Title: Lecture - Quantum Gravity, PHYS 644

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Collection/Series: Quantum Gravity (Elective), PHYS 644, March 30 - May 1 2026

Subject: Quantum Gravity

Date: April 14, 2026 - 1:00 PM

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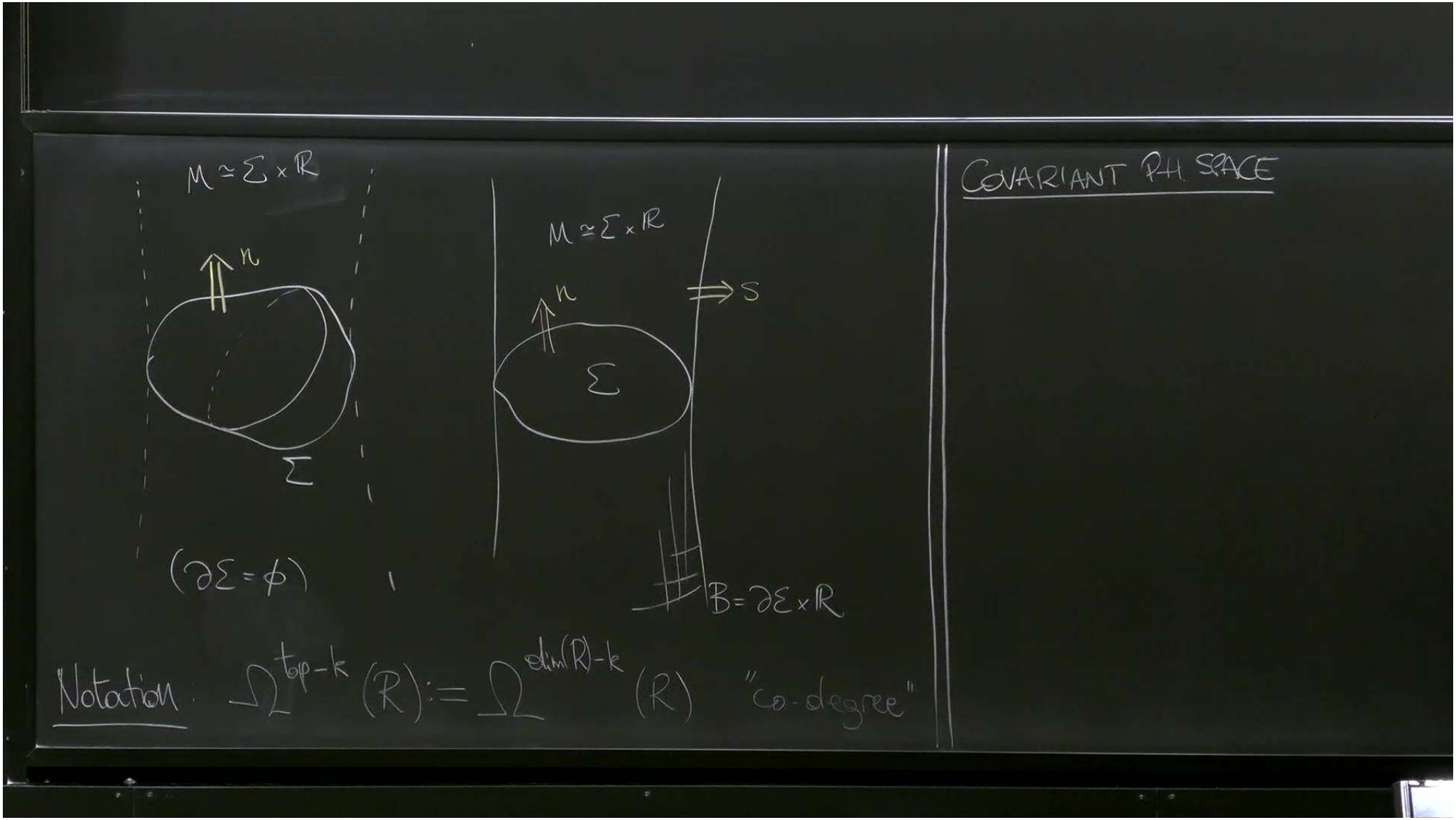

