1.793(3)	5.29(6)	0.87619(12)	0.46(8) or 0.59(9)
	4	1	−0.756(40)	0.657(9)	1.422(16)	3.9 or 3.198(6)	0.689(10)	−0.0944(28)
	5	1	≈ -0.85	0.830(10)	1.185(5)	3.0	0.569(5)	−0.075(20) or −0.0565
	6 ⁺ (MF)	1	−1	1	1	2	$\frac{1}{2}$	0
Directed percolation	1	1	0.159464(6)	0.276486(8)	2.277730(5)	0.159464(6)	1.096854(4)	0.313686(8)
	2	1	0.451	0.536(3)	1.60	0.451	0.733(8)	0.230
	3	1	0.73	0.813(9)	1.25	0.73	0.584(5)	0.12

Can consistency determine a CFT?

$$\text{OPE } \phi \times \phi \sim \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} \mathcal{O}$$

CFT data: operator dimensions, spins, and OPE coefficients.

Crossing symmetry



how much of the CFT data is fixed by crossing ?

Crossing has the geometry of a cone

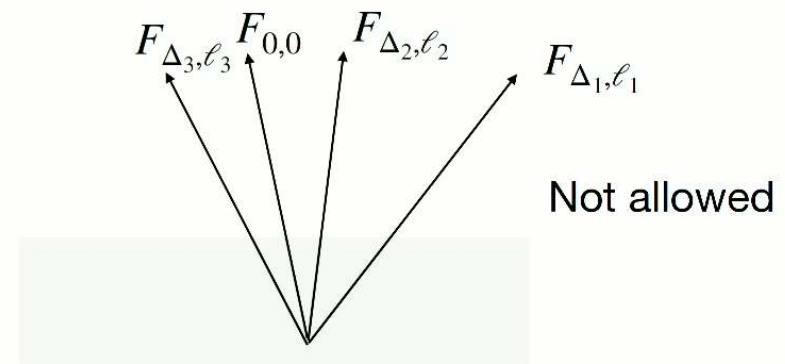
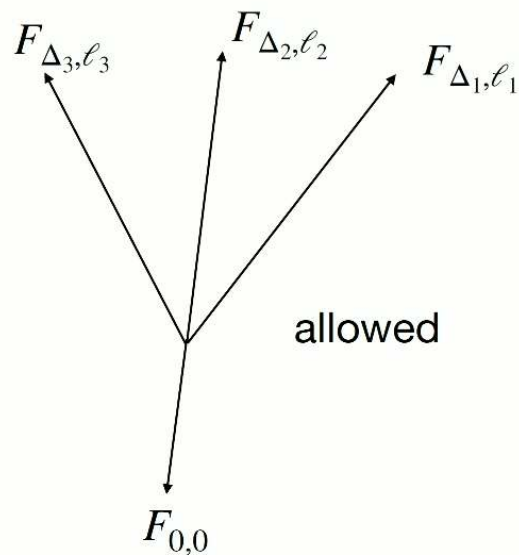
$$\sum_{\Delta, \ell} \lambda_{\Delta, \ell}^2 F_{\Delta, \ell}(z, \bar{z}) = 0$$

$$\text{Unitarity: } \lambda_{\Delta, \ell}^2 \geq 0$$

[Rattazzi, Rychkov, Toni, Vichi, 0807.0004]

Sum of functions with positive coefficients will not always give zero!!

We assume a possible set of CFT data and ask whether it can satisfy crossing with positive OPE coefficients.

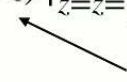


Conventional approach: derivative functionals

Taylor expand the crossing vector around crossing symmetric point.

[Rattazzi, Rychkov, Toni, Vichi, 0807.0004]

$$F_{\Delta,\ell}(z, \bar{z}) = \sum_{m,n} \left(z - \frac{1}{2}\right)^m \left(\bar{z} - \frac{1}{2}\right)^n \partial_z^m \partial_{\bar{z}}^n F_{\Delta,\ell}(z, \bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}}$$

$F_{\Delta,\ell}^{m,n}(1/2, 1/2)$


What is the maximum gap or OPE coefficient?

Linear functionals:

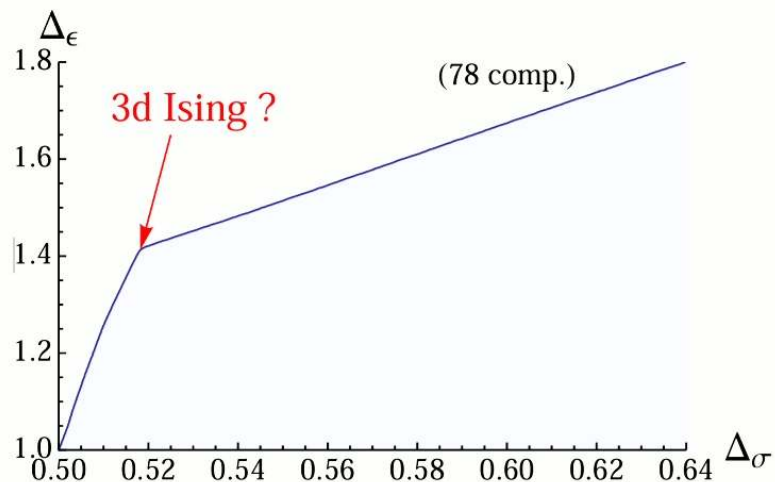
$$\alpha[F] = \sum_{m,n}^{m+n=\Lambda} a_{m,n} F_{\Delta,\ell}^{m,n}(1/2, 1/2)$$

\uparrow
 Optimization fix these numbers!

This has been the main workhorse of precision numerical bootstrap.

The remarkable success of derivative functionals

Duffin, 1612.08471



El'Showk et al., 1203.6064

Bound saturating solutions are called Extremal CFTs.

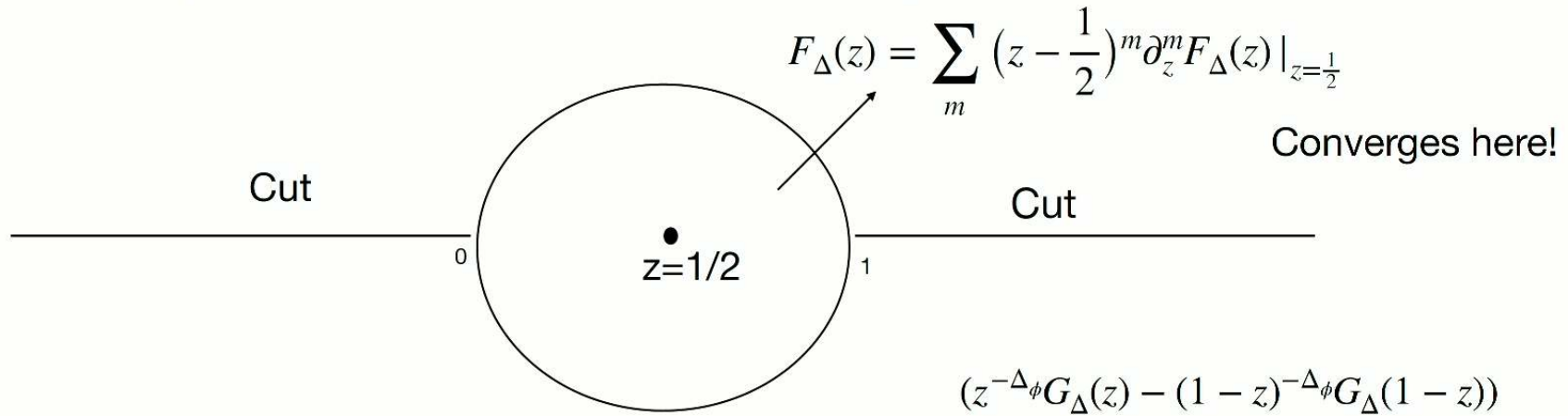
We can solve the CFT data of extremal CFTs.

