Title: Lecture - Mathematical Physics II, PHYS 777

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Subject: Mathematical physics

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Last time

Only possible OPEs for J were

$$J_a(z_1)\,J_b(z_2) \sim \frac{K\,\delta_{ab}}{(z_1-z_2)^2} + \frac{f_{ab}^{\ c}\,J_c(z_1)}{(z_1-z_2)}$$

Boundary OPE in CS is
this where the k appearing in OPE
is not necessarily same as
K_{bare} in CS action

loop corrections

Perturbatively we computed 2+3 pt fns

$$\langle J_{a_1}(z_1)\, J_{a_2}(z_2)\rangle = \frac{k_{bare}}{(z_1-z_2)^2}\,\delta_{a_1 a_2}$$

$$\langle J_{a_1}(z_1)\, J_{a_2}(z_2)\, J_{a_3}(z_3)\rangle = \frac{k_{bare}\, f_{a_1 a_2 a_3}}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}$$

As $z_3 \to z_2$, 3 pt fn
$\Rightarrow$ 2 pt fn $\times$ OPE

3 pt fn ans

$$\frac{1}{z_3 - z_2}(\quad) + \text{regular at } z_3 = z_2$$

$$\frac{-1}{(z_2 - z_3)} \frac{K_{bare} f_{a_1 a_2 a_3}}{(z_1 - z_2)^2} + \text{regular}$$

$$\langle J_{a_1}(z_1) J_{a_2}(z_2) J_{a_3}(z_3) \rangle$$

$$= \frac{1}{z_2 - z_3} f^{c}_{a_2 a_3} \langle J_{a_1}(z_1) J_c(z_2) \rangle + \text{regular terms}$$

n point fns can be computed inductively using OPE

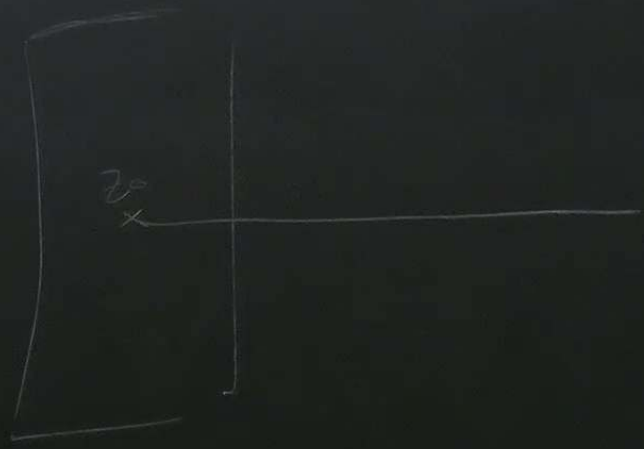
Plan:

- Hilbert space of CS on S^2 w. end points of lines can be computed using chiral bc.
- Braid group action on this can also be computed, using KZ connection

Chiral bc $\rightsquigarrow$ Knot invariants

Wilson line in a rep R of G is a QM system with Hilbert space R coupled to CS.

CS on $\mathbb{C} \times \mathbb{R}_{\geq 0}$



Place a Wilson line at z_0, end on boundary

What does this do at boundary?

At $z_0 \in \mathbb{C}$, we will have a defect in our 2d boundary system, where there are new states / operators $V_i \in R$

Generic z, operators are $J(z)$, $\partial J(z)$, $J^2(z)$ ... as before

At z_0, operators are

$V_i(z_0)$ $\quad J(z_0)V_i(z_0)$ $\quad \partial J\, V_i(z_0)$

....

= elements of R, dressed by ordinary local operators

$\partial_z V_i(z_0)$ is not an operator, as Wilson line lives only at z_0

New OPE to consider:

$J_a(z_1)\ V_i(z_0)$ as $z_1 \to z_0$

V_i is of dim Δ

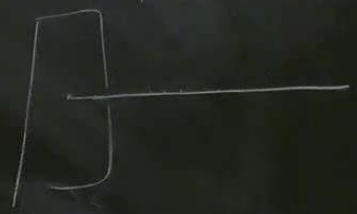
J_a is of dim 1

Only possible OPE coeffs are

$$J_a(z_1)\, V_i(z_0) \sim \frac{1}{z_1 - z_0} \rho_{ai}^{j}\, V_j(z_0)$$

G symmetry tells us that ρ_{ai}^{j} must be the matrix defining g action on R.

Perturbation theory.