$M \simeq \Sigma \times \mathbb{R}$
n
Σ
$(\partial\Sigma = \phi)$
$M \simeq \Sigma \times \mathbb{R}$
n
$\Rightarrow s$
Σ
$B = \partial\Sigma \times \mathbb{R}$
COVARIANT PH. SPACE
Notation: $\Omega^{top-k}(R) := \Omega^{dim(R)-k}(R)$ "co-degree"

$\Rightarrow S$

$B = \partial\mathcal{E} \times \mathbb{R}$

R) "co-degree"

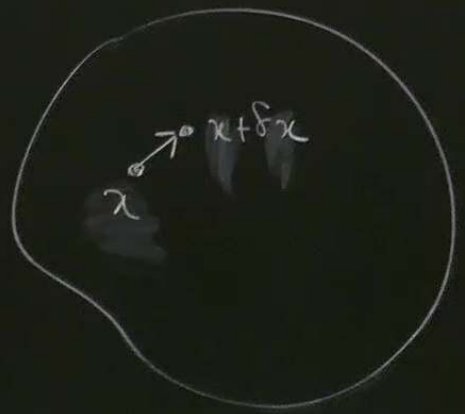
COVARIANT PH SPACE

$M \simeq \Sigma \times \mathbb{R}$ sp. time

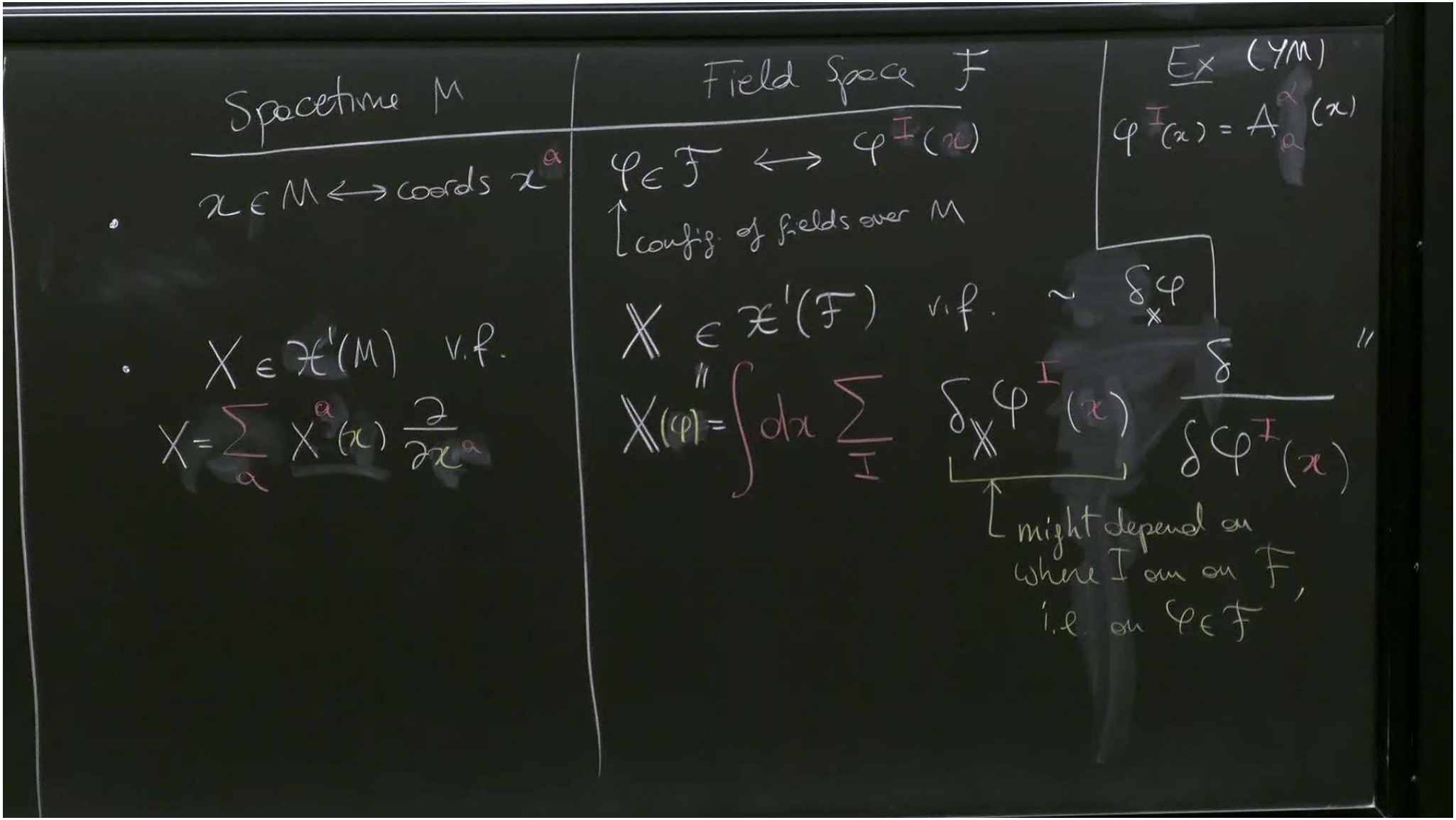
$\mathcal{F} = \Gamma(E \to M) \sim$ | tensors on M
connections on $M \sim \Omega^1(M, g)$