\mathcal{O}	\mathbb{Z}_2	ℓ	Δ	$f_{\sigma\sigma\mathcal{O}}$	$f_{\epsilon\epsilon\mathcal{O}}$
ϵ	+	0	1.412625(10)	1.0518537(41)	1.532435(19)
ϵ'	+	0	3.82968(23)	0.053012(55)	1.5360(16)
	+	0	6.8956(43)	0.0007338(31)	0.1279(17)
	+	0	7.2535(51)	0.000162(12)	0.1874(31)
$T_{\mu\nu}$	+	2	3	0.32613776(45)	0.8891471(40)
$T'_{\mu\nu}$	+	2	5.50915(44)	0.0105745(42)	0.69023(49)
	+	2	7.0758(58)	0.0004773(62)	0.21882(73)
$C_{\mu\nu\rho\sigma}$	+	4	5.022665(28)	0.069076(43)	0.24792(20)
	+	4	6.42065(64)	0.0019552(12)	-0.110247(54)
	+	4	7.38568(28)	0.00237745(44)	0.22975(10)
	+	6	7.028488(16)	0.0157416(41)	0.066136(36)
\mathcal{O}	\mathbb{Z}_2	ℓ	Δ	$f_{\sigma\epsilon\mathcal{O}}$	-
σ	-	0	0.5181489(10)	1.0518537(41)	
σ'	-	0	5.2906(11)	0.057235(20)	
	-	2	4.180305(18)	0.38915941(81)	
	-	2	6.9873(53)	0.017413(73)	
	-	3	4.63804(88)	0.1385(34)	
	-	4	6.112674(19)	0.1077052(16)	
	-	5	6.709778(27)	0.04191549(88)	

Optimal functional

[Mazac, 1611.10060]

It turns out the optimal functionals want to probe the analytic structure of the correlation function.

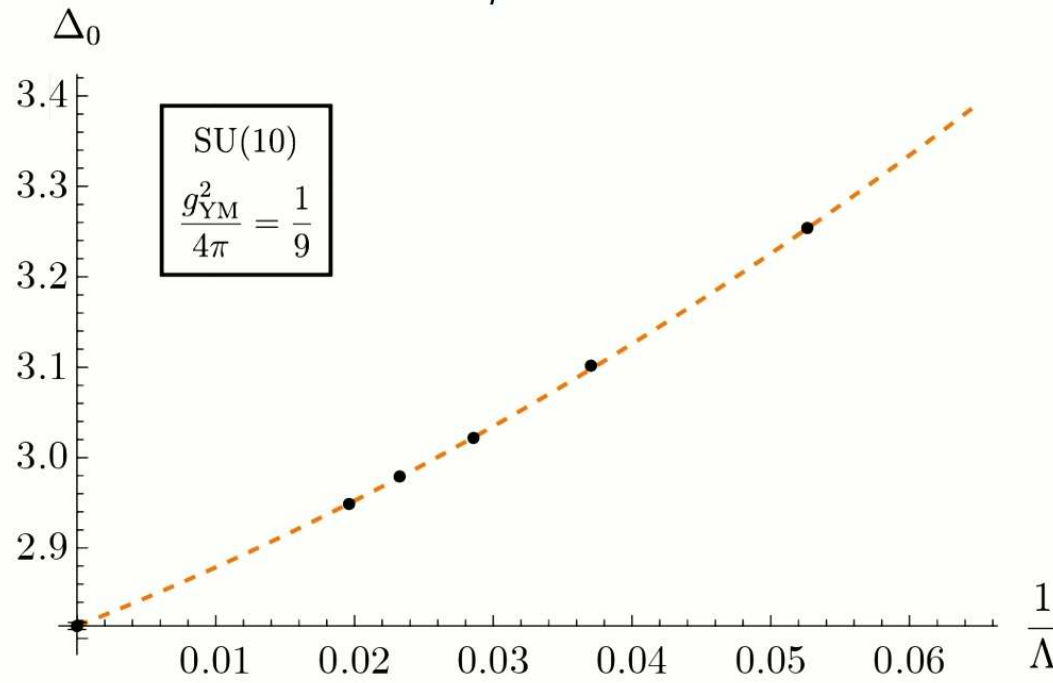


$$\alpha[F] = \sum_{m,n}^{m+n=\Lambda} a_{m,n} F_{\Delta,\ell}^{m,n}(1/2,1/2)$$

↑
Grows as we increase basis size

Limitation

$$\Delta_\phi = 2$$



Gauge theory is hard!

upper bound on the lowest scaling dimension for the SU(10) theory

[Chester, Dempsey, Pufu, 2312.12576]

Many study

[He, Rong, and Su, 2101.07262]

[Reehorst, Refinetti, and Vichhi, 2012.085233]

To obtain a bound within 1% of this extrapolated value directly from the bootstrap $\Lambda > 200$

Need a better tool!

Today's talk

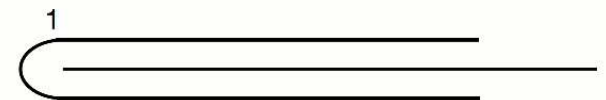
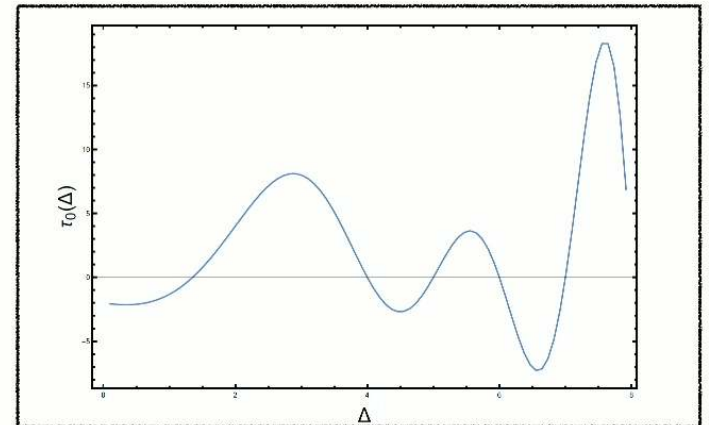
Analytic functional

