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Optimal cloning of free-fermion states

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Thm.

$$U\left(|\psi\rangle\otimes|0\rangle\right)=|\psi\rangle\otimes|\psi\rangle$$

Or, more generally:

Thm. $\nexists\ T\in CPTP\left(\mathcal{H}^{\otimes N},\mathcal{H}^{\otimes N+K}\right)$:

$$T\left(\psi^{\otimes N}\right)=\psi^{\otimes N+K},\quad \forall|\psi\rangle\in\mathcal{H}$$

What does it mean?

Fundamentally: quantum info

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Or, more generally:

Thm.

$$\nexists \; T \in CPTP(\mathcal{H}^{\otimes N}, \mathcal{H}^{\otimes N+K}):$$

$$T(\psi^{\otimes N}) = \psi^{\otimes N+K}, \quad \forall |\psi\rangle \in \mathcal{H}$$

What does it mean?

Fundamentally: quantum info cannot be copied

$T(\psi^{\cdots}) = \psi^{\cdots}, \quad \forall |\psi\rangle \in S$

What does it mean?

Fundamentally: quantum info cannot be copied

$\Rightarrow$ limit on quantum memory manipulations

$\Rightarrow$ quantum crypto tools (QKD, quantum money, ...)

What about approximate cloning?

Def. Fidelity f_T cloner T:

$$\mathrm{tr}\left(T(\psi^{\otimes N})\,\psi^{\otimes N+K}\right) \geq f_T, \ \forall |\psi\rangle$$

Thm. $\exists\, T_{opt}$ with $f_{opt} = \binom{D+N-1}{N} \cdot \binom{D+N+K-1}{N+K}^{-1}$, $D := \dim \mathcal{H}$

Werner '98

$$f_T \leq f_{opt} \quad \forall\, T \in \mathrm{CPTP}$$

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Werner '98

$$f_T \leq f_{opt} \quad \forall T \in \mathrm{CPTP}$$

Ex. $N=1,\ K=1,\ \dim \mathcal{H} = 2:\quad f_{opt} = \frac{2}{3}$

$\rightarrow$ no-cloning is robust

Ex. $N=1,\ K=1,\ \dim \mathcal{H}=2:\ f'_{opt}=\frac{5}{3}$

$\to$ no-cloning is robust

When is cloning effectively allowed?

Asking $f_{opt} \geq 1-\epsilon$

Consider $K=1$ ($N \to N+1$ cloning)

Cor. $1-\epsilon \leq \frac{N+1}{D+N} = 1-\frac{D-1}{D+N} \Rightarrow N \geq N_{min} = \Theta\left(\dim \mathcal{H}/\epsilon\right)$;

exists, but exponentially hard for n-body $\mathcal{H}$

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Cloning structured states

<u>Ex.</u> Promised $|\psi\rangle \in S = \{ |\psi_j\rangle \} \subset \mathcal{H}$, "Classical states"

i.e. $\langle \psi_j | \psi_k \rangle = 0 \ \forall j,k \implies$ Perfectly clonable

<u>Why</u>: $T(\rho) = \sum_j \mathrm{Tr}(\psi_j^{\otimes N} \rho) \cdot \psi_j^{\otimes N+K}$

valid CPTP : $D(\mathrm{span}(|\psi_j\rangle^{\otimes N})) \to D(\mathcal{H}^{\otimes N+K})$

Achieves fidelity $\mathrm{Tr}(T(\psi_j^{\otimes N}) \psi_j^{\otimes N+K}) = 1$

What about other structured

$S \subset \mathcal{H}$?

Def For $T \in CPTP\left(\mathrm{Span}(|\psi\rangle^{\otimes N}, |\psi\rangle \in S), \mathcal{H}^{\otimes N+K}\right)$,

$$f_{T,S} := \min_{|\psi\rangle \in S} \mathrm{Tr}\left(\psi^{\otimes N+K}\, T(\psi^{\otimes N})\right)$$

$$T_{opt} := \underset{T}{\operatorname{argmin}}\, f_{T,S} \quad \text{- optimal cloner}$$

What f_{opt} can be achieved ?

