Title: How to count states in gravity

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Collection/Series: Quantum Gravity

Subject: Quantum Gravity

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Abstract:

Gibbons and Hawking proposed that the Euclidean gravity path integral with periodic boundary conditions in time computes the thermal partition sum of gravity. As a corollary, they argued that a derivative of the associated free energy with respect to the Euclidean time period computes gravitational entropy. Why is this interpretation correct? That is, why does this path integral compute a trace over the Hilbert space of quantum gravity? I will show that the quantity computed by the Gibbons-Hawking path integral is equal to an a priori different object -- an explicit thermal trace over the Hilbert space spanned by states produced by the Euclidean gravity path integral. I will explain that this follows if the Hilbert space with two boundaries factorizes into a product of two single boundary Hilbert spaces. To show the latter I will develop a basis for the nonperturbative Hilbert space of quantum gravity with one asymptotic boundary. I will use this basis to show that the Hilbert space for gravity with two disconnected boundaries factorizes into a product of two copies of the single boundary Hilbert space, from which our main result will follow.

How To Count States In Gravity

VIJAY BALASUBRAMANIAN

Based largely on:

- "How to Count States In Gravity", VB & Tom Yildirim, 2506.15767
- "A Nonperturbative Toolkit For Gravity", VB & Tom Yildirim, 2504.16986
- "The Nonperturbative Hilbert Space of Quantum Gravity With One Boundary", VB & Tom Yildirim, 2506.04319
- "Observing Spacetime", VB & Tom Yildirim, 2509.09763

Also

- VB, Javier Magan, Martin Sasieta, Albion Lawrence 2212.02447, 2212.08623
- Martin Sasieta 2211.11794
- Ana Climent, R. Emparan, J. Magan, M. Sasieta, 2401.08775
- M. Knysh, J. Hernandez, D. He, M. Khramtsov, B. Craps 2510.02997, 2412.06884, 2410.00091

Parallel work

(a) JT gravity
Iliesu, Boruch, Tuviaci, Maxfield...

(b) ETH & variants
de Boer, Belin, Sasieta, Post....

(c) Earlier work on islands
PSSY, Chandra, Hartman...

A puzzle

Consider a QFT with some fields $\{\phi\}$ and an action I
The path integral in Euclidean signature $(t \to i\tau)$ with periodic time $(\tau \sim \tau + \beta)$ computes the partition function:

semiclassical

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar}\int I[\phi]\, dt} \sim e^{-\beta I[\phi_c]}$$

Saddlepoint

Identify $I[\phi_c]$ = free energy

$\Rightarrow$ S = entropy $= \left(\beta \frac{\partial}{\partial \beta} - 1\right) I[\phi_c]$

GIBBONS + HAWKING (1978)

Apply this: path integral with $t \to i\tau$; $\tau \sim \tau + \beta$ to gravity

$$I[X] = \frac{-1}{16\pi G}\int_X (R - 2\Lambda) + \frac{1}{8\pi G}\int_{\partial X} K + I_{c.t.}$$

Cosm. Const (Λ)

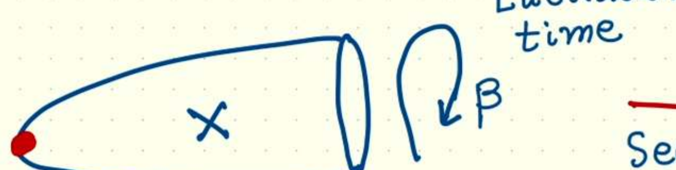
Ricci Scalar (R)

Extrinsic curvature for well-defined e.o.m. (K)

Possible counterterms to make the action finite ($I_{c.t.}$)

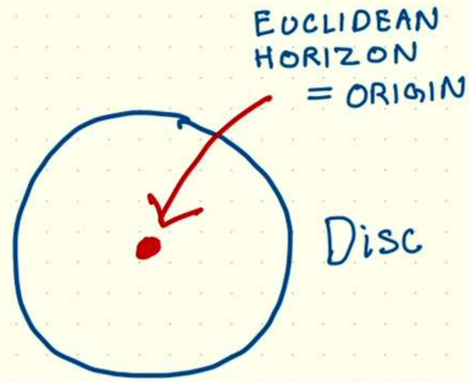
Saddlepoints

$$ds^2 = f(r)\, d\tau^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2_{d-1}$$



"Cigar geometry"

Seen from right



$$\Rightarrow \quad \boxed{S = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I[X_c] = \frac{A}{4G}}$$

- These pictures apply to $\Lambda = 0$ and $\Lambda < 0$ (asymptotically flat and AdS space)