We restrict ourselves on a line. $z = \bar{z}$

Crossing:
$$\sum_{\Delta} a_{\Delta} (z^{-\Delta\phi} G_{\Delta}(z) - (1-z)^{-\Delta\phi} G_{\Delta}(1-z)) = 0$$

$$F_{\Delta}(z) = \sum_n \tau_n(\Delta) F_{\Delta_n}(z)$$

$$\Delta_n = 2\Delta\phi + n$$



$$\alpha(\Delta) = \int_{\Gamma} dz h(z) F_{\Delta}(z)$$

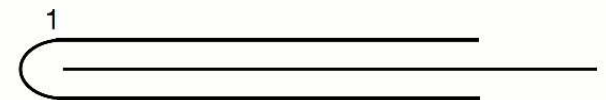
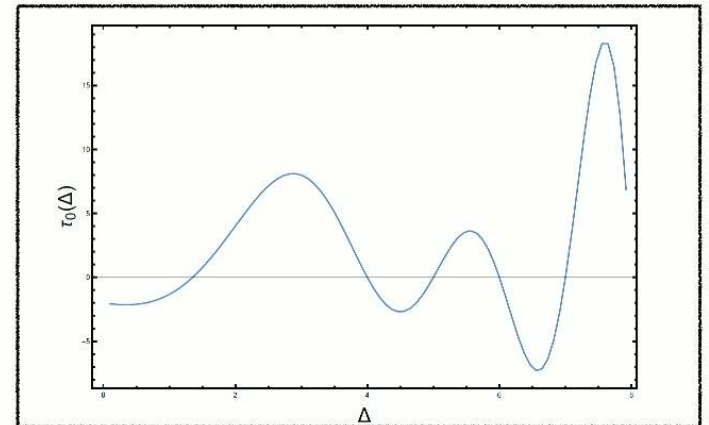
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$$\alpha(\Delta) = \int_{\Gamma} dz h(z) F_{\Delta}(z)$$

Analytic functional

[Mazac, 1611.10060]

To put bound we need a functional with double zeroes.

[Mazac, Paulos, 1811.10646]

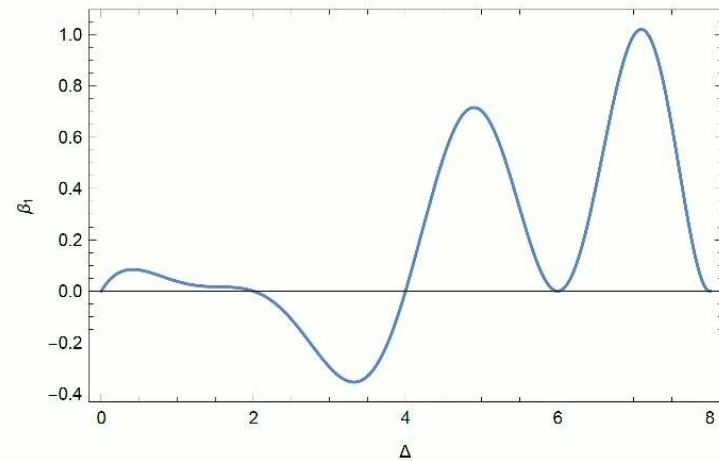
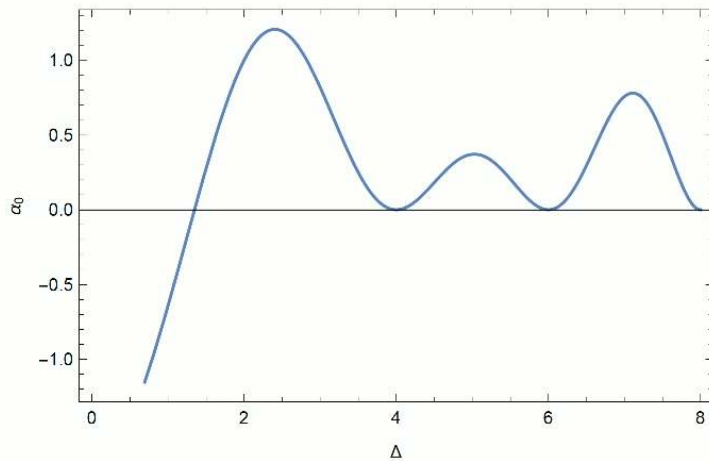
One can choose kernel suitably so that this is achieved. $F_{\Delta}^{-}(z) = \sum_n \beta_n^{-}(\Delta) \partial F_{\Delta_n}(z) + \sum_n \alpha_n^{-}(\Delta) F_{\Delta_n}(z)$

Two bases of functionals:

$$\alpha_n(\Delta_m) = \delta_{m,n}, \quad \alpha'(\Delta_m) = 0$$

$$\beta_n(\Delta_m) = 0, \quad \beta'_n(\Delta_m) = \delta_{nm}$$

$$\Delta_n^B = 2\Delta_\phi + 2n \quad \text{B=bosonic}$$

$$\Delta_n^F = 2\Delta_\phi + 2n + 1 \quad \text{F=fermionic}$$


Polyakov Bootstrap

[Gopakumar, Kaviraj, Sen, Sinha (2016)]

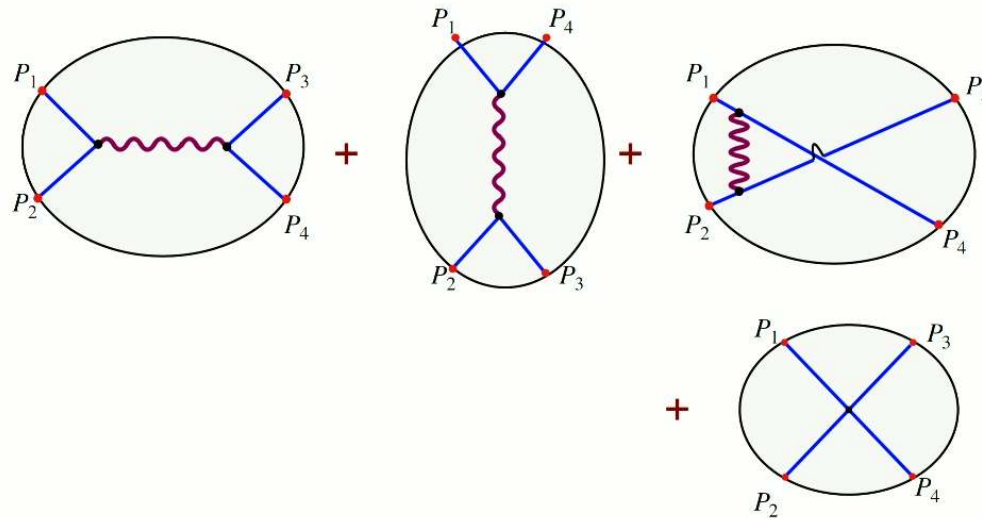
[Mazac, Paulos, 1811.10646]

[KG, Kaviraj, Paulos, 2107.00041]

[Kaviraj, 2109.02658]

[Ferrero, KG, Sinha, Zahed, 1911.12388]

Correlators can be expanded in the following basis



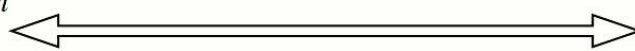
sum of exchange Witten diagrams + contact diagram

This is how practically we compute functionals for numerical applications.