For $f_{opt} = 1 - \epsilon$, can we do better than $N_{min} = \frac{\dim \mathcal{H}}{\epsilon}$?

Our target: free-fermion states

n fermion modes: $\mathcal{H} = \mathrm{Span}\left(|s\rangle,\ s \in \{0,1\}^n \right)$,

for $2n$ majorana operators c_α, $\{c_\alpha, c_\beta\} = \delta_{\alpha\beta}$,

$$|s\rangle : \quad i\, c_{2j-1} c_{2j} |s\rangle = (-1)^{s_j} |s\rangle$$

free-fermion $\quad |\psi\rangle := \text{G.S.}\left(\sum i h_{\alpha\beta}\, c_\alpha c_\beta \right)$

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$$S = \{ \text{free-fermion } |\psi\rangle \}$$

and

$$S = \{ \text{free-fermion } |\psi\rangle \text{ with particle number } q \}$$

$\begin{pmatrix} \text{Hamming} \\ \text{weight of} \\ s \end{pmatrix}$

i.e. Slater determinants

These allow for exponentially improved simulation, state learning, ground state finding — what abt cloning?

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These allow for exponentially improved simulation, state learning, ground state finding — what abt cloning?

Especially, can we go beyond $N_{min} = \frac{\dim \mathcal{H}}{\epsilon} \sim 2^n/\epsilon$ for the hardness of approximate cloning?

Cloning and learning

Basic connection

Can perfectly clone $\overset{\Rightarrow}{\Leftarrow}$ can perfectly learn

The only examples:

for $S = \{\text{mutually orthogonal}\}$, can do Both

What about imperfect?

What about imperfect?

For $|\psi\rangle \in \mathcal{S} = \{\text{stabilizers}\}$ on n qubits

Learning $N = \Theta(n + \log\frac{1}{\delta})$ copies of $\psi \xrightarrow{\text{learn}} \text{Class}(\tilde{\psi})$ s.t. $\tilde{\psi} = \psi$ with probability $1-\delta$

Montanaro 2017

$\Downarrow$

Cloning near-perfect $f_T = 1-\delta$ cloner T_{stab}

for $N = \Theta(n + \log\frac{1}{\delta})$, K = any

Result: ✓

Step 1	Learn the state $\tilde{\psi}$

Step 2	Output $\tilde{\psi}^{\otimes N+K}$

$$T_{\text{Stab}}(\psi^{\otimes N}) = (1-\delta)\,\psi^{\otimes(N+K)} + \delta\cdot\rho_{\text{trash}}$$

For $|\psi\rangle \in S = \{\text{stabilizers}\}$ on n qubits

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Is $N = \Omega(n)$ needed for near-perfect cloning? Yes!

Bansal, Caro, Mahajan '26

Cloning and learning, pt. 2: continuous $\mathcal{S}$

$\mathcal{S}$ = {free-fermions} are learnable $\Rightarrow$ gives a good cloner?

NO!

Near-perfect learner from N copies	NOT $\not\Rightarrow$ TRUE	Near-perfect $N \to N+K$ cloner

Case $\mathcal{S}$ = {all states}, $N \to N+1$ cloning

Estimator $L(\psi^{\otimes N}) = \sum_i \mathrm{tr}(\psi^{\otimes N} M_i)\, \tilde{\psi}_i$:

Case $\mathcal{S}$ = {all states}, $N \to N+1$ cloning

Estimator $\mathbb{L}(\psi^{\otimes N}) = \sum_i \mathrm{tr}(\psi^{\otimes N} M_i)\, \tilde{\varphi}_i$:

$f_{est} = \mathrm{Tr}\big(\mathbb{L}(\psi^{\otimes N})\,\psi\big) =: 1 - \epsilon_{est}$, cloning yields

f (learn then clone) $= (1-\epsilon_{est})^{N+1} =: 1 - \epsilon_{cl}$, $\epsilon_{est} \approx \dfrac{\epsilon_{cl}}{N}$

for optimal $\mathbb{L}_{opt}$ requires

$$N \geq \frac{\dim \mathcal{H}}{\epsilon_{est}} = \frac{\dim \mathcal{H}}{\epsilon_{cl}} \cdot N$$: impossible!