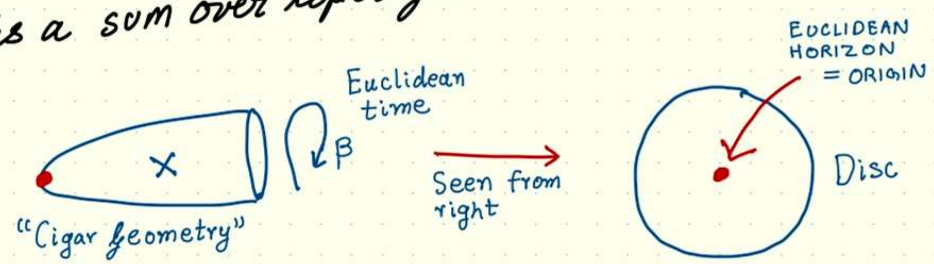
- $\Lambda > 0$ (asymptotically de Sitter) is a bit different since the Euclidean geometry has no boundary (it is a sphere)

Why is this correct?

① Why does the Euclidean gravity path integral with periodic boundary conditions compute a trace over the quantum gravity Hilbert space?

② How do the coarse-grained semiclassical saddlepoints of a path integral for the metric somehow know about the full quantum gravity Hilbert space?

③ The saddlepoint in question does not even look like a cylinder ( ). It looks like a disc ( ). So why does it compute a trace? What is more, in general there is a sum over topologies.



We will show: [Gibbons-Hawking Path Integral] = [Explicit thermal trace over Hilbert space produced by the Euclidean path integral]

This follows in two ways

(a) If the Hilbert space with two boundaries factorizes into a product of Hilbert spaces with one boundary. So we will show that this factorization happens.

(b) Via explicit resolution of the trace by a spanning basis of states.

We will focus on (a) which is an important question about quantum gravity in its own right

A proof in 8 steps

1. How to make states
2. Action of the Hamiltonian
3. Wormholes and nonperturbative overlaps
4. Using a non-orthogonal basis
5. Constructing a basis
6. Factorization of the 2-boundary Hilbert space
7. Statement of the puzzle
8. Resolution of the puzzle

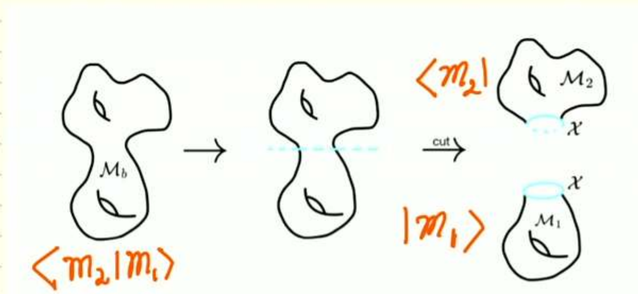
STEP 1: HOW TO MAKE STATES

$$\zeta[\Sigma_b] = \int_{g,\phi \to \Sigma_b} \mathcal{D}g\, \mathcal{D}\phi\, e^{-I_{bulk}[g,\phi]}$$

$$I_{bulk} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{g}\,(R - 2\Lambda + \mathcal{L}_{matter}) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}=\mathcal{M}_b} \sqrt{h}\, K$$