Higher d

[Gopakumar, Sinha, Zahed, 2101.09017]

To see it explicitly,

$$\begin{aligned} G(z) &= \sum_{\Delta} \lambda_{\phi\phi\Delta}^2 \left[W_{\Delta}^s(z) + W_{\Delta}^t(z) + W_{\Delta}^u(z) + c(z) \right] \\ &= \sum_{\Delta} \lambda_{\phi\phi\Delta}^2 \left[g_{\Delta}(z) \right. \\ &\quad \left. + \sum_n \left(\alpha_n g_{2\Delta_{\phi}+2n}(z) + \beta_n g'_{2\Delta_{\phi}+2n}(u, v) \right) \right] \end{aligned}$$


This should vanish!! Crossed channel contributes to only this part.

$$\sum_{\Delta} \lambda_{\phi\phi\Delta}^2 \alpha_n(\Delta) = 0$$

$$\sum_{\Delta} \lambda_{\phi\phi\Delta}^2 \beta_n(\Delta) = 0$$

General Correlators

[KG, Kaviraj, Paulos, 2307.01257]

[KG, Kaviraj, Paulos, 2107.06266]

$$P_{\Delta}^{ij,kl}(z) =$$

Anti-Crossing symmetric vector $F_{\Delta}^{+}(z) = z^{-2\Delta_{\psi}}G_{\Delta}(z) + (1 - z)^{-2\Delta_{\psi}}G_{\Delta}(1 - z)$

They also admit the following basis decomposition $F_{\Delta}^{+}(z) = \sum_n \beta_n^{+}(\Delta)F_{\Delta_n}^{+}(z) + \sum_n \alpha_n^{+}(\Delta)\partial F_{\Delta_n}^{+}(z)$

Full extremal solution in 1D

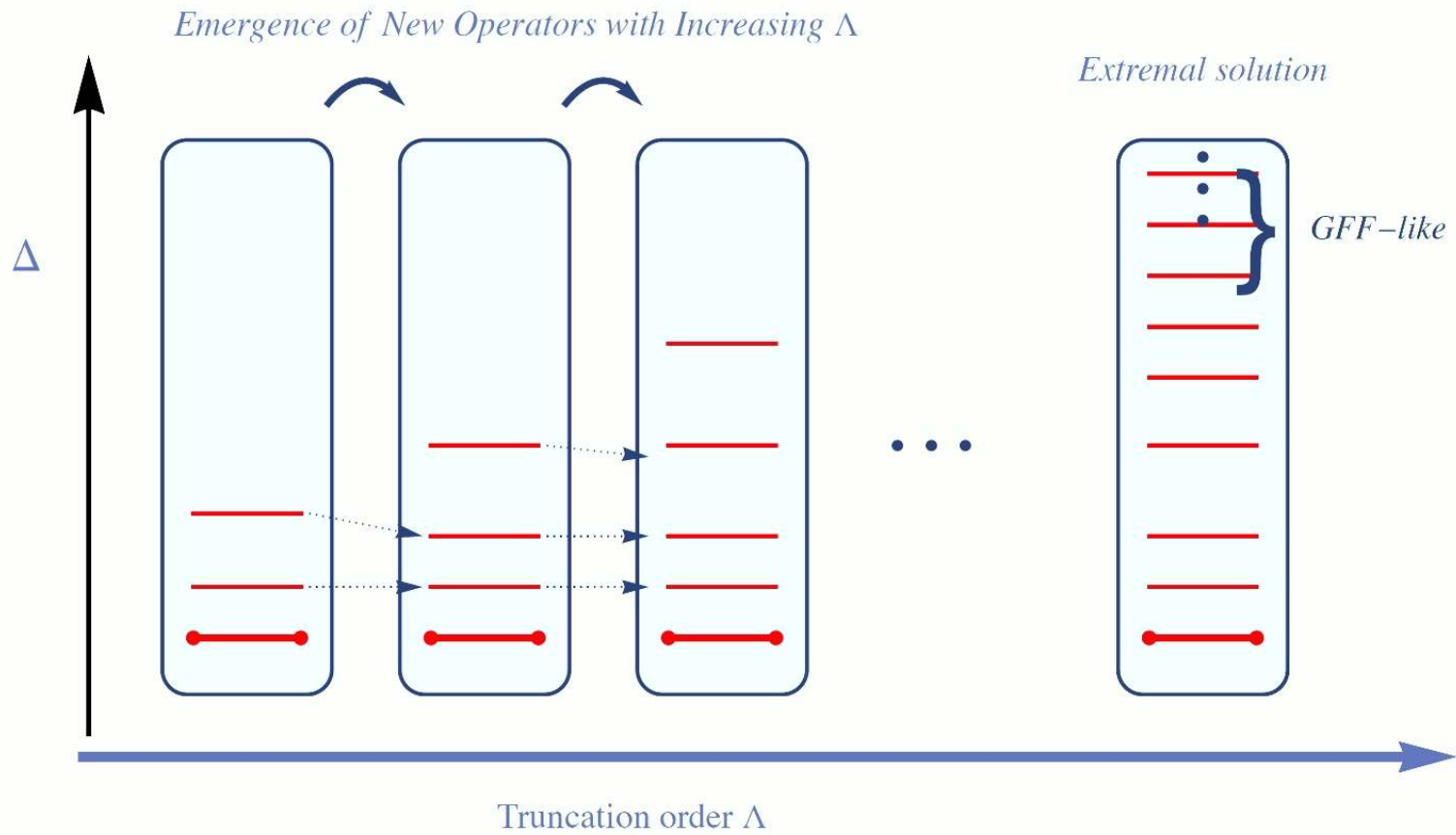
What is the spectrum of extremal CFT when Λ goes to infinity ?

[KG, Suchel, Paulos'2503.22798]

Asymptotic Freedom: Any extremal solutions to crossing equation will behave like a
Conjecture generalized free field in the UV.

$$\{\Delta_n^E, a_n^E\} \rightarrow \{\Delta_n^{B/F}, a_n^{B/F}\},$$
$$n \rightarrow \infty$$