while $N \to N+1$ optimal cloner with $f_{cl} = 1 - \epsilon_{cl}$ does exist (for $N \gtrsim \dim \mathcal{H}/\epsilon_{cl}$)

$$f\,(\text{learn then clone}) = (1-\epsilon_{est})^{N+1} =: 1-\epsilon_{cl}\,,\quad \epsilon_{est} \approx \frac{\epsilon_{cl}}{N}$$

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while $N \to N+1$ optimal cloner with $f_{cl} = 1-\epsilon_{cl}$ does exist (for $N \gtrsim \dim \mathcal{H}/\epsilon_{cl}$)

Same for free fermions : $f_{est} = 1-\epsilon_{est} \Rightarrow N = \frac{n^2}{\epsilon_{est}} \Rightarrow$ (good) learning-cloning is forbidden !

$\Rightarrow$ for stabilizers it worked due to discreteness

$\Rightarrow$ for free fermions : can we do better ?

"Nice" state families : coherent orbits

(Key technical concept for this work)

Def. $S \subset \mathcal{H}$ is a coherent orbit iff :

(1) $S = \{ \varphi(g)|\psi_0\rangle : g \in G \}$ for unitary rep φ of group G

(2) $\varphi^{\otimes N}$ is irreducible on $\mathcal{H}_S^{(N)}$, where

$$\mathcal{H}_S^{(N)} := \mathrm{Span}\left(|\psi\rangle^{\otimes N} : |\psi\rangle \in S \right) \subset \mathcal{H}^{\otimes N}, \ \forall N$$

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Ex.: $S = \{\text{all states}\}$

(1) $S = \{U|\psi_0\rangle : U \in U(\mathcal{H})\}$

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These are key properties used in Werner's optimal channel.

Now, let's generalize it!

Thm. If S = coherent orbit,

optimal $N \to N+K$ cloner is given by:

$$T_{opt}(\rho_N) := \frac{d_N}{d_{N+K}} \Pi_{N+K} \left(\rho_N \otimes \mathbb{I}_K\right) \Pi_{N+K},$$

Π_{N+K} = projector onto $\mathcal{H}_S^{(N+K)} = \mathrm{Span}\left(|\psi\rangle^{\otimes N+K}\right)$,

$d_N := \dim\left(\mathcal{H}_S^{(N)}\right)$, $\rho_N \in D\left(\mathcal{H}_S^{(N)}\right)$

Its cloning fidelity is $f_S = \dfrac{d_N}{d_{N+K}}$

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Ex.: $S = \{$all states$\}$, $d_N = \dim(\mathrm{Sym}^N(\mathcal{H})) = \binom{\dim\mathcal{H} + N - 1}{N}$

$\rightsquigarrow$ fidelity $f_S = \frac{d_N}{d_{N+K}}$, as discussed!

Ex.: $S = \{\text{all states}\}$, $d_N = \dim(\mathrm{Sym}^N(\mathcal{H})) = \binom{\dim\mathcal{H} + N - 1}{N}$

$\hookrightarrow$ fidelity $f_S = \frac{d_N}{d_{N+K}}$, as discussed!

Proof plan: ① Show $T_{opt} \in CPTP$

② Find fidelity $f_{T_{opt}, S}$

③ Show $f_{T,S} \leq f_{T_{opt}, S}$, $\forall T$

Proof sketch: $T_{opt}(\rho_N) = \frac{d_N}{d_{N+K}} \Pi_{N+K} (\rho_N \otimes \mathbb{1}_K) \Pi_{N+K}$