$\mathcal{F}$: space of sections of some (vector) bundle.

∞-dim



$X \sim \delta x$


Spacetime M
$x \in M \leftrightarrow$ coords x^a
$X \in \mathfrak{X}^1(M)$ v.f.
$X = \sum_a X^a(x) \frac{\partial}{\partial x^a}$
Field Space $\mathcal{F}$
$\varphi \in \mathcal{F} \leftrightarrow \varphi^I(x)$
[config. of fields over M
$X \in \mathfrak{X}^1(\mathcal{F})$ v.f. $\sim \delta_X \varphi$
$X(\varphi) = \int dx \sum_I \delta_X \varphi^I(x) \frac{\delta}{\delta \varphi^I(x)}$
[might depend on where I am on $\mathcal{F}$, i.e. on $\varphi \in \mathcal{F}$
Ex (YM)
$\varphi^I(x) = A_a^\alpha(x)$

Spacetime M

- $x \in M \leftrightarrow$ coords x^a
- $X \in \mathfrak{X}^1(M)$ v.f.

$$X = \sum_a X^a(x) \frac{\partial}{\partial x^a}$$

$f : M \to \mathbb{R}$

$$X(f) = \lim_{\epsilon \to 0} \frac{f(x^a + \epsilon X^a) - f(x)}{\epsilon}$$

Field Space $\mathcal{F}$

$\varphi \in \mathcal{F} \leftrightarrow \varphi^I(x)$

↑ config. of fields over M

$X \in \mathfrak{X}^1(\mathcal{F})$ v.f. $\sim \delta_X \varphi$

$$X(\varphi) = \int dx \sum_I \underbrace{\delta_X \varphi^I(x)}_{\text{might depend on where I am on } \mathcal{F}, \text{ i.e. on } \varphi \in \mathcal{F}} \frac{\delta}{\delta \varphi^I(x)}$$

$f : \mathcal{F} \to \mathbb{R}$

$$X(f) = \lim_{\epsilon \to 0} \frac{f(\varphi + \epsilon \delta_X \varphi) - f(\varphi)}{\epsilon}$$

Cartan's calculus on differential forms

$\alpha \in \Omega^k(M)$

d = exterior diff.

