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Speakers: Benjamin Lovitz

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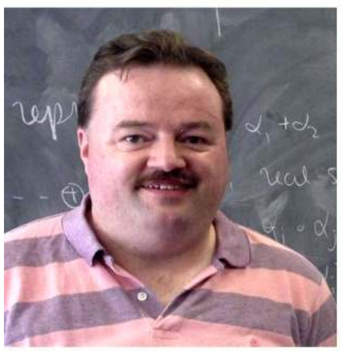
Constructive counterexamples to the additivity of minimum output Rényi entropy of quantum channels for all $p > 1$



Harm Derksen and Benjamin Lovitz

Northeastern University

Concordia University

Classical capacity of a quantum channel

Let $\Phi: D(A) \to D(B)$ be a quantum channel.

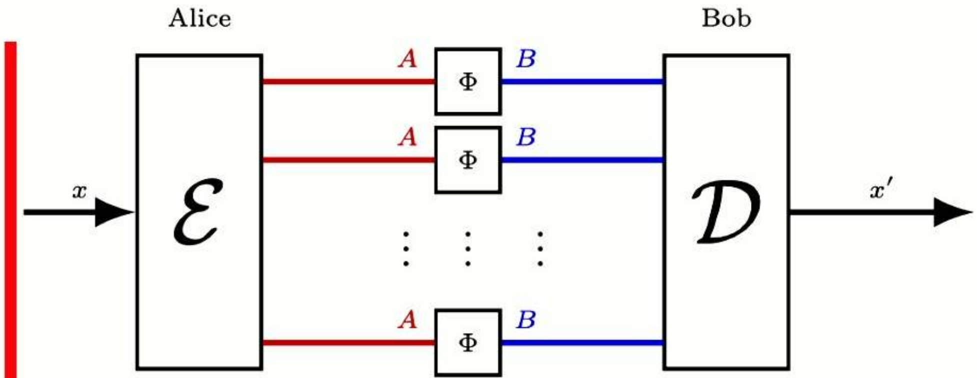
$C(\Phi) :=$ classical bits/channel use that can be sent through Φ

$= \max R$ s.t. $\exists\, m_k, \delta_k, \mathcal{E}_k, \mathcal{D}_k$ satisfying:

1. $R = \lim_{k\to\infty} \frac{m_k}{k}$

2. $\forall x \in \{0,1\}^{m_k}$,
$$\Pr\left(\left(\mathcal{D}_k \circ \Phi^{\otimes k} \circ \mathcal{E}_k\right)(x) = x\right) \geq 1 - \delta_k$$

3. $\lim_{k\to\infty} \delta_k = 0$



$$\{0,1\}^{m_k} \xrightarrow{\mathcal{E}_k} D\left(A^{\otimes k}\right) \xrightarrow{\Phi^{\otimes k}} D\left(B^{\otimes k}\right) \xrightarrow{\mathcal{D}_k} \{0,1\}^{m_k}$$

The Holevo-Schumacher-Westmoreland (HSW) Theorem

Holevo information: $\chi(\{p_i, \rho_i\}) \coloneqq H(\sum_i p_i\ \rho_i) - \sum_i p_i H(\rho_i)$, $H(\rho) \coloneqq -\mathrm{Tr}(\rho \log \rho)$

Holevo capacity: $\chi(\Phi) \coloneqq \max_{\{p_i, \rho_i\}} \chi(\{p_i, \Phi(\rho_i)\})$

Theorem: [H 98, SW 97] $\mathrm{C}(\Phi) = \lim_{k\to\infty} \frac{\chi(\Phi^{\otimes k})}{k}$

The Additivity Conjecture

Additivity Conjecture: $\chi(\Phi \otimes \Psi) = \chi(\Phi) + \chi(\Psi)$ for any Φ, Ψ (Hence $C(\Phi) = \chi(\Phi)$)
($\geq$ is obvious)

Counterexample: [Hastings 09] Random Φ satisfies $\chi(\Phi \otimes \Phi) > 2\,\chi(\Phi)$

Open question: Find explicit Φ with $\chi(\Phi \otimes \Phi) > 2\,\chi(\Phi)$
(hence $\mathrm{C}(\Phi) > \chi(\Phi)$)

Minimum Output Entropy (MOE)

Benjamin Lovitz — Concordia

Rényi p-entropy: For $p > 1$, let $H^{(p)}(\rho) := \frac{1}{1-p} \log \operatorname{Tr}(\rho^p)$

Further, let $H^{(1)}(\rho) := \lim_{p \to 1} H^{(p)}(\rho)$ Fact: $H^{(1)}(\rho) = H(\rho)$

Min output p-entropy: $H_{\min}^{(p)}(\Phi) := \min_{|\psi\rangle} H^{(p)}(\Phi(\psi))$

Theorem: [Shor 04] The following statements are equivalent.