① $T_{opt} \in CPTP$. CP - manifest, TP:

$$\mathrm{Tr}\left(T_{opt}\left(\varphi(g)^{\otimes N} \rho_N \varphi(g)^{\otimes N}\right)\right) = \mathrm{Tr}\left(T_{opt}(\rho_N)\right)$$

$$\underset{\substack{! \ \varphi^{\otimes N} = \text{irred.} \\ \text{on } \mathcal{H}_S^{(N)}}}{\Longrightarrow} \quad \mathrm{Tr}\left(T_{opt}(\rho_N)\right) = \alpha \, \mathrm{Tr}(\rho_N),$$

check for $\rho_N = \frac{\Pi_N}{d_N} \Rightarrow \alpha = 1$

② Observe: $\varphi^{\otimes N+K} \, \Pi_{N+K} (\varphi^{\otimes N} \otimes \mathbb{1}) \Pi_{N+K} = \varphi^{\otimes N+K}$

$$\Rightarrow \mathrm{Tr}\left(T_{opt}(\psi \quad)\psi \quad\right) = \frac{\quad}{d_{N+k}} = f_{T_{opt},S} \quad (\quad = |\psi\rangle\langle\psi| \,)$$

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(3) Any Better T?

- Assume $T \neq T_{opt}$, $T =$ optimal
- Can assume covariant T:

$$T = T_g := \varphi^\dagger(g)^{\otimes N+k}\, T\left(\varphi(g)^{\otimes N}(\cdot)\,\varphi^\dagger(g)\right)\varphi(g)^{\otimes N+k}$$

(otherwise, $\mathbb{E}_g T_g = T'$ has $f \geq f_T$)

- By irreducibility of $\varphi^{\otimes N+k}$ on $\mathcal{H}_S^{(N+k)}$,

$$T\left(\frac{\Pi_N}{d_N}\right) = \lambda \frac{\Pi_{N+k}}{d_{N+k}} + (1-\lambda)\rho', \quad \rho' \in \mathcal{D}\left(\left(\mathcal{H}_S^{(N+k)}\right)_\perp\right)$$

$$0 \leq \lambda \leq 1$$

Then $\forall |\psi\rangle$: $0 \leq \mathrm{Tr}\left(\psi^{\otimes N+k}\, T\left(\Pi_N - \psi^{\otimes N}\right)\right) = \lambda \frac{d_N}{d_{N+k}} - f_{T,S} \Rightarrow f_{T,S} \leq f_{T_{opt},S}$ ∎

- $S = \{\text{all states}\}$: $\mathrm{Span}(|\psi\rangle^{\otimes N}) = \mathrm{Sym}^N(\mathcal{H})$,

$$d_N = \dim(\mathrm{Sym}^N(\mathcal{H})) = \binom{\dim \mathcal{H} + N - 1}{N}, \quad f_S = \frac{d_N}{d_{N+K}} \quad \text{(Werner '98)}$$

- $S = \{\text{mutually orthogonal } |s\rangle,\ s = 0, \dots d-1\}$: orbit of $G = \langle X, Z\rangle$,

acts irreducibly on $\mathrm{Span}(|s\rangle)$ (and $\mathrm{Span}(|s\rangle^{\otimes N})$),

"classical case"

$$\dim(\mathrm{Span}(|s\rangle^{\otimes N})) = \dim(\mathrm{Span}(|s\rangle)) = d$$

$$\Rightarrow f_S = \frac{d_N}{d_{N+K}} = \frac{d}{d} = 1, \text{ as expected}$$

- $S = \{$ Bose ... $(H = \sum h_{kl}\, a_k^\dagger a_l)\}$.

"classical case"

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- $S = \left\{ \begin{array}{l}\text{Bose}\\ \text{condensates}\end{array} |\psi\rangle = q\text{-particle G.S. } \left(H = \sum h_{jl}\, a_j^\dagger\, a_l\right) \right\}$:

$$|\psi\rangle = \frac{1}{\sqrt{q!}} \left(\sum v_j\, a_j^\dagger\right)^q |0\rangle \cong v^{\vee q} \in \mathrm{Sym}^q(\mathbb{C}^n),$$

$$S = \{v^{\vee q}\} = \text{orbit of } U(\mathbb{C}^n), \text{ irred. on } \mathrm{Span}\left((v^{\vee q})^{\otimes N}\right) = \mathrm{Sym}^{qN}(\mathbb{C}^n),$$

