Title: Lecture - Mathematical Physics II, PHYS 777

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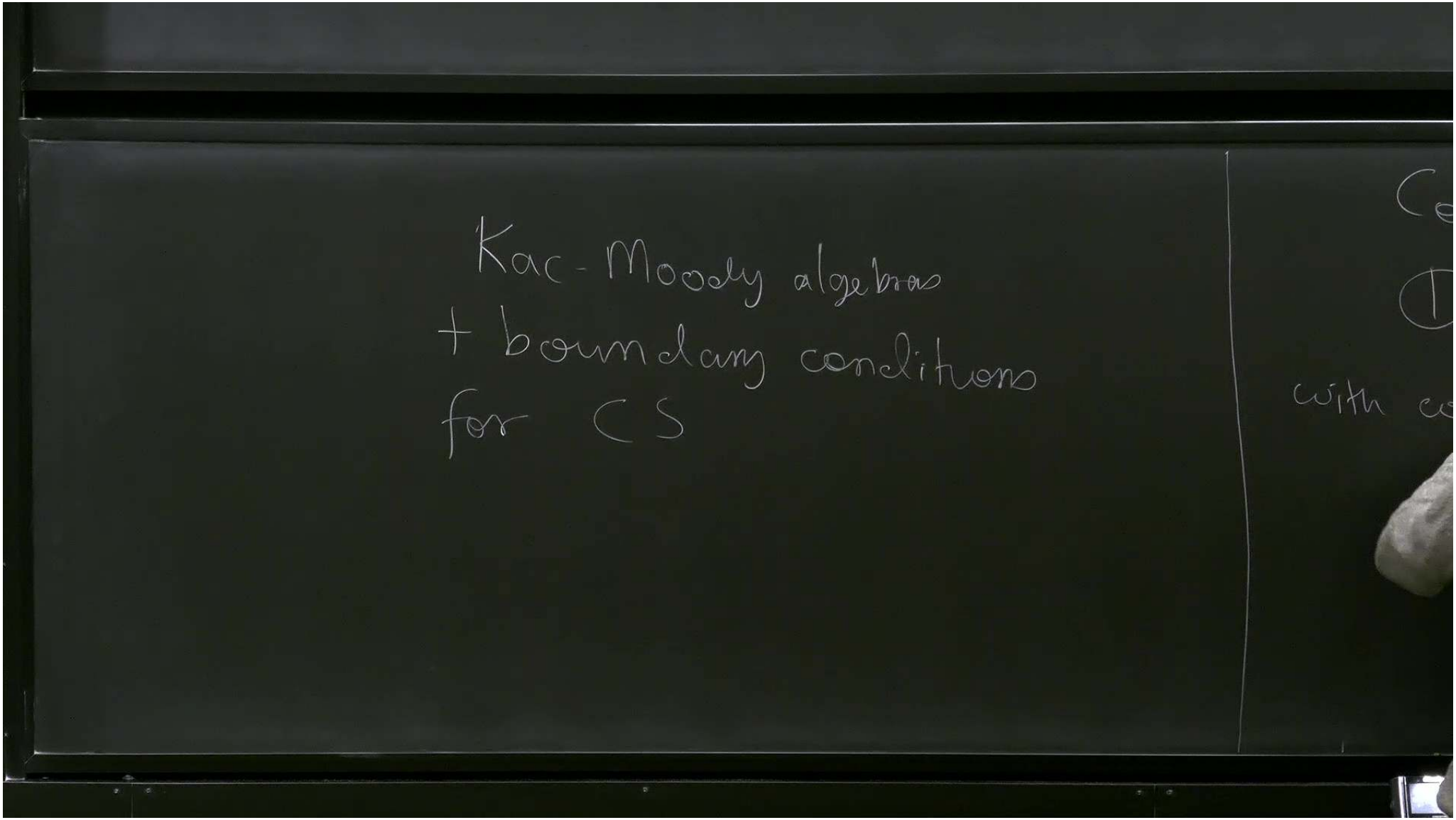