1. $\chi(\Phi \otimes \Psi) = \chi(\Phi) + \chi(\Psi)$ for all Φ, Ψ

2. $H_{\min}^{(1)}(\Phi \otimes \Psi) = H_{\min}^{(1)}(\Phi) + H_{\min}^{(1)}(\Psi)$ for all Φ, Ψ
   ($\leq$ is obvious)

Historically, additivity conjecture mostly studied through MOE (#2).

Some prior work on the additivity conjecture

Cases when the conjecture holds:

$H_{\min}^{(1)}(\Phi \otimes \Psi) = H_{\min}^{(1)}(\Phi) + H_{\min}^{(1)}(\Psi)$ whenever Φ is:

- Identity channel [Asomov-Holevo 01], Unital qubit channel [King 02], Entanglement breaking channel [Shor 02], Depolarizing channel [King 03], ...

Negative results:

Random channel Φ for which $H_{\min}^{(p)}(\Phi \otimes \Phi) < 2H_{\min}^{(p)}(\Phi)$:

- For $p > 1$ [Hayden-Winter 08]
- For p close to zero [Cubitt-Harrow-Leung-Montanaro-Winter 08]
- For $p = 1$ [Hastings 09]

Explicit channel Φ for which $H_{\min}^{(p)}(\Phi \otimes \Phi) < 2H_{\min}^{(p)}(\Phi)$:

- For $p > 4.79$ [Werner-Holevo 02]
- For $p > 2$ [Grudka-Horodecki-Pankowski 10, Szczygielski-Studziński 24]
- For $p > 1$ [Derksen-Lovitz 25]

Main result: [Derksen-Lovitz 25]

For any $p > 1$, an explicit channel Φ with in / out dimension $2^{\tilde{O}\left(\frac{1}{p-1}\right)}$ such that

$$H_{\min}^{(p)}(\Phi \otimes \Phi) < 2H_{\min}^{(p)}(\Phi).$$

Proof ingredients:

- Channels $\longrightarrow$ Subspaces
- Useful bounds on $H_{\min}^{(p)}$
- Entangled subspace construction [Beauzamy+ 90]

Channels ⟶ Subspaces

Def: For a subspace $U \subseteq B \otimes E$, let $H_{\min}^{(p)}(U) := \min_{|\phi\rangle \in U} H^{(p)}(\phi_B)$

Lemma: Let $\Phi(\rho) = \mathrm{Tr}_E\left(V\rho V^\dagger\right)$ and let $U := \mathrm{im}\,(V) \subseteq B \otimes E$.

Then $H_{\min}^{(p)}(\Phi) = H_{\min}^{(p)}(U)$

Proof:

$$H_{\min}^{(p)}(\Phi) = \min_{|\psi\rangle \in A} H^{(p)}(\Phi(\psi)) = \min_{|\psi\rangle \in A} H^{(p)}\left(\mathrm{Tr}_E(V\psi V^\dagger)\right) = \min_{|\phi\rangle \in U} H^{(p)}(\phi_B) = H_{\min}^{(p)}(U)$$ ✓

New goal: Construct subspace $U \subseteq B \otimes E$ s.t. $H_{\min}^{(p)}(U \otimes U) < 2H_{\min}^{(p)}(U)$

$$U \otimes U \subseteq (B \otimes B) \otimes (E \otimes E)$$

Main result: [D-L 25] For any $p > 1$, construct $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ s.t.

$$H_{\min}^{(p)}(U \otimes U) < 2H_{\min}^{(p)}(U) \quad \text{and} \quad n = 2^{\tilde{O}\left(\frac{1}{p-1}\right)}$$

