Title: Lecture - Quantum Matter, PHYS 777

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Collection/Series: Quantum Matter (Elective), PHYS 777, March 30 - May 1 2026

Subject: Condensed Matter

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Bardeen–Cooper–Schrieffer (BCS)

Cooper pair $\phi = c_\uparrow c_\downarrow$ $q=2$

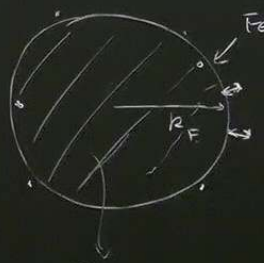
Start with Free Fermi gas.

$$H_0 = \sum_k \left(\frac{k^2}{2m} - \mu\right) c^\dagger_{k,\sigma} c_{k,\sigma}$$

Fermi energy $\mu = \frac{k_F^2}{2m}$

k_F: Fermi momentum.

G.S. $n_{k,\sigma} = \langle c^\dagger_{k,\sigma} c_{k,\sigma} \rangle = \theta(k_F - k)$.



$k = k_F + \delta k$

$\varepsilon(k) \simeq v_F \delta k$. $v_k = \frac{k_F}{m}$ Fermi velocity

Density of (single particle) states

$$\nu(\varepsilon) = \frac{N(\varepsilon+d\varepsilon) - N(\varepsilon)}{V\, d\varepsilon} = \frac{2m k_F}{\pi^2} = \nu_0 + O(\varepsilon)$$

(If CFT, e.g. Dirac fermion in $(d+1)$-d: $\nu(\varepsilon) \sim \varepsilon^{d-1}$)

X $\varepsilon \sim |k|$

$= \theta(k_F - k)$.

$k_F + \delta k$

$\simeq v_F \delta k$. $v_F = \frac{k_F}{m}$ Fermi velocity

f (single particle) states

$\nu(\varepsilon) = \frac{2m k_F}{\pi^2} + \mathcal{O}(\varepsilon) = \nu_0$

e.g. Dirac fermion in $(d+1)$, i)

$\varepsilon) \sim \varepsilon^{d-1}$

(k)

Landau Fermi liquid theory:

Stable against perturbations: local, U(1) & translation symmetric.

weak

Except for "BCS interaction" $\longrightarrow$ S.C.

(Modern RG treatment. Shankar. Rev. Mod. phys.)

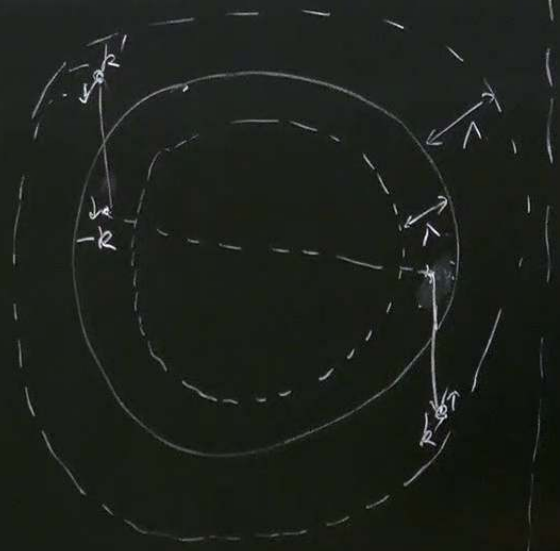
$$H_I = -\frac{g}{V} \sum_{\substack{k,k',q \\ q \in \Lambda}} c^\dagger_{k'+q,\uparrow} c^\dagger_{-k',\downarrow} c_{-k,\downarrow} c_{k+q,\uparrow}$$