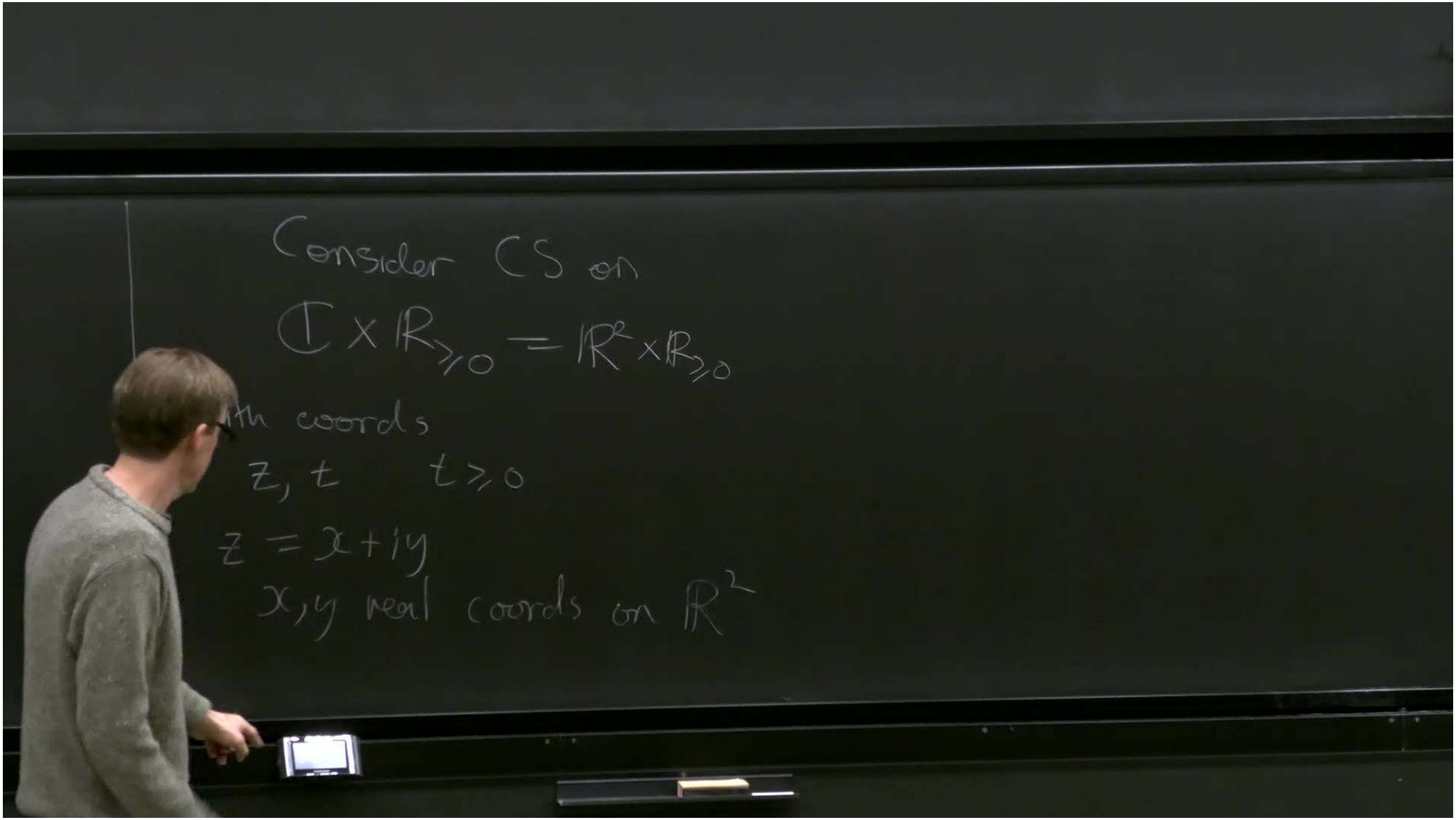
Collection/Series: Mathematical Physics II (Elective), March 30 - May 1, 2026