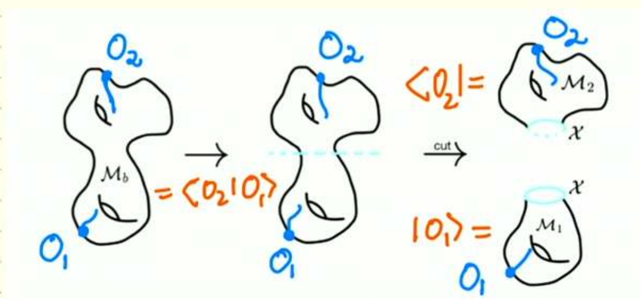
Gravitational Path Integral

Sum over all geometries and topologies subject to the boundary condition



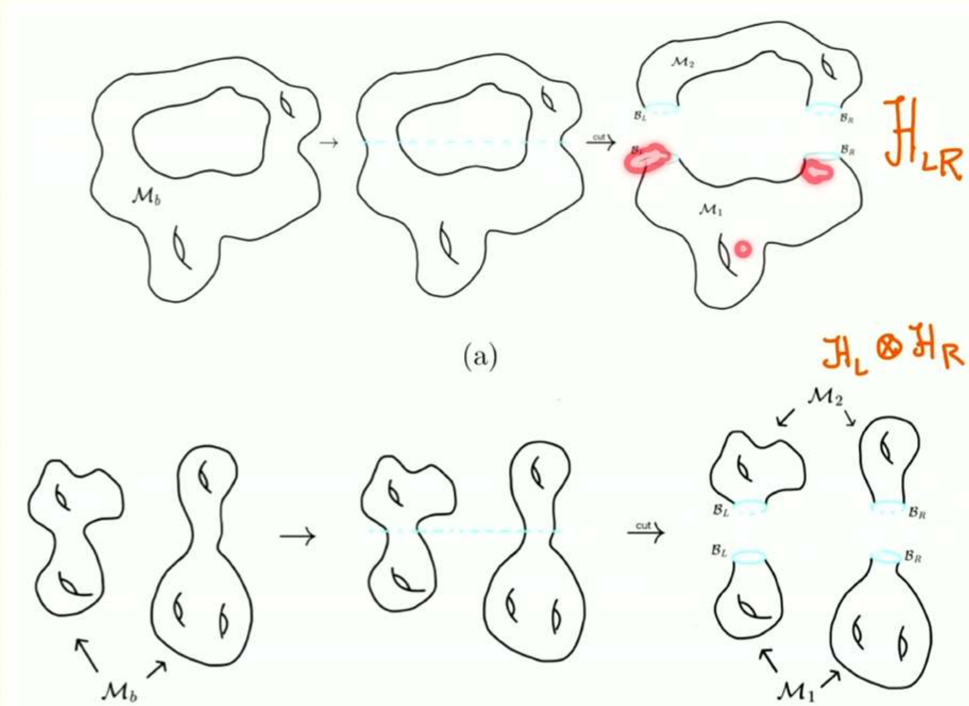
Cut open the path integral to make states

We usually evaluate in the saddlepoint approximation in which there is a fixed semiclassical topology, geometry, and matter trajectory that extremize the action



Can insert operators at the boundary to change the boundary conditions and create additional states

One and Two Boundary Hilbert Spaces



Naively,

$H_{L \cup R} = H_{LR} \cup (H_L \otimes H_R)$

We will see that

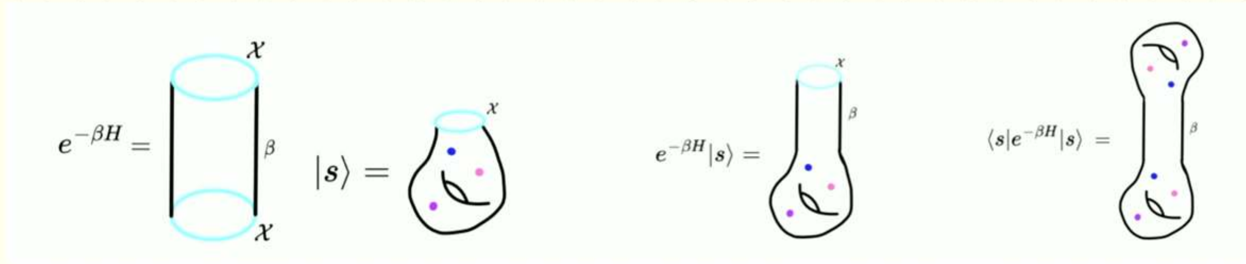
$H_{LR} = H_L \otimes H_R$

$\Downarrow$

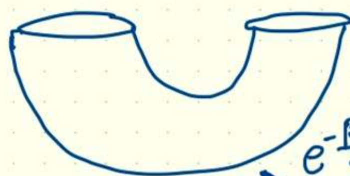
The two boundary Hilbert space is a product of the single boundary Hilbert spaces

$$\mathcal{H}_{LR} \stackrel{?}{=} \mathcal{H}_{LR} \stackrel{?}{=} \mathcal{H}_L \otimes \mathcal{H}_R$$

STEP 2: ACTION OF THE HAMILTONIAN

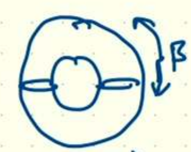


- $e^{-\beta H}$ glues a cylinder of length β to the boundary
- Equivalently:



$e^{-\frac{\beta}{2}H} \equiv |\beta\rangle$
(A Thermofield Double State)

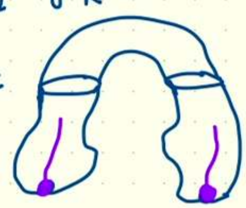
$\Longrightarrow \langle \beta | \beta \rangle =$



$= Z(\beta)$

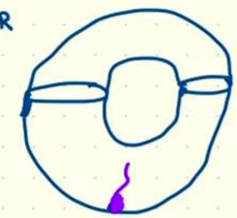
So: $\langle \beta | \, |i\rangle_L \otimes |j\rangle_R$

$=$



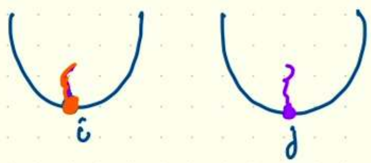
$\langle \beta | \mathcal{O} \rangle_{LR}$