$g > 0$: attractive interaction

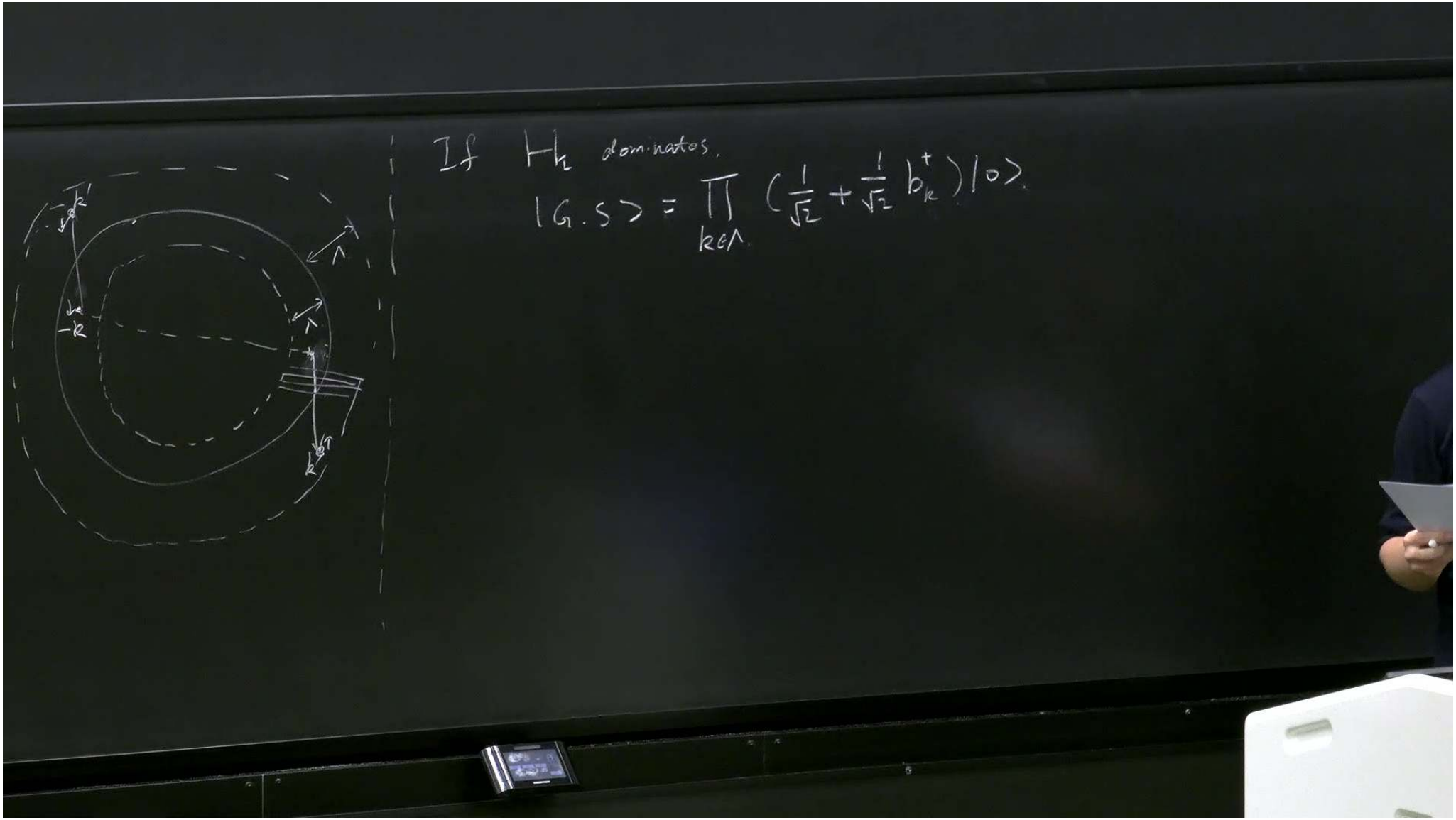
$\rightsquigarrow -g\, c^\dagger_{\uparrow} c_{\uparrow}\, c^\dagger_{\downarrow} c_{\downarrow}$