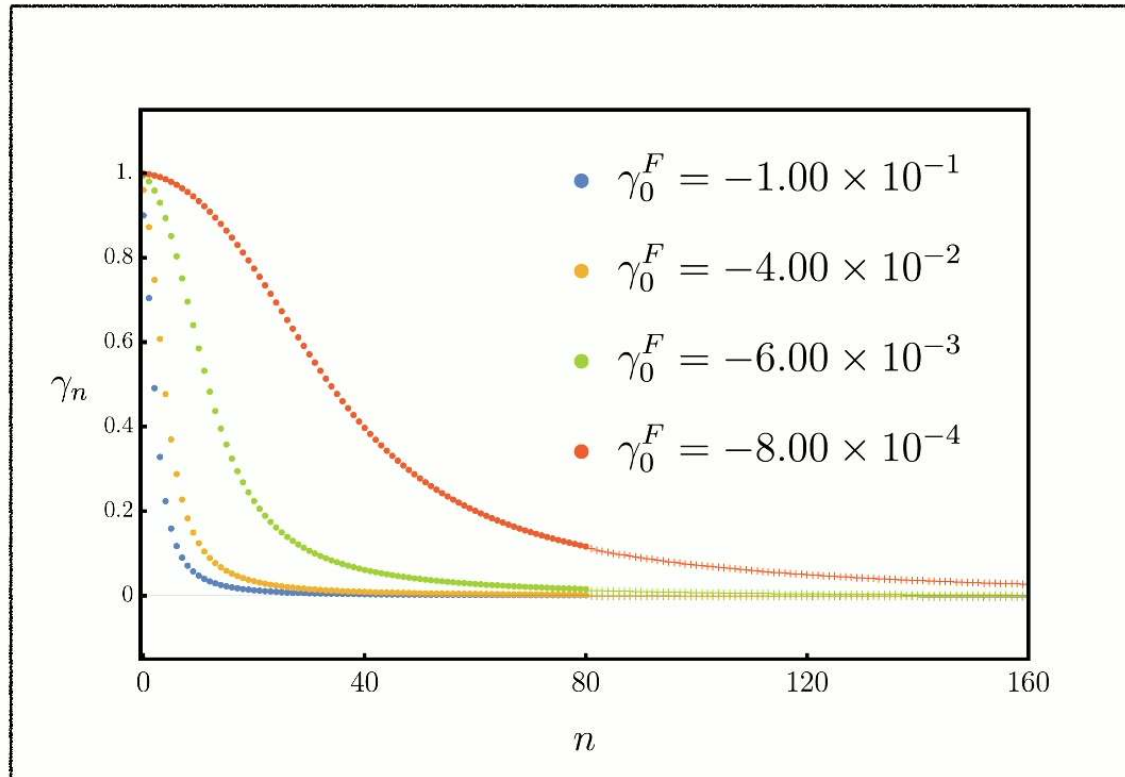
Full extremal solution in 1D



Full extremal solution in 1D

OPE Maximization

[KG, Suchel, Paulos'2503.22798]



$$\omega^{opt}(\Delta_0) = 1$$
$$\lambda_{\phi\phi\Delta_0}^2 \leq -\omega^{opt}(0), \quad \omega^{opt}(\Delta) \geq 0 \forall \Delta \geq \Delta_g$$

Product Functional

2D global conformal block takes the factorised form

$$G_{\Delta, \ell}^{(2d)}(z, \bar{z}) = \frac{1}{2} \left[G_{\tau}^{(1d)}(z) G_{\rho}^{(1d)}(\bar{z}) + G_{\rho}^{(1d)}(z) G_{\tau}^{(1d)}(\bar{z}) \right], \quad \tau = \Delta + \ell, \quad \rho = \Delta - \ell$$

Factorized crossing vector $F_{\Delta, \ell}^{(2d)}(z, \bar{z}) + (\bar{z} \rightarrow 1 - \bar{z}) = \frac{1}{4} \left[F_{\tau}^{-}(z) F_{\rho}^{+}(\bar{z}) + F_{\rho}^{-}(z) F_{\tau}^{+}(\bar{z}) \right]$

Factorized functional action $(\omega_1^{-} \otimes \omega_2^{+}) \left[F_{\Delta, \ell}^{(2d)} \right] = \frac{1}{2} \left[\omega_1^{-}(\tau) \omega_2^{+}(\rho) + \omega_1^{-}(\rho) \omega_2^{+}(\tau) \right]$

Product Functional

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Factorized functional action $(\omega_1^{-} \otimes \omega_2^{+}) \left[F_{\Delta, \ell}^{(2d)} \right] = \frac{1}{2} \left[\omega_1^{-}(\tau) \omega_2^{+}(\rho) + \omega_1^{-}(\rho) \omega_2^{+}(\tau) \right]$

Product Functional Bases

$$F_{\Delta, \ell}^{(2d)}(z, \bar{z}) = \sum_{m, n, i, j} \Omega_{mn}^{ij}(\Delta, \ell) B_{mn}^{ij}(z, \bar{z})$$

Basis elements $B_{mn}^{ij} \in \left\{ F_{\Delta_m}^-(z)F_{\Delta_n}^+(\bar{z}), \partial F_{\Delta_m}^-(z)F_{\Delta_n}^+(\bar{z}), F_{\Delta_m}^-(z)\partial F_{\Delta_n}^+(\bar{z}), \partial F_{\Delta_m}^-(z)\partial F_{\Delta_n}^+(\bar{z}) \right\}$

Menu of functionals:

Δ_m^-	Δ_n^+
B	F
F	B
B	B
F	F

$$\Delta_m^B = 2\Delta_\phi + 2m$$

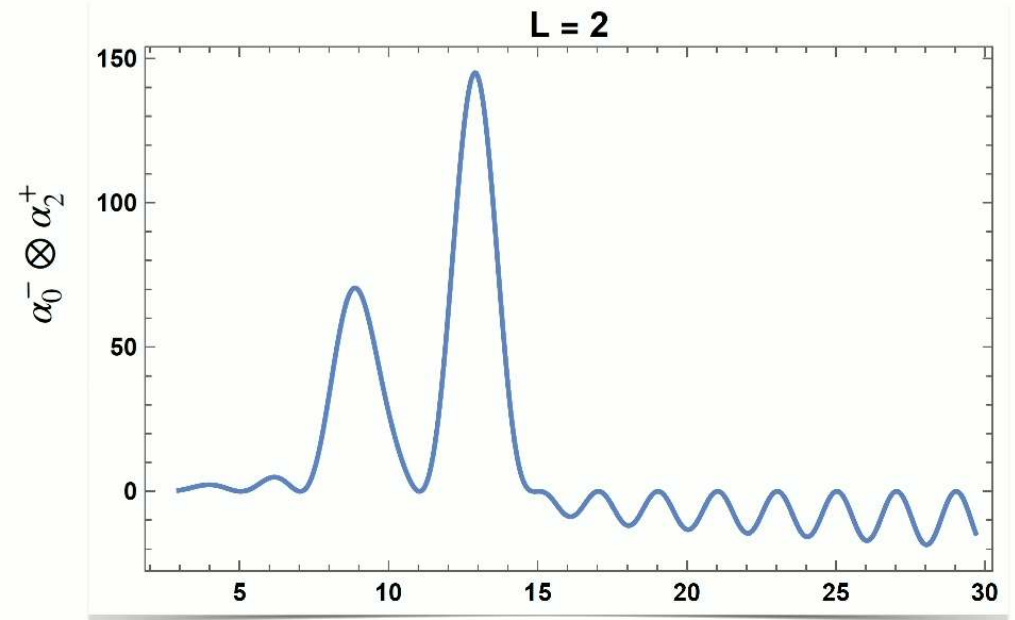
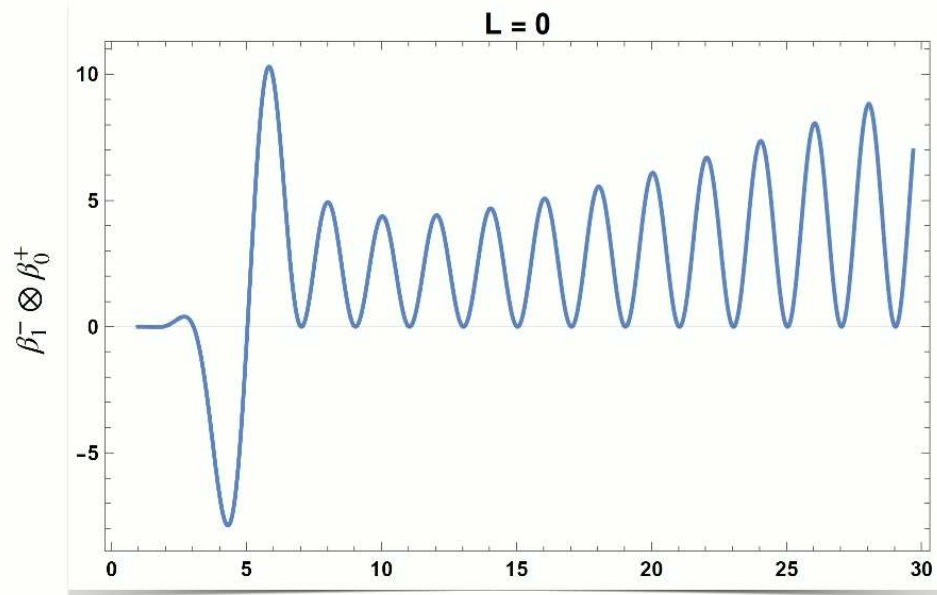
$$\Delta_n^F = 2\Delta_\phi + 2n + 1$$