$=$

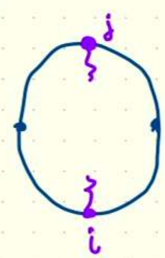


STEP 3: WORMHOLES AND NONPERTURBATIVE OVERLAPS

Suppose you create states $|i\rangle$ and $|j\rangle$



with $\overline{\langle j|i\rangle} =$

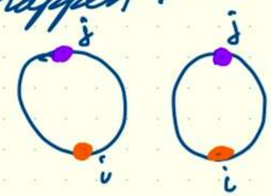


$= \delta_{ij}$

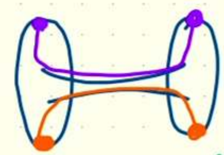
(we normalize by $\bigcirc$)

Then it can still happen that

$\overline{\langle i|j\rangle\langle j|i\rangle} =$



$+$



Nonperturbative contrib to the overlap

$= \delta_{ij}^2 + Z_{\text{WORMHOLE}}$

orthogonal in the EFT

nonperturbative fine grained contribution to overlap

We understand this as happening because the path integral coarse grains over the microscopic configurations. The effect is reproduced if assume that the states satisfy a generalized version of the Eigenstate Thermalization Hypothesis.

STEP 4 : USING A NON-ORTHOGONAL BASIS

- We will want to construct a basis of states. In the EFT there will be infinitely many apparently orthogonal states. We need to account for the nonperturbative overlaps.

- Consider a set of states $\{|i\rangle\}$ constructed by inserting operators into the Euclidean path integral and let $\mathcal{H}_K = \mathrm{Span}\{|i\rangle, i=1\cdots K\}$. We are going to let $K\to\infty$. Because of the nonperturbative overlaps it can be that $\dim \mathcal{H}_K < K$ so that basis in overcomplete

- Standard linear algebra for dealing with this

  Let $G = \begin{bmatrix} \langle 1|1\rangle & \langle 1|2\rangle & \langle 1|3\rangle & \cdots \\ \langle 2|1\rangle & \langle 2|2\rangle & \langle 2|3\rangle & \cdots \\ \langle 3|1\rangle & \langle 3|2\rangle & \langle 3|3\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ = Gram matrix

Then we can write:

ORTHOGONAL

- $|v_\gamma\rangle = G^{-1/2}_{ji}\, u_{i\gamma}\, |j\rangle$ ($G = u D u^\dagger$, D=diagonal)

- $\langle v_\gamma | \mathcal{O} | v_\xi \rangle = G^{-1/2}_{ki} G^{-1/2}_{j\ell}\, u^{-1}_{\gamma k}\, u_{\ell \xi} \langle i | \mathcal{O} | j \rangle$

- $Tr_{\mathcal{H}_K}(\mathcal{O}) = G^{-1}_{ij} \langle j | \mathcal{O} | i \rangle$

- $\mathbb{1}_{\mathcal{H}_K} = \sum_\gamma |v_\gamma\rangle\langle v_\gamma| = G^{-1}_{ij}\, |i\rangle\langle j|$

- $dim(\mathcal{H}_K) = Tr_{\mathcal{H}_K}(\mathbb{1}) = G^{-1}_{ij} \langle j | i \rangle$

$$= \lim_{n \to -1} G^n_{ij} \langle j | i \rangle = \lim_{n \to -1} Tr(G^{n+1})$$