Field A sourced by Wilson line

satisfies $\oint_{\text{around line}} A = \frac{1}{k}$

$A_{\bar{z}} = 0$ at $t=0$

Wilson line lives at $z = z_0$, wraps t direction

Field sourced by line, satisfies b.c., is

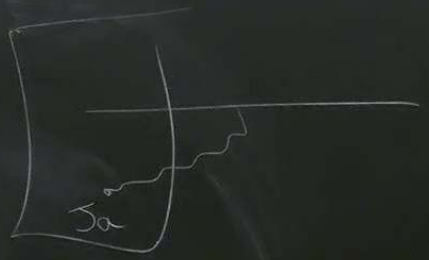
$$A = \frac{1}{2\pi i K} \frac{dz}{(z - z_0)}$$

Since $J(z)$ measures k_x z component A_z at $t = 0$

we see that

$J \sim \frac{1}{(z - z_0)}$ in presence of Wilson line.

$z = z_0$, wraps t direction



ne, satisfies b.c. , is

$$\frac{dz}{(z - z_0)}$$

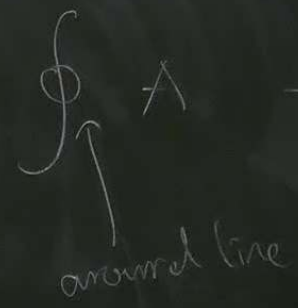
ures $k \times z$ component A_z at $t = 0$

e of Wilson line.

Only possible OPE coeffs are

Field A sourced by Wilson line

satisfies $\oint A = 1/k$



$A_{\bar{z}} = 0$ at $t=0$

Wilson li

Field so

$A =$

Since J

we see tha

$J \sim \frac{1}{(z$

Given n points $z_1 \ldots z_n \in S^2$

With representations $R_1 \ldots R_n \in S^2$

we know Hilbert space of CS is

G-invariant elements in $R_1 \otimes \ldots \otimes R_n$

(or in $(R_1 \otimes \ldots \otimes R_n)^*$ for "final time" Hilbert space)

Given

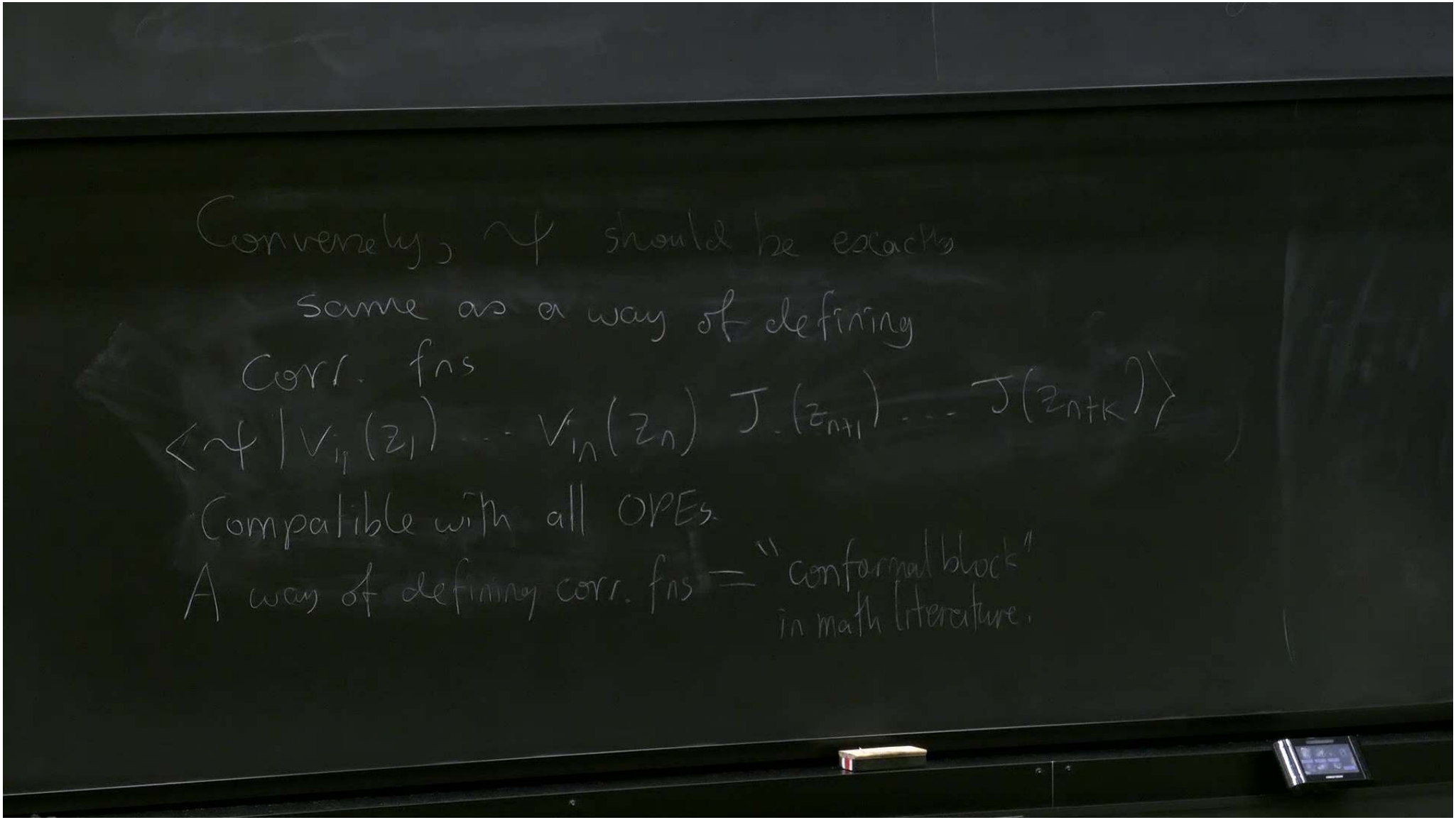
$\mathcal{V} \in \mathcal{H}(S^2, z_1 \ldots z_n)$