Proof sketch: Bounds on $H_{\min}^{(p)}(U)$

Given subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, let $\Lambda(U) := \max_{|\psi\rangle \in U} \lambda_{\max}(\psi_B)$

Suppose $|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\ |i\rangle \otimes |i\rangle$.

$$\lambda_{\max}(\psi_B) = \lambda_1 = \max_{|u\rangle, |v\rangle \in \mathbb{C}^n} |\langle \psi | u \otimes v \rangle|^2$$

Hence,

$$\Lambda(U) = \max_{|\psi\rangle \in U} \max_{|u\rangle, |v\rangle \in \mathbb{C}^n} |\langle \psi \mid u \otimes v \rangle|^2 = \max_{|u\rangle, |v\rangle \in \mathbb{C}^n} ||\Pi_U(|u\rangle \otimes |v\rangle)||^2$$

For later!

So: $\Lambda(U) = 1 \quad \Leftrightarrow \quad U$ contains a product state

Proof sketch: Bounds on $H_{\min}^{(p)}(U)$

Given subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, let $\Lambda(U) := \max_{|\psi\rangle \in U} \lambda_{\max}(\psi_B)$

$$\frac{1}{1-p}\log\Big(\Lambda(U)^p + \big(1-\Lambda(U)\big)^p\Big) \underset{1}{\leq} H_{\min}^{(p)}(U) \underset{2}{\leq} \frac{p}{1-p}\log(\Lambda(U))$$

Proofs: Let $\rho \in D(B)$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$

2: $$H^{(p)}(\rho) = \frac{1}{1-p}\log\left(\sum_{i=1}^n \lambda_i^p\right) \leq \frac{1}{1-p}\log(\lambda_1^p) = \frac{p}{1-p}\log(\lambda_1)$$

1: $$H^{(p)}(\rho) = H^{(p)}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \geq H^{(p)}\begin{pmatrix} \lambda_1 & & \\ & 1-\lambda_1 & \\ & & \mathbf{0} \end{pmatrix} = \frac{1}{1-p}\log\big(\lambda_1^p + (1-\lambda_1)^p\big)$$

Schur-concavity of $H^{(p)}$

Proof sketch: Bounds on $H_{\min}^{(p)}(U)$

Given subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, let $\Lambda(U) := \max_{|\psi\rangle \in U} \lambda_{\max}(\psi_B)$

$$\frac{1}{1-p}\log\Big(\Lambda(U)^p + \big(1-\Lambda(U)\big)^p\Big) \underset{1}{\leq} H_{\min}^{(p)}(U) \underset{2}{\leq} \frac{p}{1-p}\log(\Lambda(U))$$

<u>Lemma:</u> [Hayden 07] $\Lambda(U \otimes U) \geq \frac{\dim(U)}{n^2}$

Proof sketch: Choose ONB $|u_1\rangle, \dots, |u_\ell\rangle$ of U.

$$\Lambda(U \otimes U) = \max_{\substack{|\mu\rangle \in U \otimes U \\ |\phi\rangle \in B \otimes B \\ |\psi\rangle \in E \otimes E}} |\langle \mu | \phi \otimes \psi \rangle|^2 .$$

Let $|\mu\rangle = \frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} |u_i\rangle \otimes |u_i\rangle$,

$|\phi\rangle = |\psi\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |jj\rangle$

Proof sketch: Bounds on $H_{\min}^{(p)}(U)$

Given subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, let $\Lambda(U) := \max_{|\psi\rangle \in U} \lambda_{\max}(\psi_B)$

$$\frac{1}{1-p}\log\left(\Lambda(U)^p + \left(1-\Lambda(U)\right)^p\right) \underset{1}{\leq} H_{\min}^{(p)}(U) \underset{2}{\leq} \frac{p}{1-p}\log(\Lambda(U))$$

<u>Lemma:</u> [Hayden 07] $\Lambda(U \otimes U) \geq \frac{\dim(U)}{n^2}$

$$H_{\min}^{(p)}(U \otimes U) \underset{2}{\leq} \frac{p}{1-p}\log\left(\frac{\dim(U)}{n^2}\right) \overset{!}{<} \frac{2}{1-p}\log\left(\Lambda(U)^p + \left(1-\Lambda(U)\right)^p\right) \underset{1}{\leq} 2H_{\min}^{(p)}(U)$$

Note: For ! to hold, need $\Lambda(U)$ small and $\dim(U)$ large.