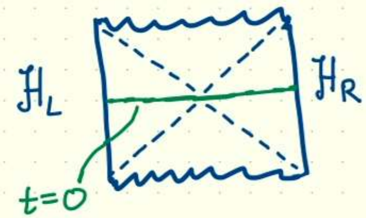
So the key object is

$$G^{-1}_{ij} = \lim_{n \to -1} G^n$$

STEP 5: CONSTRUCTING A BASIS

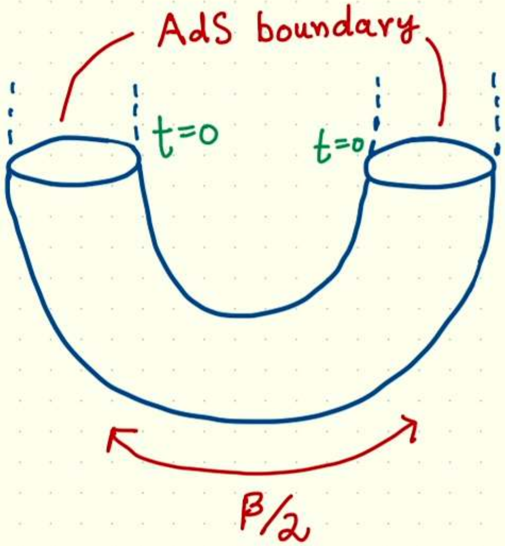
Let's start with the 2 boundary setting

CONSIDER $\Lambda < 0$ AND THE STANDARD AdS/CFT CORRESPONDENCE

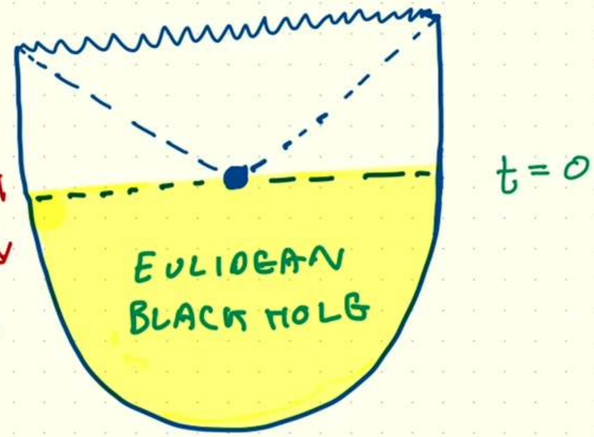


= *Thermofield Double State in $\mathcal{H}_L \otimes \mathcal{H}_R$*

$$|TFD\rangle_{CFT} = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta E_n}{2}} |n_L\rangle \otimes |n_R\rangle$$

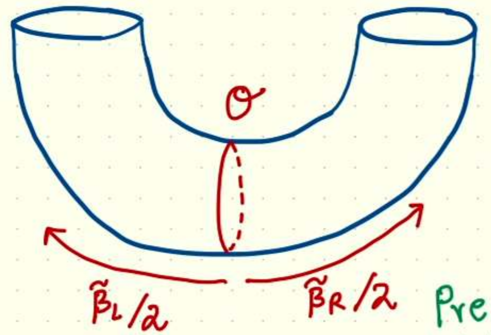


Euclidean evolution to $t=0$. Then match to Lorentzian.



Drawn here for $\Lambda < 0$. Could also do $\Lambda = 0$.

- Make a different state



$\Rightarrow$ $|\psi_{\mathcal{O}}\rangle = \frac{1}{\sqrt{2}} \sum_{n,m} e^{-\tilde{\beta}_L E_n/2} e^{-\tilde{\beta}_R E_m/2} \times \mathcal{O}_{nm} |n,m\rangle$

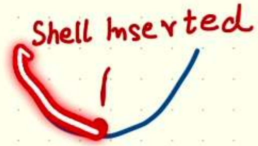
$\mathcal{O}$ = cloud of local operators

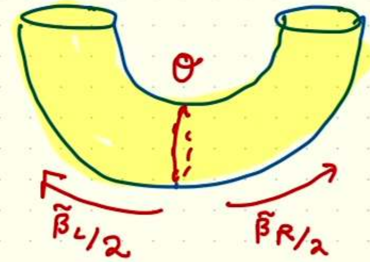
Preparation temperatures can be picked to keep the total energy/temperature fixed

- Choose $\mathcal{O}$ to have a semiclassical dual description
  - e.g.; create a spherical cloud of dust particles
  - e.g., if ϕ is an operator $\dim(\phi) = \Delta_\phi$ then $\mathcal{O} \sim \prod_{i=1}^{n} \phi(x_i) \Rightarrow \dim(\mathcal{O}) \sim n\Delta_\phi$
  - Pick $\Delta_\phi \sim 1$ and $n \sim 1/G_N$ to have large backreaction

Gravitational Description

- Past Euclidean boundary

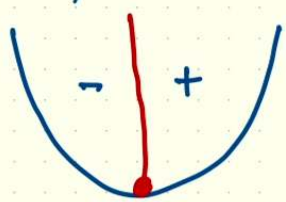




FILL IN THE BULK

- Evolve forward in time

Saddlepoint = solution to GR + shell of dust



= state with a good semiclassical description

$$T_{\mu\nu} = \sigma\, u_\mu u_\nu$$

$$m = \sigma\, V_\Omega\, r^{d-1}$$

Geometry on either side of the shell = Euclidean black hole

$$ds_\pm^2 = f_\pm(r)\, d\tau_\pm^2 + \frac{dr^2}{f_\pm(r)} + r^2\, d\Omega_{d-1}^2$$

$$f_\pm(r) = \frac{r^2}{\ell^2} + 1 - \frac{16\pi G M_\pm}{(d-1) V_\Omega\, r^{d-1}} \quad (\text{for } d > 2)$$

$$\left(\frac{dR}{dT}\right)^2 + V_{\text{eff}}(R) = 0$$

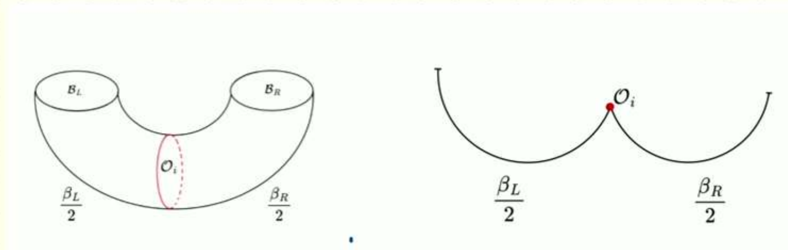
R = shell position
equation of motion from the Israel junction conditions