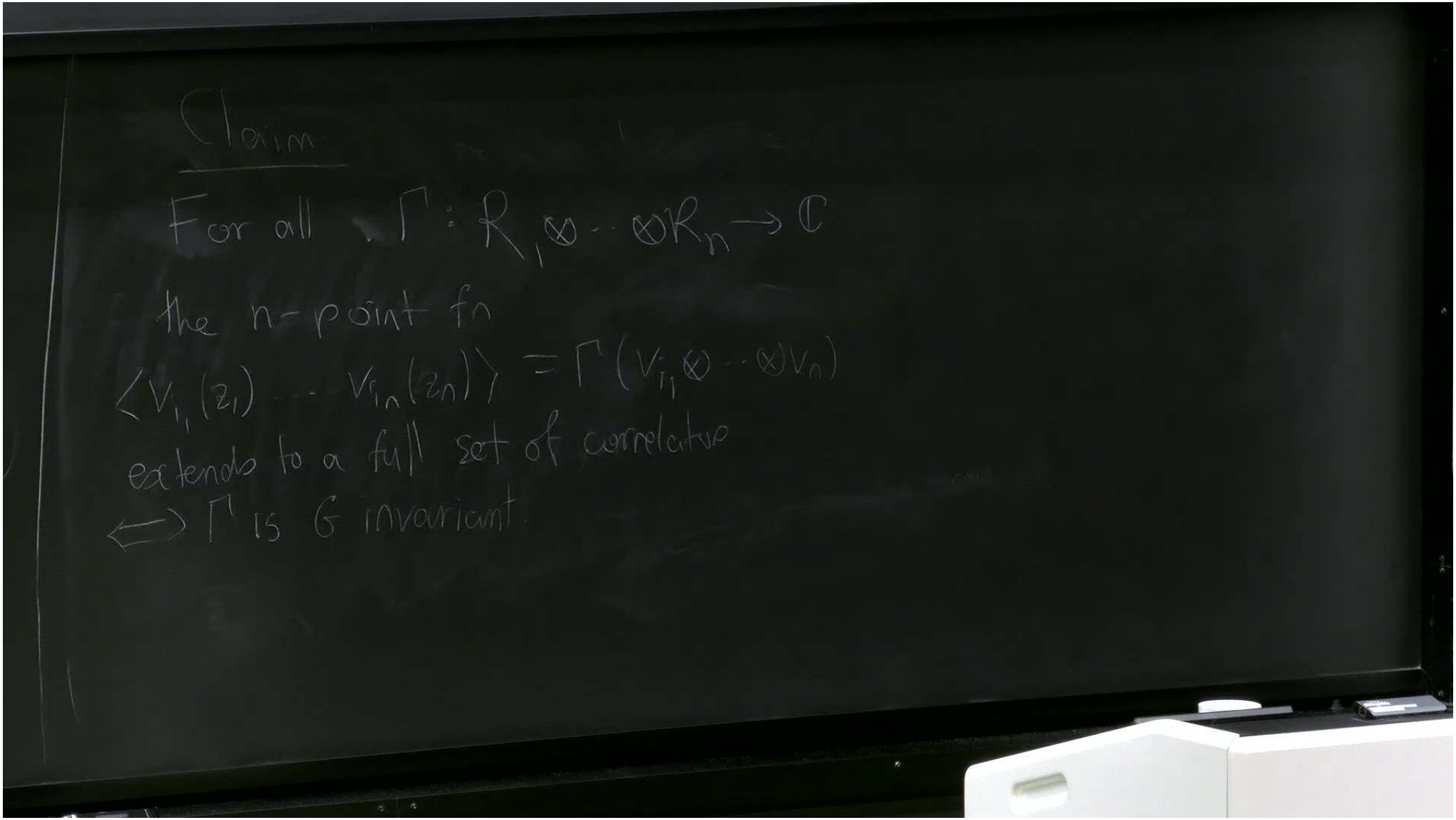
we can define correlation fns

for the boundary alg. w. defects

$\mathcal{V}$

boundary alg w/ defects


Conversely, $\mathcal{V}$ should be exactly
same as a way of defining
Corr. fns
$\langle \psi | V_{i_1}(z_1) \cdots V_{i_n}(z_n)\, J_{\cdot}(z_{n+1}) \cdots J(z_{n+k}) \rangle$
Compatible with all OPEs.
A way of defining corr. fns = "conformal block" in math literature.


Claim
For all $\Gamma : R_1 \otimes \cdots \otimes R_n \to \mathbb{C}$
the n-point fn
$\langle V_{i_1}(z_1) \cdots V_{i_n}(z_n) \rangle = \Gamma(V_{i_1} \otimes \cdots \otimes V_{i_n})$
extends to a full set of correlators
$\Leftrightarrow$ Γ is G invariant.

J picks up dz component of A,

$dz \longrightarrow \frac{-1}{z^2} dz$ under $z \to \frac{1}{z}$

when we work near ∞, corr. fn of $J(z) \longrightarrow \frac{1}{z^2}$ as $z \to \infty$

Equivalently

$\langle V_1(z_1) \cdots$

$$\langle V_1(z_1) \cdots V_n(z_n)\, \overline{J}(z_{n+1}) \rangle \, dz_{n+1}$$

is a meromorphic 1 form for $z_{n+1} \in \mathbb{C}\mathbb{P}^1$ with poles only at $z_{n+1} = z_i$

$$\oint_{z_{n+1}=\infty} \langle V_1(z_1) \cdots V_n(z_n)\, J_a(z_{n+1}) \rangle = 0$$