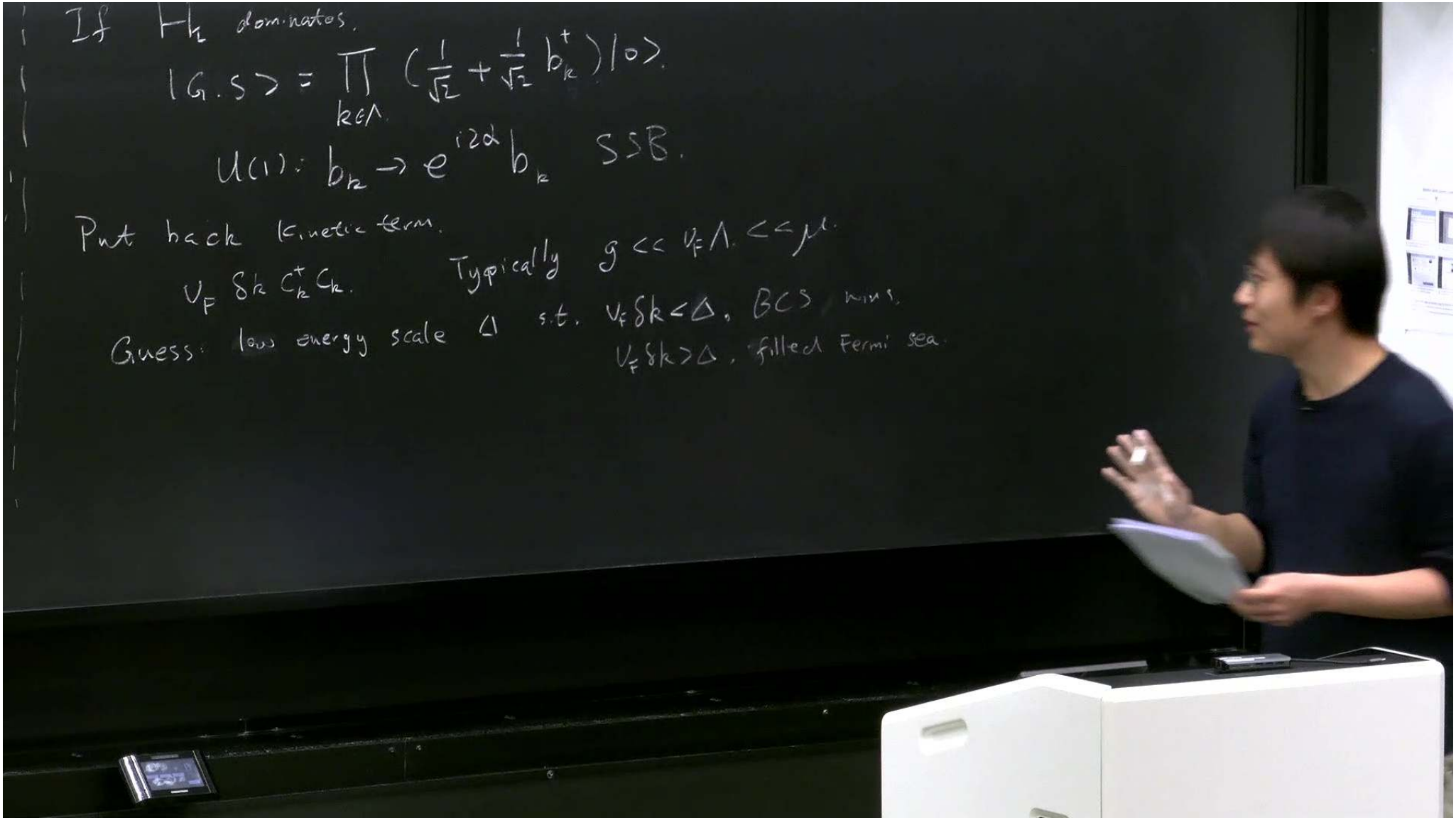
Focus on $q = 0$: $b_k := c_{-k,\downarrow} c_{k,\uparrow}$