$L_X \alpha = i_X d\alpha + d i_X \alpha$

↑ Lie derivative.

Cartan's calc over $\mathcal{F}$

$\alpha \in \Omega^k(\mathcal{F})$

d = exterior diff. $(d^2 = 0)$

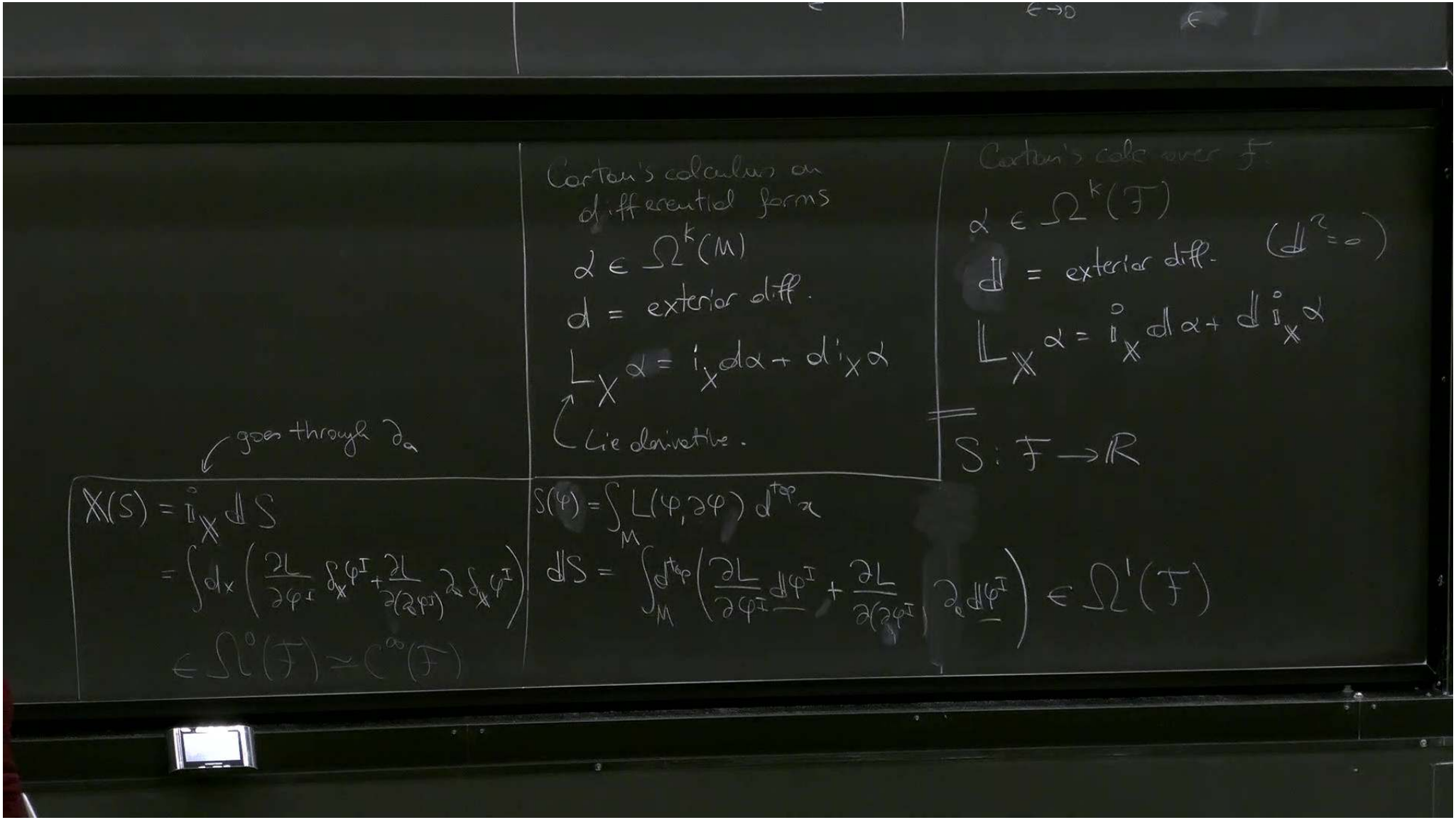
$L_X \alpha = i_X d\alpha + d i_X \alpha$