$$= \sum_{i=1}^{n} \oint_{z_{n+1}=z_i} dz_{n+1} \langle V_1(z_1) \cdots V_n(z_n)\, J_a(z_{n+1}) \rangle$$

Perturbation theory

Only possible OPE coeffs are

$$\oint_{z_{n+1}=z_1} \langle V_1(z_1) \cdots V_n(z_n)\, J_a(z_{n+1}) \rangle$$

$$= \langle V_1(z_1) \cdots (t_a V_i)(z_i) \cdots V_n(z_n) \rangle$$

where t_a is Lie alg. generator acting on my representation

comes from OPE

$$\sum_{i=1}^{S_0} \langle V_1(z_1) \cdots (t_a V_i)(z_i) \cdots V_n(z_n) \rangle = 0$$

But g acts on $R_1 \otimes \cdots \otimes R_n$ by saying a generator t_a acts by $t_a \otimes 1 \cdots \otimes 1 + 1 \otimes t_a \otimes \cdots \otimes 1 + \cdots + (1 \otimes \cdots \otimes 1) \otimes t_a$

Conclude, $\langle V_1(z_1) \cdots V_n(z_n) \rangle$ must be a g inv map

$R_1 \otimes \cdots \otimes R_n \to \mathbb{C}$

The $n+k$ point fns

$$\langle V_1(z_1) \cdots V_n(z_n) J_{-}(z_{n+1}) \cdots J(z_{n+k}) \rangle$$

all exist + are unique, given n, and $n+1$ point functions exist.

$R_1 \otimes \cdots \otimes R_n \to \mathbb{C}$

Correlators of chiral alg. with defects

$=$ G inv. elements of $(R_1 \otimes \cdots \otimes R_n)^*$

$=$ States on S^2 with defects