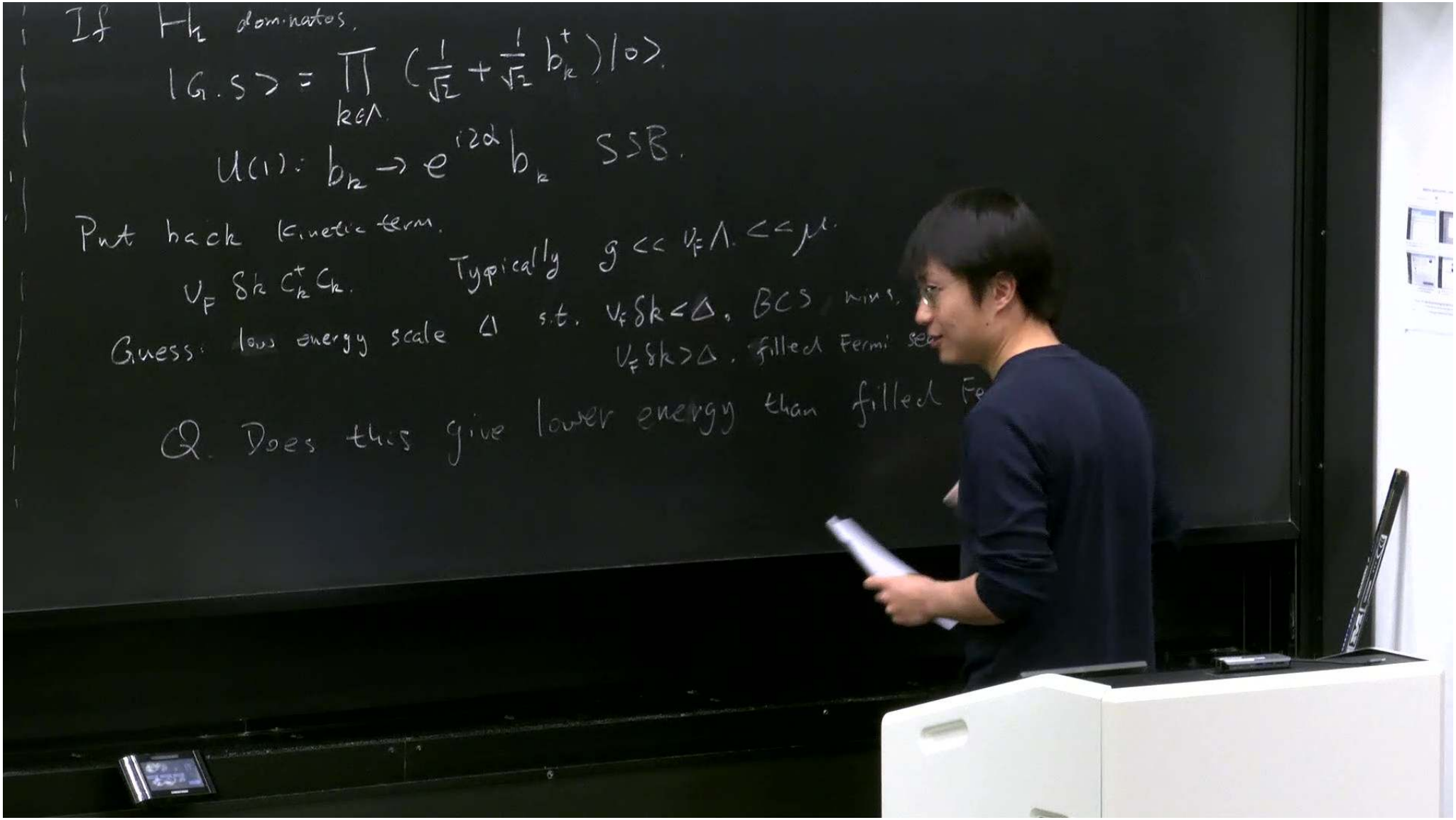
$$H_I \rightsquigarrow -\frac{g}{V} \sum_{k,k'} b^\dagger_{k'} b_k$$



If H_I


If H_2 dominates,
$|G.S\rangle = \prod_{k \in \Lambda} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} b_k^\dagger\right)|0\rangle$.
k'
$-k$
Λ
Λ
k
Λ


If H_k dominates.
$|G.S\rangle = \prod_{k\in\Lambda} (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} b_k^\dagger)|0\rangle$.
$U(1): b_k \to e^{i2\alpha} b_k$ SSB.
Put back kinetic term.
$v_F \delta k\, c_k^\dagger c_k$. Typically $g \ll v_F\Lambda \ll \mu$.
Guess: low energy scale Δ s.t. $v_F\delta k < \Delta$, BCS wins.
$v_F\delta k > \Delta$, filled Fermi sea.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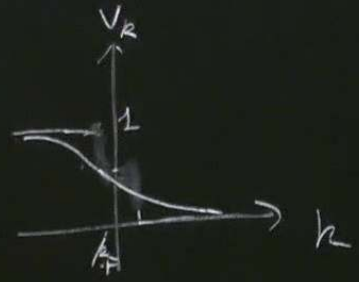
Q. Does this give lower energy than filled Fe

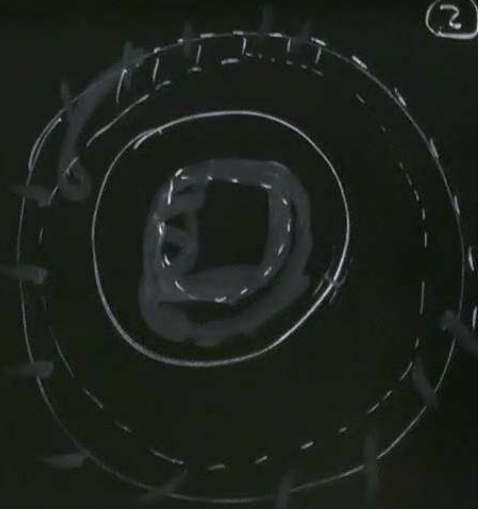
$$\delta E = -\frac{g}{V}\frac{1}{2}\left(\frac{\nu(0)\,\Delta\, V}{2}\right)^2 + \Delta\cdot(\delta\,\nu(0)\,V) = \Delta^2\,\nu(0)\,V\cdot\left(-\frac{g\,\nu(0)}{8}+1\right)$$