Subject: Mathematical physics

Date: April 21, 2026 - 2:15 PM

URL: https://pirsa.org/26040031


Kac-Moody algebras
+ boundary conditions
for CS


Consider CS on
$\mathbb{C} \times \mathbb{R}_{\geq 0} = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$
th coords
$z, t \quad t \geq 0$
$z = x + iy$
x, y real coords on $\mathbb{R}^2$

Chiral Dirichlet boundary conditions

set $A_{\bar{z}} = 0$ at $t=0$

And gauge trans. $=0$ at $t=0$.

$$A = A_x dx + A_y dy + A_t dt$$

$$= A_z dz + A_{\bar{z}} d\bar{z} + A_t dt$$

$dz = dx + i dy$

$d\bar{z} = dx - i dy$

CS has a 1st order kinetic term, so a b.c. should set ½ of fields to zero

Let's check this is gauge invariant (Abelian gauge group)

If $\chi(z,t)$ is a param. of gauge transformation. Our b.c. says $\chi = 0$ at $t=0$

Consider $\delta A = d\chi$

Under this gauge transformation,

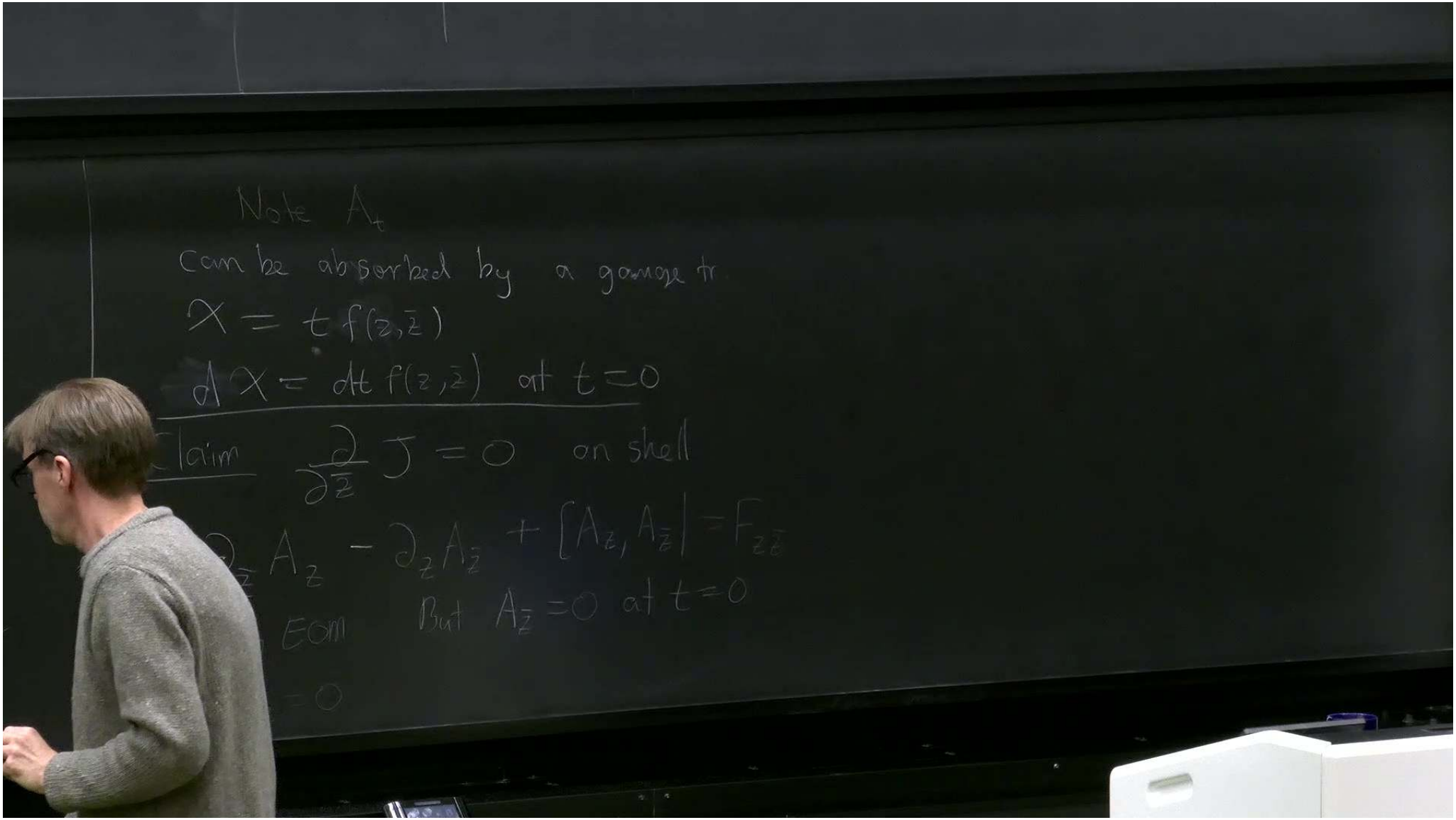
$CS(A) = AdA$ is sent to $d\chi dA$

Finite gauge transformation is

$$e^{\chi} = 1d \text{ at } t=0$$

Since on the boundary, $t=0$, we have no gauge transformations, now we have boundary local operators

$J(z) = kA_z$ as, A_z does not vanish on boundary


Note A_t
can be absorbed by a gauge tr.
$\alpha = t f(z,\bar{z})$
$d\alpha = dt\, f(z,\bar{z})$ at $t=0$
Claim $\frac{\partial}{\partial \bar{z}} J = 0$ on shell
$\partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} + [A_z, A_{\bar{z}}] = F_{z\bar{z}}$
EOM
But $A_{\bar{z}} = 0$ at $t=0$
$= 0$

on boundary

$\partial_z A_z = 0$

All boundary operators are built from

$J_a(z)$ (a Lie algebra index)

General operator at $z=0$ is of form

$$\left(\partial_z^{k_1} J_{a_1}(0)\right)\left(\partial_z^{k_2} J_{a_2}(0)\right)\cdots\left(\partial_z^{k_n} J_{a_n}(0)\right)$$

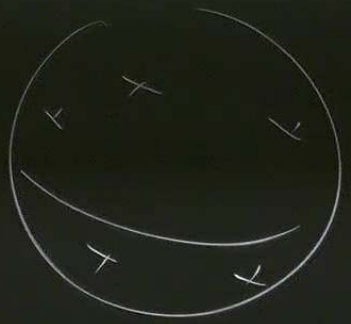
This is a classical expression; at quantum level use normally ordered product

All of these operators are holomorphic

So, if we study e.g. CS on B^3 with Dirichlet b.c., we can define correlation fns

$\langle \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) \rangle$ of boundary local ops.

These are meromorphic, poles at $z_i = z_j$



First: What is

$\langle J_a(z_1)\, J_b(z_2) \rangle$?