Orthonormality

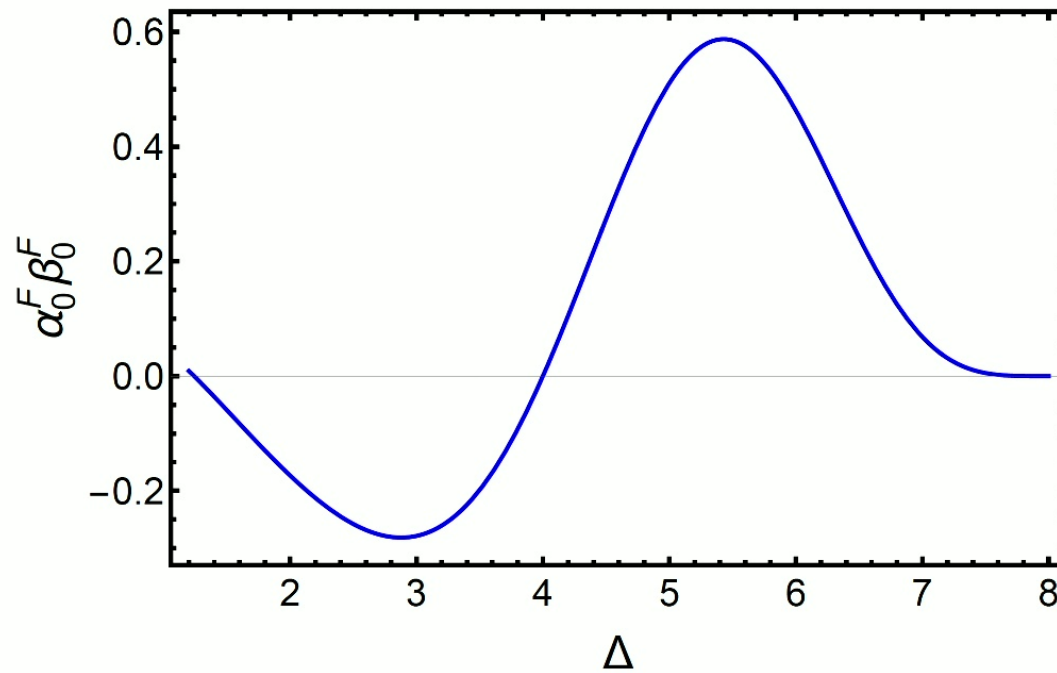
The product functionals also satisfy orthonormality conditions:

But not in conformal dimensions but in twist.

All of them has double zero at GFF locations except few places.

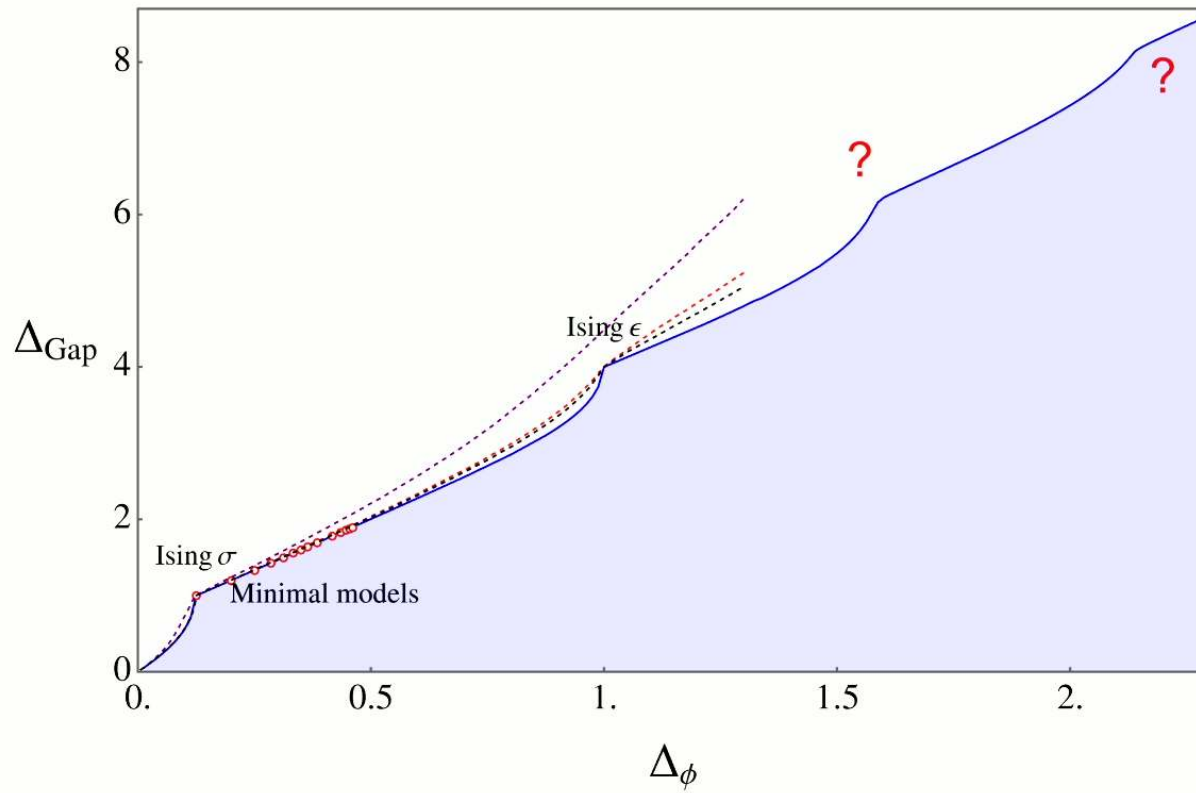


Extremal Functional



This is saturated by energy correlator of 2D Ising model.

2D gap maximisation



Dimensional reduction: 3D blocks as towers of 2D blocks

[Hoggervost, 1604.08913]

$$G_{\Delta, \ell}^{(3d)}(z, \bar{z}) = \sum_{n=0}^{\infty} \sum_j A_{n,j}^{\Delta, \ell} G_{\Delta+2n,j}^{(2d)}(z, \bar{z})$$

n : labels 2d multiplets with $\Delta_{2d} = \Delta + 2n$,

j : 2d spin from $SO(3) \rightarrow SO(2)$ $j = \ell, \ell - 2, \dots$

Three dimensional functional action as sum of 2D product functional:

$$\Omega^{(3d)}(\Delta, \ell) = \sum_{n=0}^{\infty} \sum_j A_{n,j}^{\Delta, \ell} \Omega^{(2d)}(\Delta + 2n, j)$$

Bottleneck: the dimensional reduction tail

$$\Omega^{3d} = \sum_{n,j} A_{n,j} \Omega^{2d}(\Delta + 2n, j)$$

Good news is the sum is convergent $\sim \frac{1}{n^{4\Delta_\phi+6}}$

$$\text{Large pre-factor} \sim \frac{\Delta^{4\Delta_\phi+\frac{11}{2}}}{n^{4\Delta_\phi+6}}$$

The tail is convergent, but high Δ evaluation becomes expensive.