BCS wavefunction

$$|\Psi_{BCS}\rangle = \prod_k (u_k + v_k b_k^\dagger)|0\rangle$$

$u^2+v^2=1$



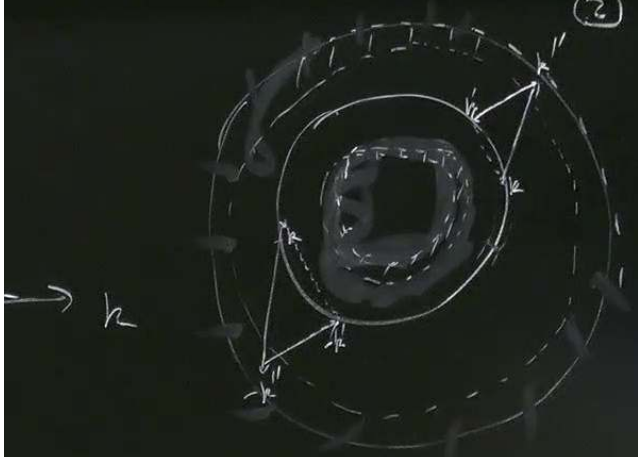


Q. Does this give lower energy than filled Fermi sea.

$$\Delta^2 \nu(0) V \cdot \left(-\frac{g\,\nu(0)}{8} + 1 \right)$$

① sgn doesn't depend on Δ. ⇐ BCS interaction is marginal.

② $g\nu(0)$ has to $> O(1)$ critical value ⇐ WRONG!



$\underbrace{\Lambda,\ \Lambda - d\Lambda}$ integrate out:

$$\delta g = (-\underline{V}) \cdot \left(\frac{g}{\underline{V}}\right)^2 \overbrace{\sum_{\Lambda - d\Lambda < k'' < \Lambda} \underbrace{\frac{1}{2\varepsilon_k - 2\varepsilon_{k''}}}_{= -\frac{1}{\Lambda \cdot V_F}}}^{\nu(0)\, d\Lambda \cdot \underline{V}} \simeq \frac{\nu(0) g^2}{4} \frac{d\Lambda}{\Lambda}$$

$$g = \frac{g_0}{1 + \frac{\nu_0 g_0}{4} \ln \frac{\Lambda}{\Lambda_0}}$$

Eventually wins at scale.

$$\Delta = v_F \Lambda = v_F \Lambda_0 e^{-\frac{\alpha}{\nu_0 g_0}}, \quad \alpha = O(1)$$

g: marginally relevant

Once Cooper pairs condense,

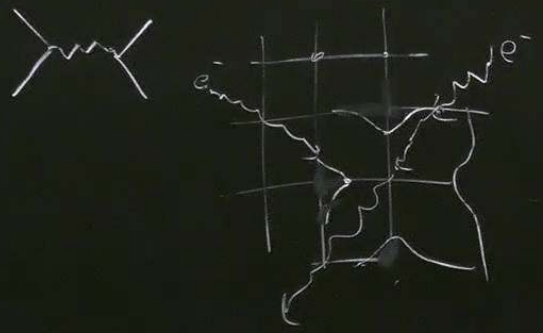
$$H_1^{MF} \simeq -\frac{g}{V} \left(\sum_{k'} \langle c^\dagger_{k',\uparrow} c^\dagger_{k',\downarrow} \rangle \right) \sum_k c_{-k\downarrow} c_{k\uparrow} + h.c.$$

$$:= \Delta \sum_k c_{-k\downarrow} c_{k\uparrow} + h.c.$$

$\Rightarrow$ Gapped

$\varepsilon(k)$

Coulomb $g_{Coul} < 0$, $\longrightarrow \dfrac{J}{1 + \# \cdot \ln \frac{\mu}{\omega_D}}$ at ω_D attractive $g_{ph} > 0$.

ω_D → Debye scale

For many metal $g_{ph} + g_{Coul}^{renormalized} < 0$.