Proof sketch: Bounds on $H_{\min}^{(p)}(U)$

Given subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, let $\Lambda(U) := \max_{|\psi\rangle \in U} \lambda_{\max}(\psi_B)$

$$\frac{1}{1-p}\log\left(\Lambda(U)^p + \left(1-\Lambda(U)\right)^p\right) \underset{1}{\leq} H_{\min}^{(p)}(U) \underset{2}{\leq} \frac{p}{1-p}\log(\Lambda(U))$$

Lemma: [Hayden 07] $\Lambda(U \otimes U) \geq \frac{\dim(U)}{n^2}$

$$H_{\min}^{(p)}(U \otimes U) \underset{2}{\leq} \frac{p}{1-p}\log\left(\frac{\dim(U)}{n^2}\right) \overset{!}{<} \frac{2}{1-p}\log\left(\Lambda(U)^p + \left(1-\Lambda(U)\right)^p\right) \underset{1}{\leq} 2H_{\min}^{(p)}(U)$$

Thm*: [D-L 25] For any $\epsilon > 0$, explicit $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ with

$\Lambda(U) \leq 1-\epsilon$ and $\dim(U) = (1-\epsilon)n^2$

[D-L 25] $\Rightarrow$! :

$$p\log(1-\epsilon) \overset{!}{>} 2\log(\epsilon^p + (1-\epsilon)^p) \quad \Longleftarrow \quad (1-\epsilon)^p \overset{!}{>} (\epsilon^p + (1-\epsilon)^p)^2$$ ✓ for $\epsilon \to 0^+$

Proof sketch: The construction

Thm*: [D-L 25] For any fixed $\epsilon > 0$, explicit $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ with

$$\Lambda(U) \leq 1 - \epsilon \quad \text{and} \quad \dim(U) = (1-\epsilon)n^2$$

Symmetric subspace: $S^d(\mathbb{C}^a) \subseteq (\mathbb{C}^a)^{\otimes d}$

Set of $|\psi\rangle$ for which $|\psi\rangle_{i_1,\ldots,i_d} = |\psi\rangle_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$ for any permutation $\sigma \in \mathfrak{S}_d$

Example: $S^2(\mathbb{C}^a) = \{|\psi\rangle \in \mathbb{C}^a \otimes \mathbb{C}^a \ : \ |\psi\rangle_{ij} = |\psi\rangle_{ji}$ for all $i, j = 1, \ldots, n\}$

Note: $S^{2d}(\mathbb{C}^a) \subseteq S^d(\mathbb{C}^a) \otimes S^d(\mathbb{C}^a).$

Proof sketch: The construction

Thm*: [D-L 25] For any fixed $\epsilon > 0$, explicit $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ with
$\Lambda(U) \leq 1 - \epsilon$ and $\dim(U) = (1-\epsilon)n^2$

Symmetric subspace: $S^d(\mathbb{C}^a) \subseteq (\mathbb{C}^a)^{\otimes d}$
Set of $|\psi\rangle$ for which $|\psi\rangle_{i_1,\ldots,i_d} = |\psi\rangle_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$ for any permutation $\sigma \in \mathfrak{S}_d$

Let $U = S^{2d}(\mathbb{C}^a)^{\perp} \subseteq S^d(\mathbb{C}^a) \otimes S^d(\mathbb{C}^a)$

$B = E = S^d(\mathbb{C}^a)$

Lemma: [Beauzamy+ 90] $\Lambda(U) \leq 1 - \binom{2d}{d}^{-1}$

Thm pf sketch: *Assuming $\epsilon = \binom{2\mathrm{d}}{\mathrm{d}}^{-1}$ for some d:

Note $n = \dim\left(S^d(\mathbb{C}^a)\right) = \binom{a+d-1}{d}$.