Infinitely many solutions of this kind with the same exterior mass

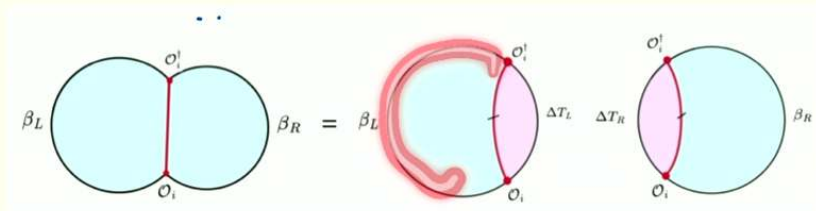
$\Downarrow$

microstates?

2-Sided State Preparation

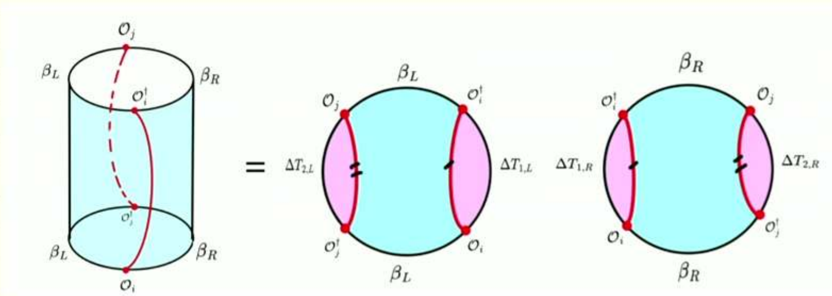


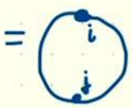
Saddlepoint for overlap $\langle i|i\rangle$

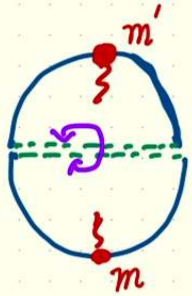


We will take the masses m_i of the shells to to be $O(1/G_N)$ and different so that $\langle i|j\rangle$

Wormhole Contribution To $\langle i|j\rangle\langle j|i\rangle$



=  ≈ 0



$$= \langle \psi_{m'}|\psi_m\rangle = \delta_{mm'} + O(e^{-(m-m')})$$

These computations simplify as $m\to\infty$:  (Kissing Discs)