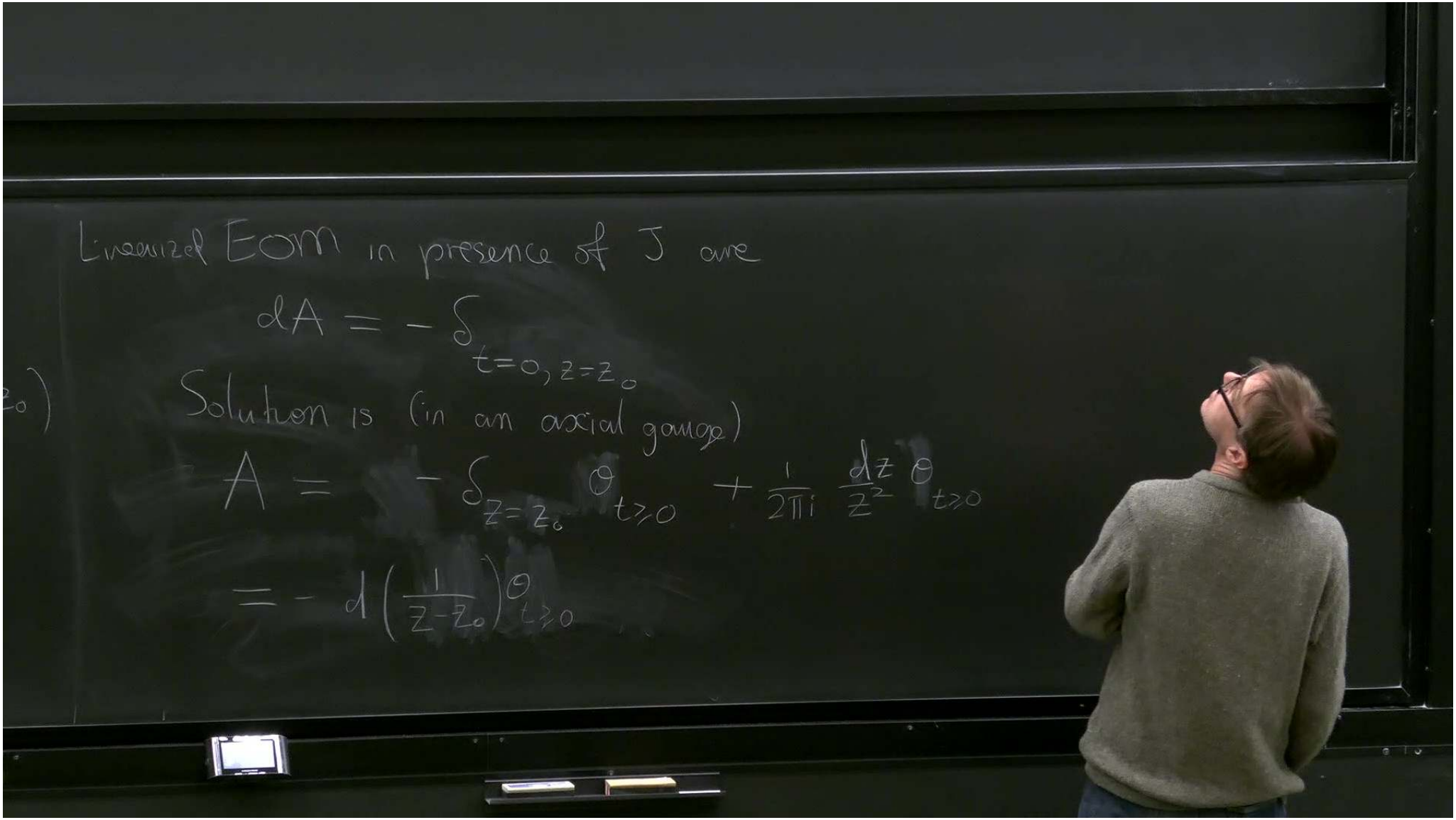
Answer will be $\dfrac{-K\,\delta_{ab}}{(z_1 - z_2)^2}$

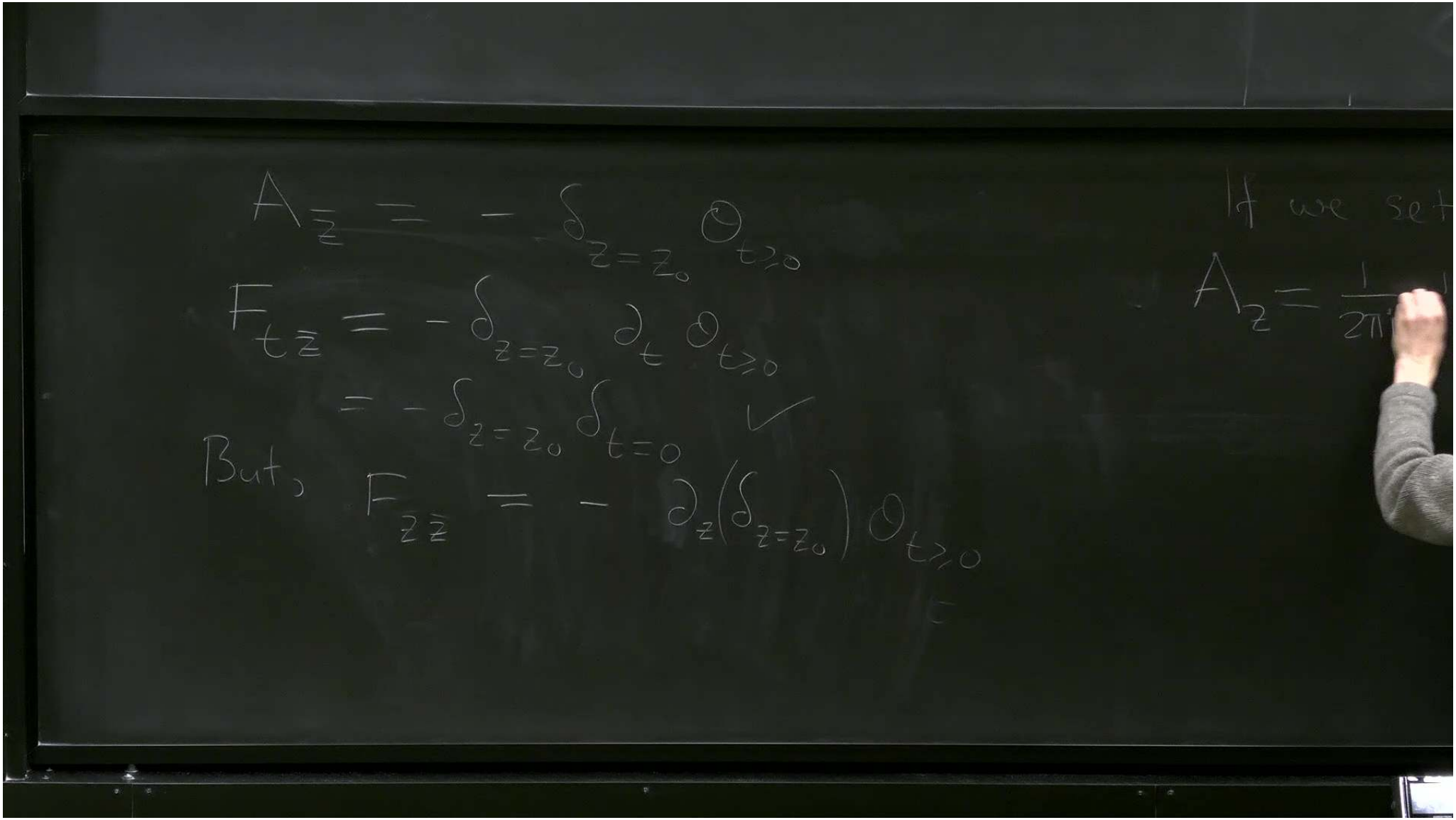
We will compute this by
Feynman diagrams