Saalschützian / very-well-poised hypergeometric structure

$$A_{n,j}^{\Delta,\ell} \Omega^{2d}(\Delta + 2n, j) \sim \Gamma(\dots) L(a, b, c, d; e; f, g)$$

[Mishev,1008.1011]

$$L(a, b, c, d; e; f, g) \equiv \text{very-well-poised } {}_7F_6(1)$$

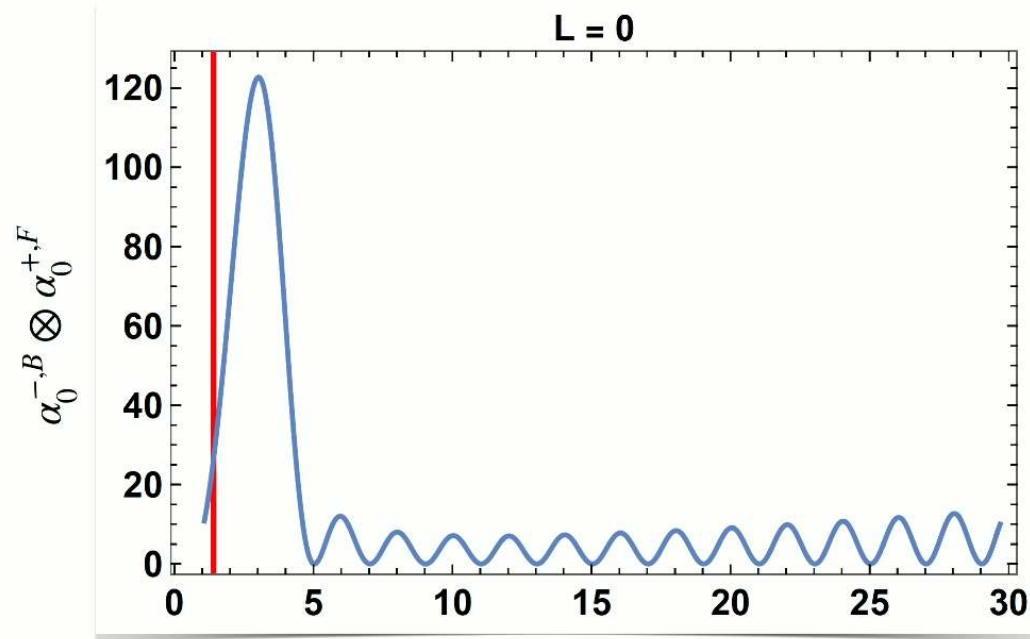
Coxeter group invariance

$$L(\vec{x}) = L(w\vec{x}), \quad w \in W(D_5)$$

The tail becomes a systematic 1/n expansion.

We subtract as many terms as we want and then add the piece summing those terms analytically.

Instant Bound

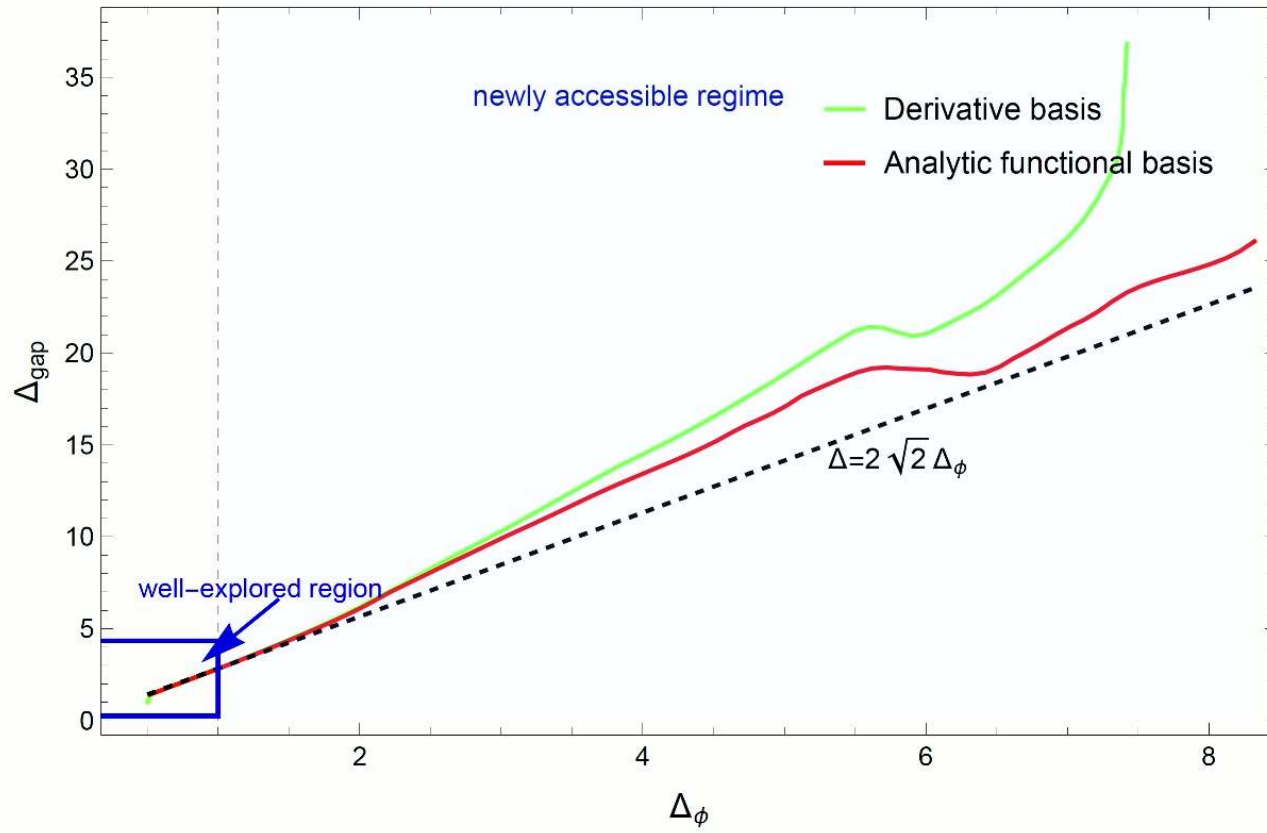


This functional provides the following OPE bound for 3D Ising model

$$C_{\phi\phi\epsilon} \leq 1.3219 \text{ (1.1064)}$$

Gap Maximization in 3D

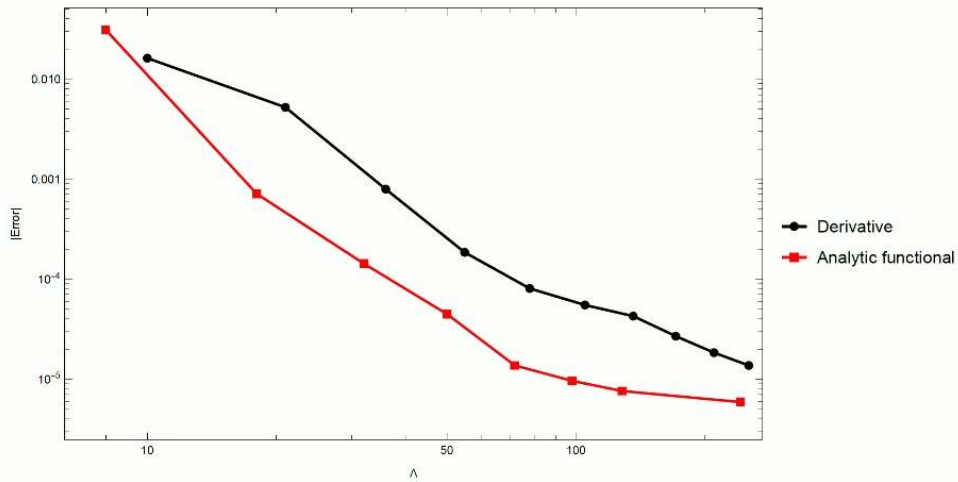
[KG, Zheng, 2605.xxxx]