(Actually, follows from Werner)

$$f_s = \frac{d_N}{d_{N+K}} \text{ for } d_N = \binom{n+qN-1}{qN}; \quad f_s = 1 - \epsilon_{cl} \text{ for } N \to N+1$$

achieved for $N \simeq \frac{n}{\epsilon_{cl}} \ll \frac{\dim \mathcal{H}_{q,n}}{\epsilon_{cl}}$

• $S = \{\text{products } |\psi\rangle = |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle, |\psi_j\rangle \in \text{coherent orbit } S_j\}$

orbit of $G = G_1 \times G_2 \dots \times G_n$,

$$S = \{ (\varphi_1(g_1) \otimes \dots \otimes \varphi_n(g_n)) |0\rangle \otimes \dots \otimes |0\rangle \}$$

Could use optimal T on each factor; why is this optimal?

Action of $G^{\otimes N}$ is irreducible on

$$\mathrm{Span}(|\psi\rangle^{\otimes N}) = \mathcal{H}_{S_1}^{(N)} \otimes \mathcal{H}_{S_2}^{(N)} \otimes \dots \otimes \mathcal{H}_{S_n}^{N},$$

Because each G_j is irreducible (fact in rep. theory)

$$\Rightarrow F = \frac{d_N^{(1)}}{d_{N+K}^{(1)}} \cdot \frac{d_N^{(2)}}{d_{N+K}^{(2)}} \cdots \frac{d_N^{(n)}}{d_{N+K}^{(n)}}$$

$$\Rightarrow f = \frac{d_N^{(1)}}{d_{N+K}^{(1)}} \cdot \frac{d_N^{(2)}}{d_{N+K}^{(2)}} \cdots \frac{d_N^{(n)}}{d_{N+K}^{(n)}}$$

- A non-example : $S = \{\text{stabilizer states}\}$,

  orbit of G = Clifford group :

  G = irreducible on $\mathcal{H} = \mathrm{Span}(S)$,

  but $G^{\otimes 4}$ = reducible on $\mathrm{Span}(|\psi\rangle^{\otimes 4}, |\psi\rangle = \mathrm{Stab.})$

  (experts say...)

- What about free fermions?

Free fermion states form coherent orbits

Prop. 1: $S = \{$ free-fermion on n modes with q particles $\}$ is a coherent orbit,

with $d_N = \dim(N\text{-copy span})$

$$= \prod_{i=1}^{q} \prod_{j=1}^{N} \frac{n+j-i}{N+q-i-j+1}$$, giving

optimal $f_{\infty} = \prod^{q} \binom{n-i+N}{N} \cdot \binom{q-i+N+K}{N+K}$ of $N \to N+K$

Proof sketch: Prop. 1 (q-particle)

① $S \ni |\psi\rangle \cong v_1 \wedge v_2 \ldots \wedge v_q \in \Lambda^q \mathbb{C}^n$,

$\mathrm{Span}(|\psi\rangle) = \Lambda^q \mathbb{C}^n$ = rep. φ of $U(n)$,

$\varphi(U)\,|\psi\rangle = (Uv_1)\wedge(Uv_2)\wedge\ldots\wedge(Uv_n)$

$\Rightarrow S$ = orbit of $|\psi_0\rangle = e_1 \wedge e_2 \wedge \ldots \wedge e_n$ ← one-hot vectors

② Coherent? Observe that $|\psi_0\rangle$ = highest weight vector of φ

Young diag. of φ:

φ = highest weight rep. w/ $\omega_\varphi = (\underbrace{1, 1, \ldots 1}_{q}, 0, 0, \ldots 0)$

(single particle occupations highest = "Fermi Sea", Sketch

① $\mathcal{S} \ni |\psi\rangle \cong v_1 \wedge v_2 \dots \wedge v_q \in \Lambda^q \mathbb{C}^n$,

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Invoking "Cartan product" thm:

$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is highest weight if $|\psi_{1,2}\rangle$ are highest weight;