Lorentzian description

β_{HP} = Inverse Hawking-Page Temperature

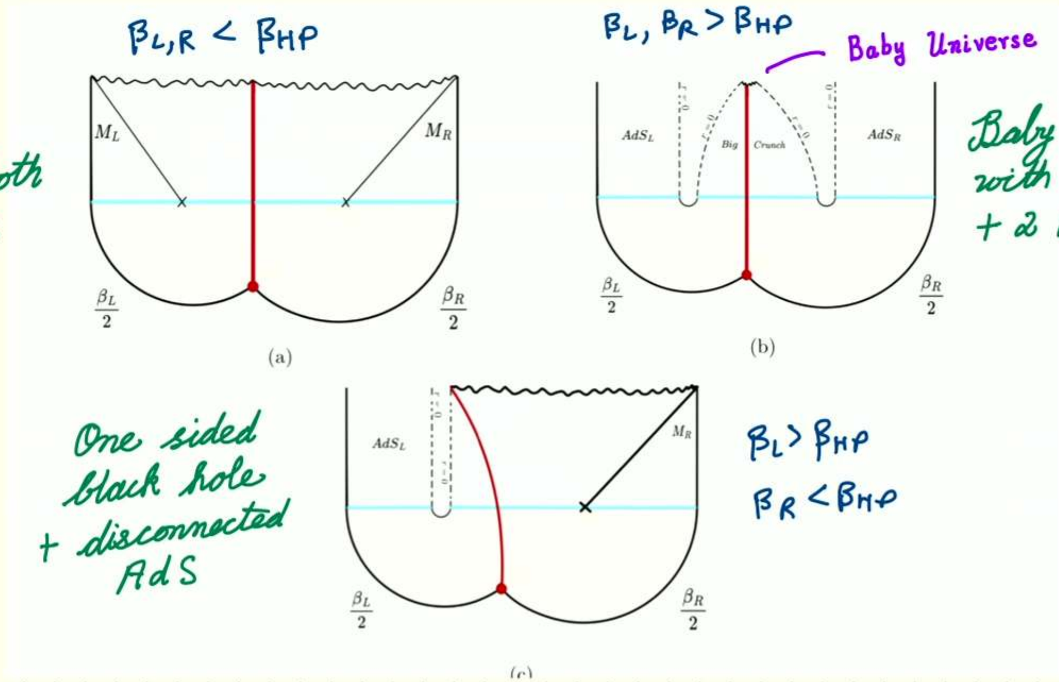
$\beta_{L,R} < \beta_{HP}$

$\beta_L, \beta_R > \beta_{HP}$

Baby Universe

Shell behind both horizons

Baby universe with a shell + 2 AdS universes



One sided black hole + disconnected AdS

$\beta_L > \beta_{HP}$

$\beta_R < \beta_{HP}$

- How to show that these shell states at fixed temperature span the 2-sided Hilbert space?

Remember

$$G = \begin{bmatrix} \langle 1|1\rangle & \langle 1|2\rangle & \langle 1|3\rangle \\ \langle 2|1\rangle & \langle 2|2\rangle & \cdots \\ \cdots & & \cdots \end{bmatrix}$$

- You need: $1_{\mathcal{H}_{LR}} \stackrel{?}{=} \prod_{\mathcal{H}_{shell}} \equiv G^{-1}_{ij}\, |i\rangle\langle j|$

overline $\Rightarrow$ grav. path integral calculation

$$\Rightarrow \overline{Z(\beta)} = \overline{\langle\beta|\beta\rangle} = \overline{\langle\beta|1_{\mathcal{H}_{LR}}|\beta\rangle} \stackrel{?}{=} \overline{\langle\beta|\prod_{\mathcal{H}_{shell}}|\beta\rangle} = \overline{G^{-1}_{ij}\langle\beta|i\rangle\langle j|\beta\rangle}$$

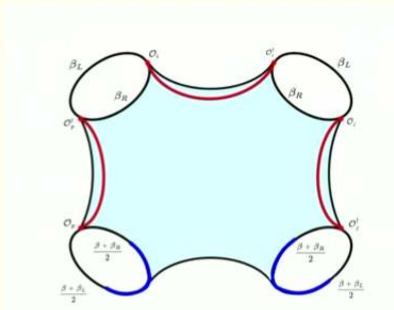
$$= \lim_{n\to -1} \underbrace{\overline{G^{n}_{ij}\langle\beta|i\rangle\langle j|\beta\rangle}}_{\text{summed over } i \text{ and } j}$$

- Each term is a product of overlaps

Each overlap has a contribution 

But the product also includes many kinds of wormholes between these diagrams.

• The topological sum simplifies drastically when $\{|i\rangle,\ i=1\cdots K\}$ with $K\to\infty$. You need to do this to have a basis for the full Hilbert space. If you are working in a microcanonical band this basis is overcomplete but the G^{-1} above takes care of that



• When $K\to\infty$ only the maximally connected wormhole contributes

Example for
$G^2\langle\beta|i\rangle\langle j|\beta\rangle$

$$\overline{Z(\beta)}=\overline{\langle\beta|\beta\rangle}=\overline{\langle\beta|\mathbb{1}_{\mathcal{H}_{LR}}|\beta\rangle}\overset{?}{=}\overline{\langle\beta|\prod_{\mathcal{H}_{shell}}|\beta\rangle}=\overline{G^{-1}_{ij}\langle\beta|i\rangle\langle j|\beta\rangle}$$

$$=\lim_{n\to-1}\overline{G^{n}_{ij}\langle\beta|i\rangle\langle j|\beta\rangle}$$

$$\lim_{n\to-1}\overline{G^{n}_{ij}\langle\beta|i\rangle\langle j|\beta\rangle}=\lim_{n\to-1}K^{n+1}\frac{\overline{Z}((n+1)(\beta_R+\beta_L)+\beta)}{\overline{Z}(\beta_L)^{n+1}\,\overline{Z}(\beta_R)^{n+1}}=\overline{Z(\beta)}\ \checkmark$$