$=$

$S : \mathcal{F} \to \mathbb{R}$

$$S = \int_M L(\varphi, \partial\varphi)\, d^{top}x$$

$$dS = \int_M d^{top}\left(\frac{\partial L}{\partial \varphi^I} d\varphi^I + \frac{\partial L}{\partial(\partial_\mu \varphi^I)} \partial_\mu d\varphi^I \right) \in \Omega^1(\mathcal{F})$$


Cartan's calculus on differential forms
$\alpha \in \Omega^k(M)$
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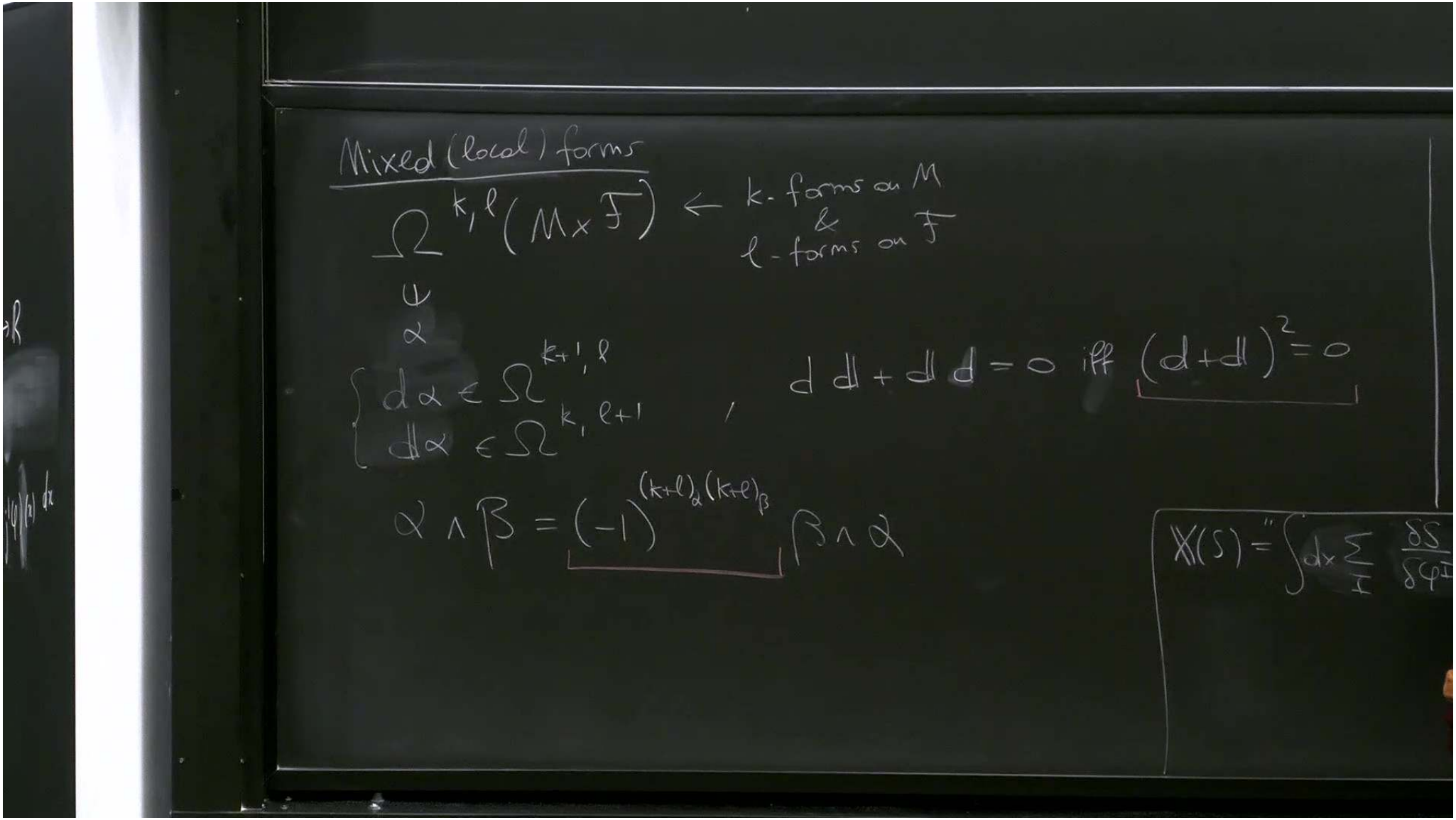
$L_X \alpha = i_X d\alpha + d i_X \alpha$
Lie derivative.
Cartan's calc over $\mathcal{F}$
$\alpha \in \Omega^k(\mathcal{F})$
d = exterior diff. $(d^2 = 0)$
$L_X \alpha = i_X d\alpha + d i_X \alpha$
$S : \mathcal{F} \to \mathbb{R}$
goes through ∂_σ
$X(S) = i_X dS$
$= \int d_\Sigma \left(\frac{\partial L}{\partial \varphi^I} \delta_X \varphi^I + \frac{\partial L}{\partial(\partial \varphi^I)} \partial \delta_X \varphi^I \right)$
$\in \Omega^0(\mathcal{F}) \simeq C^\infty(\mathcal{F})$
$S(\varphi) = \int_M L(\varphi, \partial\varphi)\, d^{\dim \varphi} x$
$dS = \int_M d^{\dim \varphi} x \left(\frac{\partial L}{\partial \varphi^I} d\varphi^I + \frac{\partial L}{\partial(\partial \varphi^I)} \partial d\varphi^I \right) \in \Omega^1(\mathcal{F})$