$|\psi_{12}\rangle$ sits in an irreducible component of $\Phi_1 \otimes \Phi_2$

Weights add: $w_{12} = w_1 + w_2$

Here: $|\psi_0\rangle^{\otimes N}$ sits in h.w. w_N component of $\Phi^{\otimes N}$,

$\mathrm{Spin}(2n) = \left\langle \left\{ U_{e^A} = e^{\sum A_{\alpha\beta} c_\alpha c_\beta} \right\} \right\rangle$ this as our G – not $O(2n)$

$e^A \in so(2n)$

② $S = \underset{\circlearrowleft \mathrm{Spin}}{S_+} \underset{\text{reflections}}{\cup} \underset{\circlearrowleft \mathrm{Spin}}{S_-}$: parity sectors (eigs of $P = c_1 \cdots c_{2n}$)

Spin(2n) acts irreducibly on each $\mathcal{H}_\pm = \mathrm{Span}(S_\pm)$,

with lowest weight rep of $w_\pm = -\frac{1}{2}(\pm 1, 1, \ldots, 1)$, ← eigs of $\frac{1}{2} i c_{2j-1} c_{2j}$

lowest weight vectors $|\psi_0\rangle = |0\rangle$ and $|\psi_1\rangle = |10\ldots0\rangle$

(annihilated by all "root vectors", morally – as before)

!! $S_\pm = \mathrm{Spin}(2n) \circ |\psi_{0,1}\rangle$: orbits of lowest-weight vecs!

③ By Cartan product thm, N-copy spans of $S_\pm = \mathcal{H}^{(N)}_{S_\pm}$

$\widetilde{\quad}_{\text{reflections}}$

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!! $S_\pm = \mathrm{Spin}(2n) \circ |\psi_{0,1}\rangle$: orbits of lowest-weight vecs!

(3) By Cartan product thm, N-copy spans of $S_\pm = \mathcal{H}^{(N)}_{S_\pm}$ define irreps of Spin(2n) of weights $w_{N,\pm} = N \cdot w_\pm$

(4) $\mathcal{H}^{(N)}_S = \mathcal{H}^{(N)}_{S_+} \oplus \mathcal{H}^{(N)}_{S_-}$: why does Pin act irreducibly?

Key: $\mathcal{H}_{S_\pm}$ are ineq. irreps of Spin(2n). $\Rightarrow$ coherent orbit $\Rightarrow d_N = d(w_{N,+}) + d(w_{N,-})$ $\blacksquare$

<u>Cor. 1</u> For $S = \{q\text{-part. free-fermions}\}$, near-exact cloning is achieved for

$$N > N_{min} = \Theta\left(\frac{q(n-q)}{\epsilon}\right)$$

<u>Why:</u>
$$\epsilon = -\log f_S = \sum_{j=1}^{q} \log \frac{N+n-q+j}{N+j} \overset{N \gg n}{\simeq} \sum_{j=1}^{q} \frac{n-q}{N+j} \simeq \frac{q(n-q)}{N}$$

← Myste-rious?

<u>Cor. 2</u> For $S = \{\text{free-fermions}\}$, $N_{min} = \Theta\left(\frac{n(n-1)}{2\epsilon}\right)$ wait for it!

(Similar derivation) Both cases: $N_{min} \ll \frac{2^n}{\epsilon}$, exponential improvement

Cloning vs. learning : revisited

Even though near-perfect learning-cloning is impossible, optimal performances for learning and cloning are related:

Prop. For any coherent orbit S,

$$f(N \to 1 \text{ estimation}) = \frac{d_N}{d} \quad \left(= f(N \to N+1 \text{ cloning}) ! \right)$$

for learning and cloning are ...

Prop. For any coherent orbit S,

$$f(N \to 1 \text{ estimation}) = \frac{d_N}{d_{N+1}} \quad \left(= f(N \to N+1 \text{ cloning}) ! \right)$$

(2-line proof)

For unstructured states, this was known long time (eg see Harrow "Church of Symmetric Subspace")

For $S = \{\text{free-fermions}\}$ $f_{opt} = \frac{d_N}{\ldots}$ was

* but not cloning or results for general coherent orbit S, incl. q-particle f.-f.