STEP 6: FACTORISATION OF THE TWO BOUNDARY HILBERT SPACE

PROJECTOR ONTO 2-SIDED SHELL STATES

We need to show:

$$① \langle \psi_\otimes | \prod_{\mathcal{H}_{2S}} | \psi_\otimes \rangle = \langle \psi_\otimes | \psi_\otimes \rangle$$

state in the product of 1-sided Hilbert spaces

$$② \langle \Phi_{2S} | \prod_{\mathcal{H}_L} \prod_{\mathcal{H}_R} | \Phi \rangle_{2S} = \langle \Phi_{2S} | \Phi_{2S} \rangle$$

state in the 2-sided Hilbert space

PROJECTOR ONTO ONE-SIDED SHELL STATES

$$\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2S}} |i\rangle_L |j\rangle_R} = \lim_{n\to -1} \overline{G^n_{2S,k\ell} \langle i|_L \langle j|_R |k\rangle_{2S} \langle \ell|_{2S} |i\rangle_L |j\rangle_R}$$

$$\overline{\langle p| \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} |p\rangle_{2S}} = \lim_{n,m\to -1} \overline{G^n_{L,ac} G^m_{R,i\ell} \langle p|_{2S} a\rangle_L |i\rangle_R \langle c|_L \langle \ell|_R |p\rangle_{2S}}$$

Use the Gram matrix to insert the identity in the 2 sided or product of 1 sided Hilbert spaces

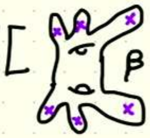
Every term is a product of overlaps. Use same techniques as we described above $\Rightarrow$

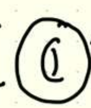
ⓐ heavy shells $\Rightarrow$ saddlepoint geometries are kissing discs

ⓑ having many shells limits the topologies to maximally connected ones

$\Rightarrow$ both ① and ② are true $\Rightarrow$ $\boxed{H_{L\cup R} = H_{LR} = H_L \otimes H_R}$

STEP 7: STATEMENT OF THE PUZZLE

$$\overline{Tr_{\mathcal{H}_\lambda}(e^{-\beta H})} = \zeta[$$  $$\beta] + \zeta[$$  $$\beta] + \zeta[$$  $$\beta] + \cdots \Big\}$$ Thermal Trace

$$\overline{Z(\beta)} = \zeta[$$  $$\beta] \Big\}$$ Gibbons-Hawking

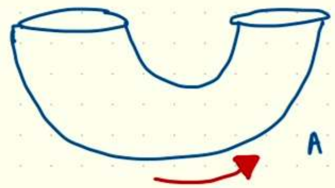
Why are these equal?

Step 8: Resolution of the Puzzle

Identity in the 1-sided Hilbert space

Thermal Trace: $\overline{\underbrace{Tr_{\mathcal{H}_\chi}(e^{-\beta H})}_{\text{Thermal Trace in 1-boundary Hilbert space}}} = \overline{Tr_{\mathcal{H}_\chi}(e^{-\frac{\beta H}{2}}\, 1_\chi\, e^{-\frac{\beta H}{2}})} = \lim_{n,m\to -1} \overline{G^n_{ij}\, G^m_{kl} \langle \alpha_i^* | e^{-\frac{\beta H}{2}} | \alpha_k\rangle \langle \alpha_l | e^{-\frac{\beta H}{2}} | \alpha_j^* \rangle}$

Now:



$e^{-\frac{\beta}{2}H} \equiv |\beta\rangle$

A Thermofield Double State

Identity in the 2-sided Hilbert space

So: $\overline{Tr_{\mathcal{H}_\chi}(e^{-\beta H})} = \lim_{n,m\to -1} \overline{G^n_{ij}\, G^m_{kl} \langle\beta|\alpha_i\rangle_L |\alpha_k\rangle_R \langle\alpha_j|_L \langle\alpha_l|_R|\beta\rangle} = \overline{\langle\beta| 1_{\chi_L}\otimes 1_{\chi_R} |\beta\rangle}$

BUT: $1_{\chi_L}\otimes 1_{\chi_R} = 1_{\chi_L\cup\chi_R}$

Gibbons-Hawking path integral

$\Longrightarrow \overline{Tr_{\mathcal{H}_\chi}(e^{-\beta H})} = \overline{\langle\beta|1_{\chi_L}\otimes 1_{\chi_R}|\beta\rangle} = \overline{\langle\beta|1_{\chi_L\cup\chi_R}|\beta\rangle} = \overline{\langle\beta|\beta\rangle} \equiv \overline{Z(\beta)}$

So we have shown that the Gibbons-Hawking path integral in fact computes the thermal trace in the one-sided theory.

CONCLUSION

- Our computations are valid in ANY dimension.
- Our computations work for $\Lambda = 0$ and $\Lambda < 0$ (asymp. flat & AdS).
- Since the gravitational path integral is computing some kind of coarse-grained average, we should also calculate the variance of each quantity. These variances turn out to vanish, which means that the results hold at a fine-grained level.
- We took several limits: $m_{\text{shell}} \to \infty$, $\kappa \to \infty$ etc.
  You have to be careful about limits. We were careful.

There is now a very powerful set of tools to study questions in quantum gravity using the resolution of the identity by states of the kind we described.