$\dim(U) = n^2 - \binom{a+2d-1}{2d} \underset{a \to \infty}{\longrightarrow} (1-\epsilon)\, n^2$ ✓

Proof sketch: The construction

Thm*: [D-L 25] For any fixed $\epsilon > 0$, explicit $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ with
$\Lambda(U) \leq 1 - \epsilon$ and $\dim(U) = (1-\epsilon)n^2$

Symmetric subspace: $S^d(\mathbb{C}^a) \subseteq (\mathbb{C}^a)^{\otimes d}$
Set of $|\psi\rangle$ for which $|\psi\rangle_{i_1,\ldots,i_d} = |\psi\rangle_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$ for any permutation $\sigma \in \mathfrak{S}_d$

Let $U = S^{2d}(\mathbb{C}^a)^{\perp} \subseteq S^d(\mathbb{C}^a) \otimes S^d(\mathbb{C}^a)$

$B = E = S^d(\mathbb{C}^a)$

Lemma: [Beauzamy+ 90] $\Lambda(U) \leq 1 - \binom{2d}{d}^{-1}$

Where does this lemma come from?

$S^d(\mathbb{C}^a) \cong \mathbb{C}[x_1, \ldots, x_a]_d$

$|\psi\rangle \mapsto p_\psi(x) = \langle \psi | x^{\otimes d} \rangle$

[Beauzamy+ 90] original statement: $\|p \cdot q\| \geq \binom{2d}{d}^{-1/2} \|p\| \, \|q\|$ for all $p, q \in \mathbb{C}[x_1, \ldots, x_a]_d$

Proof of Beauzamy+ lemma

Lemma: [Beauzamy+90] Let $U = S^{2d}(\mathbb{C}^a)^{\perp} \subseteq S^d(\mathbb{C}^a) \otimes S^d(\mathbb{C}^a)$.

Then $\Lambda(U) \leq 1 - \binom{2d}{d}^{-1}$

Proof sketch:

Let $\Pi_{2d} := \text{proj}\left(S^{2d}(\mathbb{C}^a)\right) = I - \Pi_U$

Note that:

$$\min_{|\phi\rangle,|\psi\rangle \in S^d(\mathbb{C}^a)} ||\Pi_{2d}(|\phi\rangle \otimes |\psi\rangle)||^2 = 1 - \max_{|\phi\rangle,|\psi\rangle \in S^d(\mathbb{C}^a)} ||\Pi_U(|\phi\rangle \otimes |\psi\rangle)||^2$$

$$= 1 - \Lambda(U)$$

So, equivalent to prove: $\min_{|\phi\rangle,|\psi\rangle \in S^d(\mathbb{C}^a)} ||\Pi_{2d}(|\phi\rangle \otimes |\psi\rangle)||^2 \geq \binom{2d}{d}^{-1}$

Proof of Beauzamy+ lemma

Equivalent to prove: $\min_{|\phi\rangle,|\psi\rangle \in S^d(\mathbb{C}^a)} ||\Pi_{2d}(|\phi\rangle \otimes |\psi\rangle)||^2 \geq \binom{2d}{d}^{-1}$

For $\sigma \in \mathfrak{S}_{2d}$, define $P_\sigma \in U\left((\mathbb{C}^a)^{\otimes 2d}\right)$ as

$$P_\sigma |i_1 \cdots i_{2d}\rangle = |i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(2d)}\rangle$$

Note: $\Pi_{2d} = \frac{1}{(2d)!} \sum_{\sigma \in \mathfrak{S}_{2d}} P_\sigma$

Let $|x\rangle := |\phi\rangle \otimes |\psi\rangle$. Lemma: $\langle x|P_\sigma|x\rangle \geq 0$ for all $\sigma \in \mathfrak{S}_{2d}$.

Hence, $||\Pi_{2d}(|\phi\rangle \otimes |\psi\rangle)||^2 = \frac{1}{(2d)!} \sum_{\sigma \in \mathfrak{S}_{2d}} \langle x|P_\sigma|x\rangle \geq \frac{1}{(2d)!} \sum_{\sigma \in \mathfrak{S}_d \times \mathfrak{S}_d} \langle x|P_\sigma|x\rangle$

$$= \frac{1}{(2d)!} \sum_{\sigma \in \mathfrak{S}_d \times \mathfrak{S}_d} \langle x|x\rangle = \frac{d!d!}{(2d)!} = \binom{2d}{d}^{-1}$$

Comparison to randomized constructions

Thm: [Hayden-Winter 08] For any $p > 1$, randomized $V \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ s.t.

$$2H_{\min}^{(p)}(V) - H_{\min}^{(p)}(V \otimes V) \;=\; \log n - \mathcal{O}(1)$$

By contrast, our explicit $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ achieves only constant gap $\mathcal{O}(1)$:

$$U = S^{2d}(\mathbb{C}^a)^{\perp} \subseteq S^d(\mathbb{C}^a) \otimes S^d(\mathbb{C}^a) \quad \text{contains} \quad |u\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle^d|1\rangle^d - |1\rangle^d|0\rangle^d\right)$$

$|u\rangle$ has Schmidt rank 2, so $H_{\min}^{(p)}(U) \leq 1$.