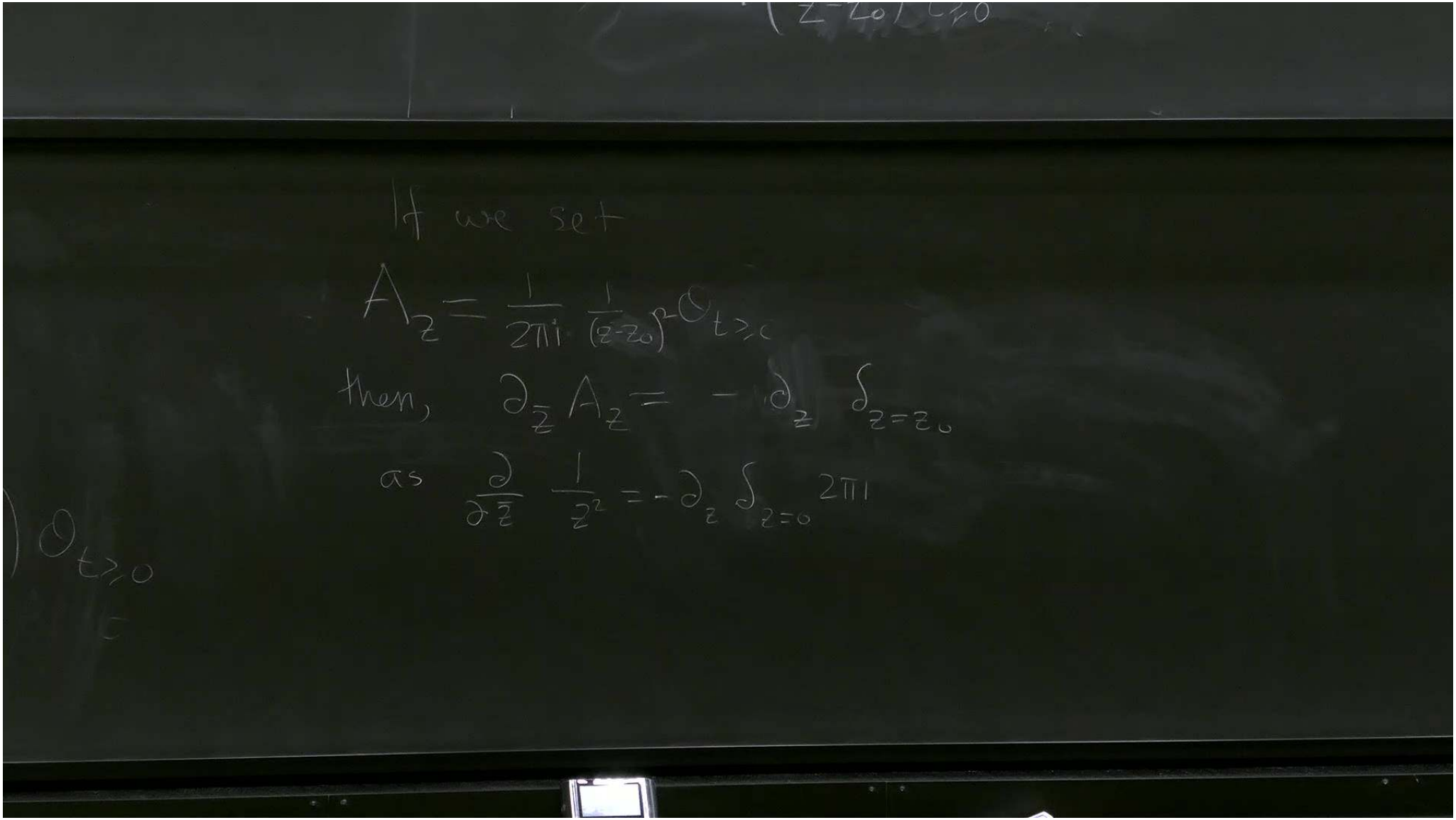
We need to find field sourced by $J(z_0)$