We have a mystery on our hands

Both for $N \to N+1$ cloning and $N \to 1$ learning, there is a mystery:

$$N_{min} = \frac{\dim \mathcal{H}}{\epsilon} \quad \dots \quad \frac{q(n-q)}{\epsilon} \quad \dots \quad \frac{n(n-1)}{2\epsilon} \quad \dots \quad \frac{n}{\epsilon}$$

Bose condensates

↳ why is it always $\dim(\text{manifold})/\epsilon$?
(just numbers coming from rep theory...)

learning, there is a mystery.

$$N_{min} = \frac{\dim \mathcal{H}}{\epsilon} \;\dots\; \frac{q(n-q)}{\epsilon} \;\dots\; \frac{n(n-1)}{2\epsilon} \;\dots\; \frac{n}{\epsilon}$$ ← Bose condensates

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First guess: no. of parameters? (to learn)

No! By Holevo, can get n bits per copy,
so would expect $N_{min}^{Hol} \sim \frac{\dim}{n}$ → n for free fermions not n^2!

↳ indeed, $N_{min}(\text{Stab}) \sim n$, although "dim" $= n^2$

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This furnishes a hint; maybe
continuity of the family is important?

Resolution of the mystery:

N th moment saddle points!

Sketch: $f = \frac{d_N}{d_{N+1}}$; goal: show $f = 1 - \frac{\dim S}{N}$

$$\mathrm{tr}\ \mathbb{E}_{|\psi\rangle\in M} \psi^{\otimes N} = 1 \Rightarrow \pi_N = d_N \cdot \mathbb{E}_S \psi^{\otimes N}$$

$$\Rightarrow 1 = \mathrm{tr}\ \pi_N \varphi^{\otimes N} \quad \forall \varphi \in S,$$

coherent orbit

$$\Rightarrow d_N = \frac{1}{\mathbb{E}_{\psi\sim S} (|\langle\psi|\varphi\rangle|^{2N})}$$

$\psi \sim S$

$$\Rightarrow \frac{d_N}{d_{N+1}} = \frac{\mathbb{E}_{\psi\sim S} |\langle\psi|\varphi\rangle|^{2N+2}}{\mathbb{E}_{\psi\sim S} |\langle\psi|\varphi\rangle|^{2N}} = \mathbb{E}_{p(\psi)=\frac{|\langle\psi|\varphi\rangle|^{2N}}{\#}} \left(|\langle\psi|\varphi\rangle|^2 \right)$$

S = high. weight $|\psi_0\rangle$ orbit of Lie G

$p(\psi) \propto |\langle\psi|\varphi\rangle|^{2N}$; $\Rightarrow$ coordinates $z \in \mathbb{C}^{\dim S/2}$;

around φ, $|\langle\psi|\varphi\rangle|^2 \simeq 1 - |z|^2 + O(|z|^4)$

$$\Rightarrow p(z) \sim \left(1 - |z|^2 + O(|z|^4)\right)^N \simeq \exp(-N \cdot |z|^2)$$

$$\Rightarrow f \simeq \mathbb{E}_{z\sim\text{Gauss}} (1 - |z|^2) = 1 - \frac{\int e^{-N|z|^2} |z|^2 dz}{\#} = 1 - \frac{\dim S}{2N} \quad \checkmark$$

(but not fully rigorous)

Other results

- Stinespring dilation (unitary realization) for T_{opt}, in terms of Schur transform & Clebsh-Gordan unitary → however, not efficient yet, C-G coefs not even given for Spin ↳ future direction
However, reasons to believe it's possible

(Beyond $N_{min} = \Theta(n)$)

- Extensions to naively non-compact scenarios: squeezed states seem a "coherent" orbit of $Sp(n, \mathbb{R})$, but need regularization ($\dim = \infty$)
- Applications (quantum crypto? distributed computations?)

"don't make your quantum coins Gaussian..."