Does our construction satisfy $H^{(1)}_{\min}(U \otimes U) < 2\, H^{(1)}_{\min}(U)$?

Typical randomized construction: [Hastings 09, Brandao-Hastings 10, Fukuda-King 10, Aubrun-Szarek-Werner 11]

Construct random $V \subseteq \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ s.t. $n_2 = \alpha\, n_1^2$ and $\dim(V) = \beta n_2$ for const. α, β

satisfying $H^{(1)}_{\min}(V) \geq \log(n_1) - \mathcal{O}\left(\frac{1}{n_1}\right)$.

For our construction, $H^{(1)}_{\min}(U) \leq 1$ 😭

Then

$$\begin{aligned} H^{(1)}_{\min}(V \otimes V) &\leq \big(1 - \Lambda(V \otimes V)\big) \log(n_1^2 - 1) + h\big(\Lambda(V \otimes V)\big) && \text{by Schur-concavity} \\ &\leq 2\left(1 - \frac{\beta}{\alpha n_1}\right) \log(n_1) + h\left(\frac{\beta}{\alpha n_1}\right) && \text{because } \Lambda(V \otimes V) \geq \frac{\dim(V)}{n_1 n_2} \\ &= 2\log(n_1) - \Omega\left(\frac{\log n_1}{n_1}\right) \end{aligned}$$

So

$$2H^{(1)}_{\min}(V) - H^{(1)}_{\min}(V \otimes V) \;\geq\; \Omega\left(\frac{\log n_1}{n_1}\right) - \mathcal{O}\left(\frac{1}{n_1}\right) \;>\; 0.$$

Conclusion

Main result: [Derksen-Lovitz 25]

For any $p > 1$, an explicit channel Φ with in / out dimension $2^{\tilde{O}\left(\frac{1}{p-1}\right)}$

such that

$$H_{\min}^{(p)}(\Phi \otimes \Phi) < 2H_{\min}^{(p)}(\Phi).$$

Question: Does this construction also work for $p = 1$?

From here, the strict inequality (6) follows easily since $\varepsilon \leq 1/2$. Note that for this construction we have $n = \binom{a+d-1}{d} \leq (a+d-1)^d \leq 2^{\mathcal{O}(-(p-1)^{-1}\log(p-1))}$. This completes the proof. □

Remark 2.5. For particular values of p we can optimize d and a so that $n = \binom{a+d-1}{d}$ is as small as possible subject to the inequality (3) holding. We record a loosely optimized table of sufficient values for d and a, along with $n = \binom{a+d-1}{d}$ and $\dim(\mathcal{U}) = n^2 - \binom{a+2d-1}{2d}$, for a few values of p:

p	d	a	n	$\dim(\mathcal{U})$
2	2	7	28	574
1.5	2	13	91	6461
1.25	3	44	15,180	$\approx 2.16 \cdot 10^8$
1.125	5	240	$\approx 6.92 \cdot 10^9$	$\approx 4.76 \cdot 10^{19}$
1.0625	10	889	$\approx 8.94 \cdot 10^{22}$	$\approx 7.99 \cdot 10^{45}$

3 Constructing entangled subspaces in other parameter regimes

In this section we explicitly construct subspaces of $\mathcal{H} = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m}$ with high geometric measure of entanglement. Let $S = [n_1] \times \cdots \times [n_m]$. For a tuple of coefficients $C = (C_\alpha)_{\alpha \in S}$, let

$$\mathcal{U}_C := \left\{ \psi \in \mathcal{H} : \sum_{\alpha \in S} C_\alpha \, \psi_\alpha = 0 \quad \text{for all} \quad k \right\},$$