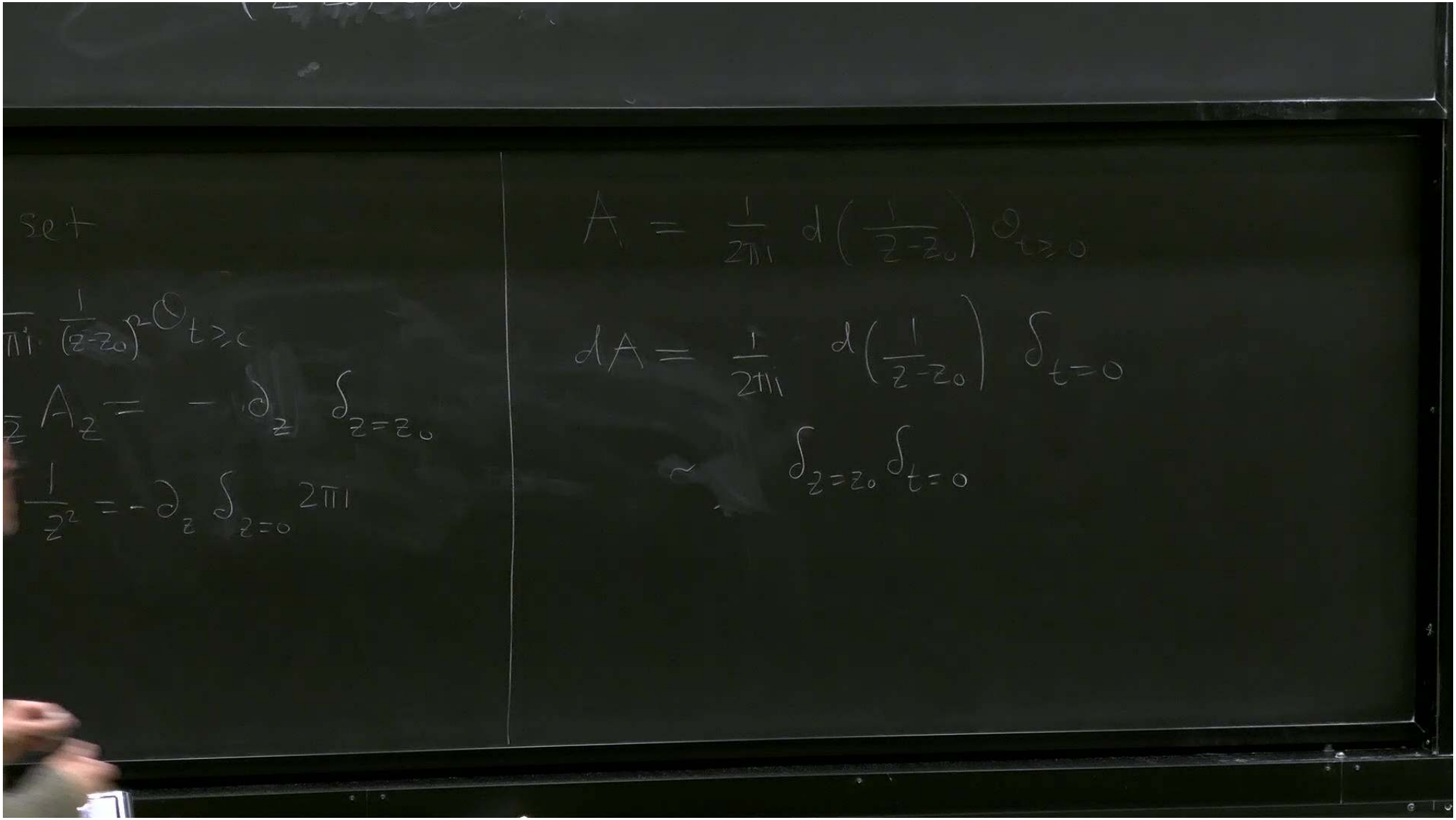
$\Leftrightarrow$ bulk-boundary propagator.

$$\int_{\mathbb{C}\times\mathbb{R}_{z>0}} A\,dA + A_z(z_0)$$


Linearized EOM in presence of J are
$dA = -\delta_{t=0,\,z=z_0}$
Solution is (in an axial gauge)
$A = -\delta_{z=z_0}\,\Theta_{t\geq 0} + \frac{1}{2\pi i}\frac{dz}{z^2}\,\Theta_{t\geq 0}$
$= -d\left(\frac{1}{z-z_0}\right)\Theta_{t\geq 0}$


$A_{\bar{z}} = -\delta_{z=z_0}\,\theta_{t>0}$
$F_{t\bar{z}} = -\delta_{z=z_0}\,\partial_t\,\theta_{t>0}$
$= -\delta_{z=z_0}\,\delta_{t=0}$
But, $F_{z\bar{z}} = -\partial_z(\delta_{z=z_0})\,\theta_{t>0}$
If we set
$A_z = \frac{1}{2\pi}$


If we set
$A_z = \frac{1}{2\pi i} \frac{1}{(z-z_0)^2} \mathcal{O}_{t>0}$
then, $\partial_{\bar{z}} A_z = -\partial_z \delta_{z=z_0}$
as $\frac{\partial}{\partial \bar{z}} \frac{1}{z^2} = -\partial_z \delta_{z=0} 2\pi i$


set
$\frac{1}{\pi i}\frac{1}{(z-z_0)^2}\theta_{t>c}$
$\partial_z A_z = -\partial_z \delta_{z=z_0}$
$\frac{1}{z^2} = -\partial_z \delta_{z=0} 2\pi i$
$A = \frac{1}{2\pi i} d\left(\frac{1}{z-z_0}\right)\theta_{t>0}$
$dA = \frac{1}{2\pi i} d\left(\frac{1}{z-z_0}\right)\delta_{t=0}$
$\sim \delta_{z=z_0}\delta_{t=0}$