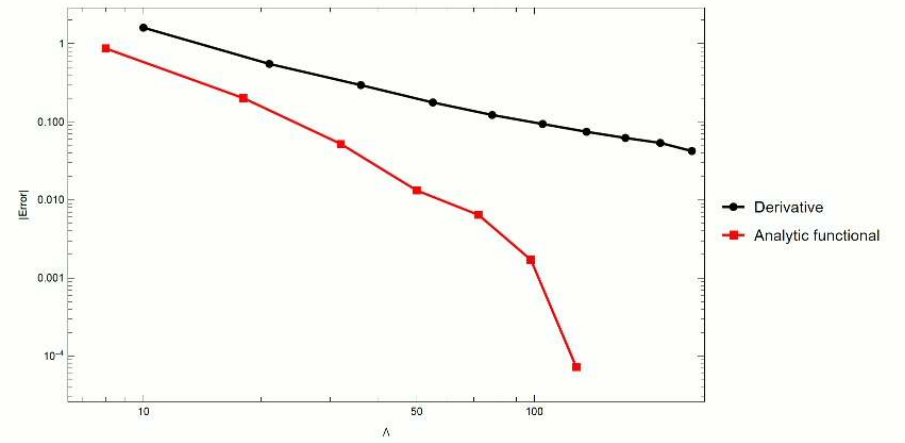
Convergence

LogLog Plot

[KG, Zheng, 2605.xxxx]



$$\Delta_\phi = 0.51814$$



$$\Delta_\phi = 1.475$$

Comparison of error as we increase the basis size

Extremal Spectrum

Asymptotic Freedom: Light cone bootstrap predicts that at large spin and fixed twist the spectrum and OPE coefficients always become that of generalized free fields.

[Fitzpatrick et al, 2012]

[Duffin et al, 2012]

[Van C Rees, 2412.06907]

We expect to see $\phi \partial_{\mu_1} \partial_{\mu_2} \dots \partial_{\mu_\ell} (\partial^2)^n \phi$ What else?

Let's ask: Is it possible to have a consistent solution where we only give anomalous dimension to generalized free field without introducing any new states?

[Gopakumar, Sinha, Zahed, 2101.09017]

Polyakov Condition:

$$\sum_{\Delta, \ell} \lambda_{\Delta, \ell}^2 \alpha_{n, J}(\Delta, \ell) = 0$$

$$\sum_{\Delta, \ell} \lambda_{\Delta, \ell}^2 \beta_{n, J}(\Delta, \ell) = 0$$

Locality equations:

$$\sum_{\Delta, \ell} \lambda_{\Delta, \ell}^2 \omega_{n, J}(\Delta, \ell) = 0$$

We may expect another tower appearing in the extremal solutions unlike 1D.

Extremal Spectrum

[KG, Zheng, 2605.xxxx]

Spin	Twist
4	2.556
6	2.733
8	2.855
10	2.956
12	3.006
14	2.995
16	2.993
18	2.986

Stable operators

$$(\epsilon\epsilon)_0 \quad 2.82\dots$$

O(2) $\Lambda = 216$

CFT data	Analytic functional	Derivative mixed-correlator
Δ_S	1.512030	1.51136(22)
Δ_T	1.23351	1.23629
C_T/C_T^{free}	0.945144	0.944056(15)
C_J/C_J^{free}	0.903282	0.904395(28)
$\Delta_{S'}$	3.79875	3.794(8)
$\Delta_{T'}$	3.66679	3.650(2)

O(3) $\Lambda = 96$

CFT data	Analytic functional
Δ_S	1.599638
Δ_T	1.2277
C_T/C_T^{free}	0.9529
C_J/C_J^{free}	0.9020
$\Delta_{S'}$	3.76746
$\Delta_{T'}$	3.6435

Takeaway I: Extremal Functional

In the conventional approach, the search space is built from derivatives

$$\mathcal{A}_\Lambda^{\text{der}} = \text{span} \left\{ \partial_z^m \partial_{\bar{z}}^n \Big|_{z=\bar{z}=1/2} : m+n \leq \Lambda \right\} \quad B_{m,n} = \frac{1}{m!n!} (z-1/2)^m (\bar{z}-1/2)^n$$

In our case, the search space is built from product functionals

$$\mathcal{A}_\Lambda^{\text{prod}} = \text{span} \left\{ \Omega_{mn}^{ij} : m+n \leq \Lambda, \quad i, j = \alpha, \beta \right\} \quad B_{mn}^{ij} \in \left\{ F_{\Delta_m}^-(z) F_{\Delta_n}^+(\bar{z}), \partial F_{\Delta_m}^-(z) F_{\Delta_n}^+(\bar{z}), F_{\Delta_m}^-(z) \partial F_{\Delta_n}^+(\bar{z}), \partial F_{\Delta_m}^-(z) \partial F_{\Delta_n}^+(\bar{z}) \right\}$$

We observe that product analytic functionals approximate the optimal functional more efficiently than derivative functionals.

It doesn't struggle to find optimal bound when dimension of the external operator is high.

This removes one important convergence bottleneck for attacking harder theories.

Takeaway II: Extremal solutions

In one dimension, the full extremal spectrum asymptotes to a single GFF.

$$[\phi\phi]_n \sim \phi (\partial^2)^n \phi, \quad \tau_n \rightarrow 2\Delta_\phi + 2n$$

In higher d it could be twice of GFF. Some evidence in favour of it.

$$[\phi\phi]_{n,\ell} \sim \phi \partial_{\mu_1} \cdots \partial_{\mu_\ell} (\partial^2)^n \phi, \quad \tau_{n,\ell} \rightarrow 2\Delta_\phi + 2n$$

$$[\epsilon\epsilon]_{n,\ell} \sim \epsilon \partial_{\mu_1} \cdots \partial_{\mu_\ell} (\partial^2)^n \epsilon, \quad \tau_{n,\ell} \rightarrow 2\Delta_\epsilon + 2n$$

This is also good for gauge theory.

Outlook

Easy Targets:

Long range Ising model

Single correlator bound for Gauge theory

Single correlator bound in 4d: is it featureless boring?

Algorithm:

Optimal truncation of the basis (m,n) and type of functionals (B/F)

Mixed correlator bootstrap: The story is exactly same. But we need to design a semi definite solver that will be well adapted for analytic functionals.

More serious target 3D QED, 3D QCD.....