Mixed (local) forms
$\Omega^{k,\ell}(M \times \mathcal{F})$ ← k-forms on M & ℓ-forms on $\mathcal{F}$
α
$d\alpha \in \Omega^{k+1,\ell}$, $d\alpha \in \Omega^{k,\ell+1}$
$d\,d_l + d_l\,d = 0$ iff $(d + d_l)^2 = 0$
$\alpha \wedge \beta = (-1)^{(k+\ell)_\alpha (k+\ell)_\beta} \beta \wedge \alpha$
$\mathcal{X}(S) = \int dx \sum_i \frac{\delta S}{\delta \varphi^i}$

iff $(d+\delta)^2 = 0$

$\alpha_1(\varphi, x) = \varphi(x)\, d\delta\varphi(x)$

$= -\varphi(x)\, \partial_a\, \delta\varphi(x) \wedge dx^a$ $\quad \in \Omega^{1,1}$

$\alpha_2(\varphi, x) = d\varphi(x) \wedge \delta\varphi(x)$

$\alpha = \alpha_1 + \alpha_2 = d\big(\varphi(x)\, \delta\varphi(x)\big)$

goes through ∂_a

$\mathbb{X}(S) = \int dx \sum_I \frac{\delta S}{\delta \varphi^I} (\delta_{\mathbb{X}} \varphi^I)$

$\mathbb{X}(S) = i_{\mathbb{X}}\, \delta S$

$= \int dx \left(\frac{\partial L}{\partial \varphi^I} \delta_{\mathbb{X}} \varphi^I + \frac{\partial L}{\partial(\partial \varphi^I)} \partial \delta_{\mathbb{X}} \varphi^I \right)$