set

$$\frac{1}{\pi i}\cdot\frac{1}{(z-z_0)^2}\,\theta_{t>c}$$

$$\bar{z}\,A_z = -\partial_z\,\delta_{z=z_0}$$

$$\frac{1}{z^2} = -\partial_z \int_{z=0} 2\pi i$$

$$A = \frac{1}{2\pi i}\,d\left(\frac{1}{z-z_0}\right)\theta_{t>0}$$

$$dA = \frac{1}{2\pi i}\,d\left(\frac{1}{z-z_0}\right)\delta_{t=0}\,dt$$

$$\sim \delta_{z=z_0}\,\delta_{t=0}\,d\bar{z}\,dt$$

on boundary

$\partial_{\bar{z}} A_z = 0$

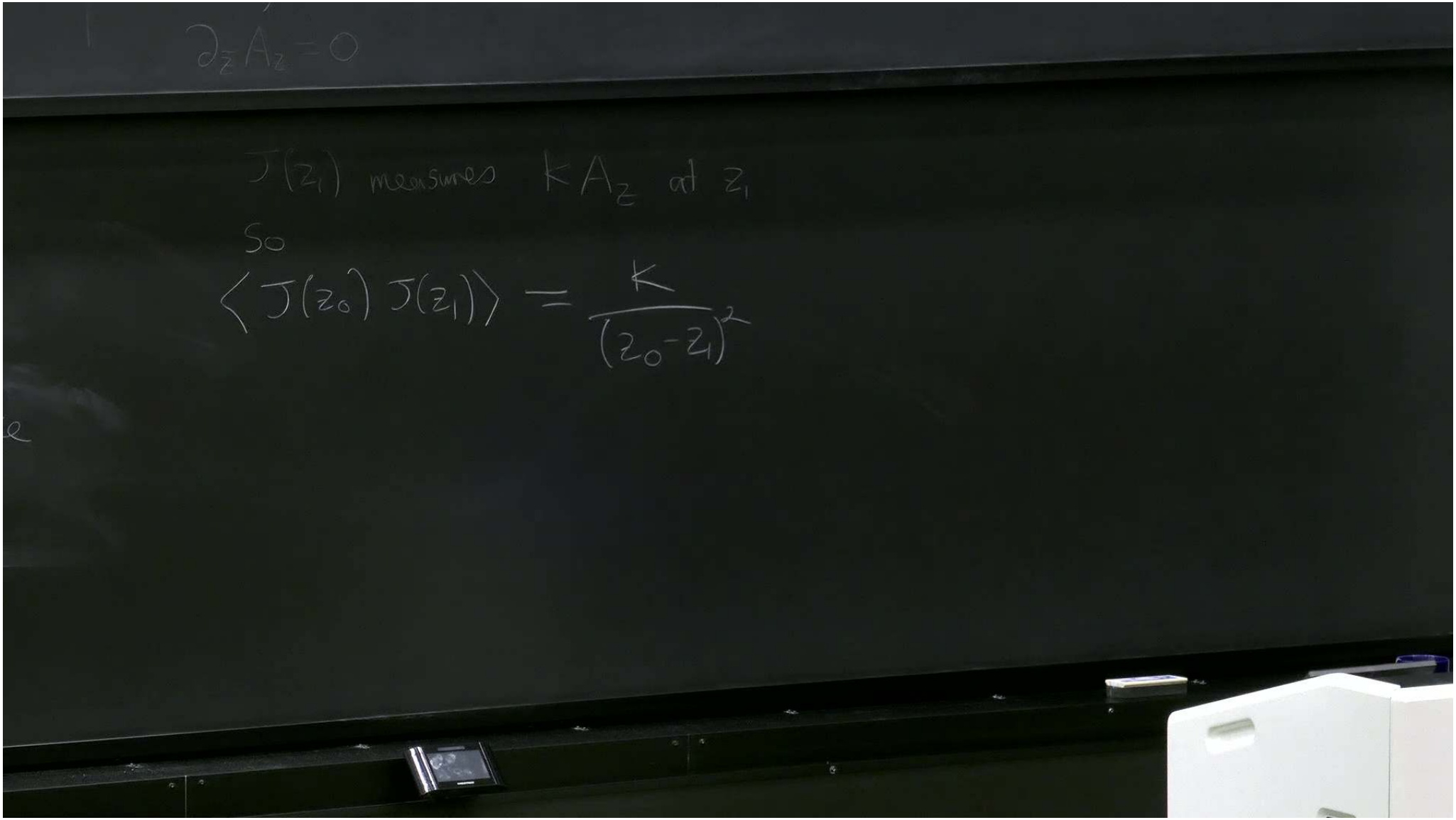
$J(z_1)$ measures

We have learned that

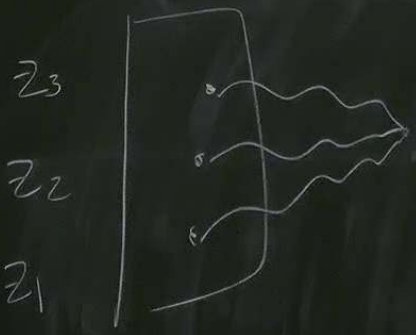
if A is field sourced by $J(z_0)$

then A_z at $t=0$ looks like

$$\frac{1}{(z-z_0)^2}$$


$\partial_{\bar{z}} A_z = 0$
$J(z_1)$ measures $k A_z$ at z_1
So
$\langle J(z_0)\, J(z_1) \rangle = \frac{k}{(z_0 - z_1)^2}$

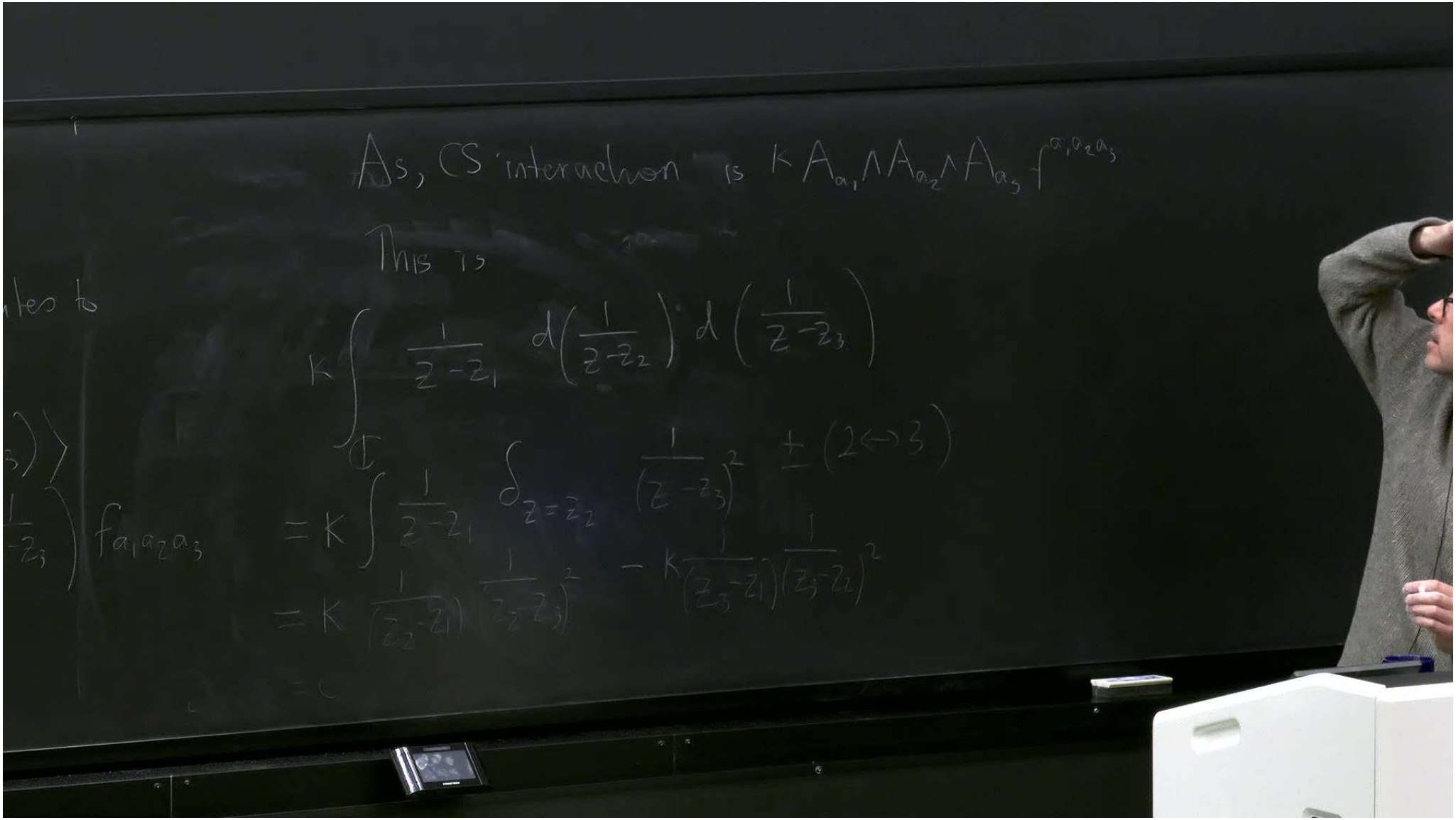
What is 3 point fn?



This diagram contributes to 3 point fn:

$$\langle J_{a_1}(z_1)\, J_{a_2}(z_2)\, J_{a_3}(z_3)\rangle$$

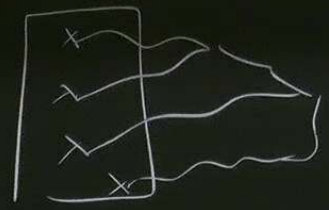
$$= \int_{\mathbb{C}\times R_{>0}} d\left(\frac{1}{z-z_1}\right) d\left(\frac{1}{z-z_2}\right) d\left(\frac{1}{z-z_3}\right) f_{a_1 a_2 a_3}$$


As, CS interaction is $K A_{a_1} \wedge A_{a_2} \wedge A_{a_3} f^{a_1 a_2 a_3}$
This is
$K \int_C \frac{1}{z-z_1} d\left(\frac{1}{z-z_2}\right) d\left(\frac{1}{z-z_3}\right)$
$= K \int \frac{1}{z-z_1} \delta_{z=z_2} \frac{1}{(z-z_3)^2} \pm (2 \leftrightarrow 3)$
$= K \frac{1}{(z_2-z_1)} \frac{1}{(z_2-z_3)^2} - K \frac{1}{(z_3-z_1)} \frac{1}{(z_3-z_2)^2}$

$\mathbb{C}^\times/\mathbb{R}_{>0}$

$= K \overline{(z_2 - z_1)}$

$= f_{a_1 a_2 a_3} \dfrac{K}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}$

$\sum$  $\rightsquigarrow \mathrm{Tr}(\ ) \dfrac{K}{(z_1 - z_2) \cdots (z_n - z_1)}$