$\in \Omega^0(\mathcal{F}) \simeq C^\infty(\mathcal{F})$

Cartan's diff

$\alpha \in$

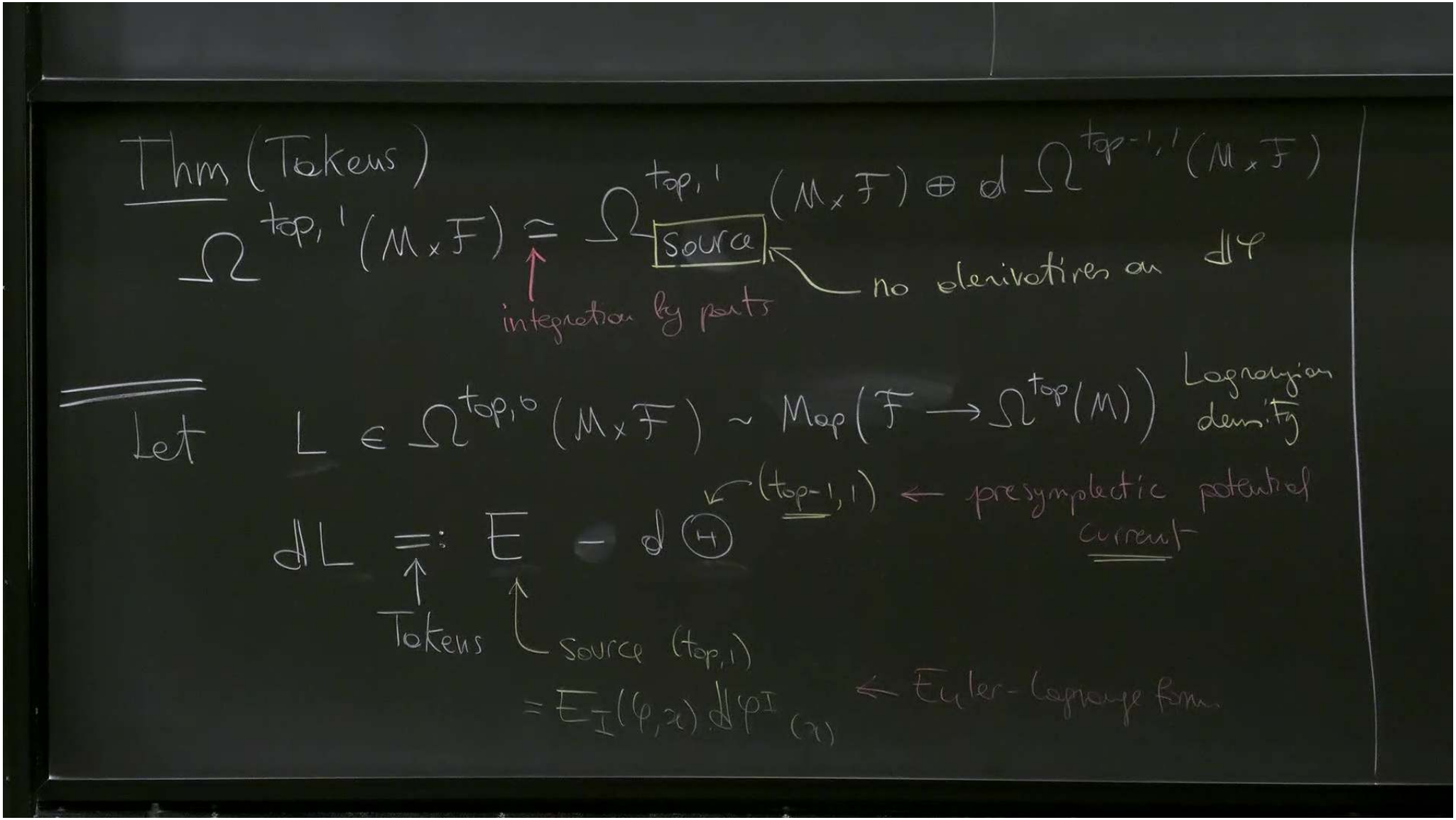
$d =$

$L_X \alpha$

Lie d

$S(\varphi) = \int_M L$

$dS =$


Thm (Tokens)
$\Omega^{top,1}(M\times\mathcal{F}) \cong \Omega^{top,1}_{Source}(M\times\mathcal{F}) \oplus d\,\Omega^{top-1,1}(M\times\mathcal{F})$
integration by parts
no derivatives on $d\varphi$
Let $L \in \Omega^{top,0}(M\times\mathcal{F}) \sim Map(\mathcal{F}\to\Omega^{top}(M))$ Lagrangian dens. Fj
$dL =: E - d\Theta$ $(top-1,1)$ ← presymplectic potential current
Tokens
Source $(top,1)$
$= E_I(\varphi,x)\,d\varphi^I(x)$ ← Euler-Lagrange form

Thm (Takens)

$$\Omega^{top,1}(M\times\mathcal{F}) \cong \Omega^{top,1}_{\boxed{\text{source}}}(M\times\mathcal{F}) \oplus d\,\Omega^{top-1,1}(M\times\mathcal{F})$$

integration by parts

no derivatives on $d\varphi$

Let $L \in \Omega^{top,0}(M\times\mathcal{F}) \sim \mathrm{Map}(\mathcal{F} \to \Omega^{top}(M))$ Lagrangian density

$$\boxed{dL =: E - d\Theta}$$

$(top-1,1)$ ← presymplectic potential current

Takens

source $(top,1)$

$= E_I(\varphi,x)\, d\varphi^I(x)$ ← Euler-Lagrange form

$\Omega^{\text{top-1},1}(M \times F)$

otives an $d\varphi$

$^{\text{top}}(M))$ Lagrangian density

symplectic potential current

Lagrange form

$$L = \mathcal{L}\, d^{\text{top}}x$$

$$E = \left[\frac{\partial \mathcal{L}}{\partial \varphi^I} - \frac{\partial}{\partial x^a}\left(\frac{\partial \mathcal{L}}{\partial(\partial_a \varphi^I)}\right)\right] d\varphi^I \wedge d^{\text{top}}x$$

$$\Theta = \left[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_a \varphi^I)}}_{\sim \pi^a}\, d\varphi^I\right] d^{\text{top-1}}x_a$$

DEF $\Omega := d\Theta \in \Omega^{\text{top-1},2}(M \times F)$

"(pre)symplectic current"

$\epsilon^{top} X$

$$d^{top} x = \frac{1}{n!} \epsilon_{a_1 \cdots a_n} dx^{a_1} \wedge \cdots \wedge dx^{a_n}$$

$$d^{top-1} x_a = \frac{1}{(n-1)!} \epsilon_{a a_1 \cdots a_{n-1}} dx^{a_1} \wedge \cdots dx^{a_{n-1}}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \varphi^I} - \frac{\partial}{\partial x^a}\left(\frac{\partial \mathcal{L}}{\partial(\partial_a \varphi^I)}\right)\right] d\varphi^I \wedge d^{top}x$$

$$d^{top}x = \frac{1}{n!}\epsilon_{a_1 \cdots a_n} dx^{a_1} \wedge \cdots \wedge dx^{a_n}$$

$$d\varphi^I \Big] \; d^{top-1}x_a$$

$$d^{top-1}x_a = \frac{1}{(n-1)!}\epsilon_{a\, a_1 \cdots a_{n-1}} dx^{a_1} \wedge \cdots dx^{a_{n-1}}$$

(17) $\in \Omega^{top-1,2}(M \times \mathcal{F})$

symplectic current"

DEF "the shell" $\overline{\mathcal{F}} = \{\varphi \in \mathcal{F} \mid E(\varphi) = 0\}$

= Space of phys config-

Notation: "Going on shell" = pulling back from $\mathcal{F}$ to $\overline{\mathcal{F}}$

THEOREM

The presympl. current is conserved on-shell –

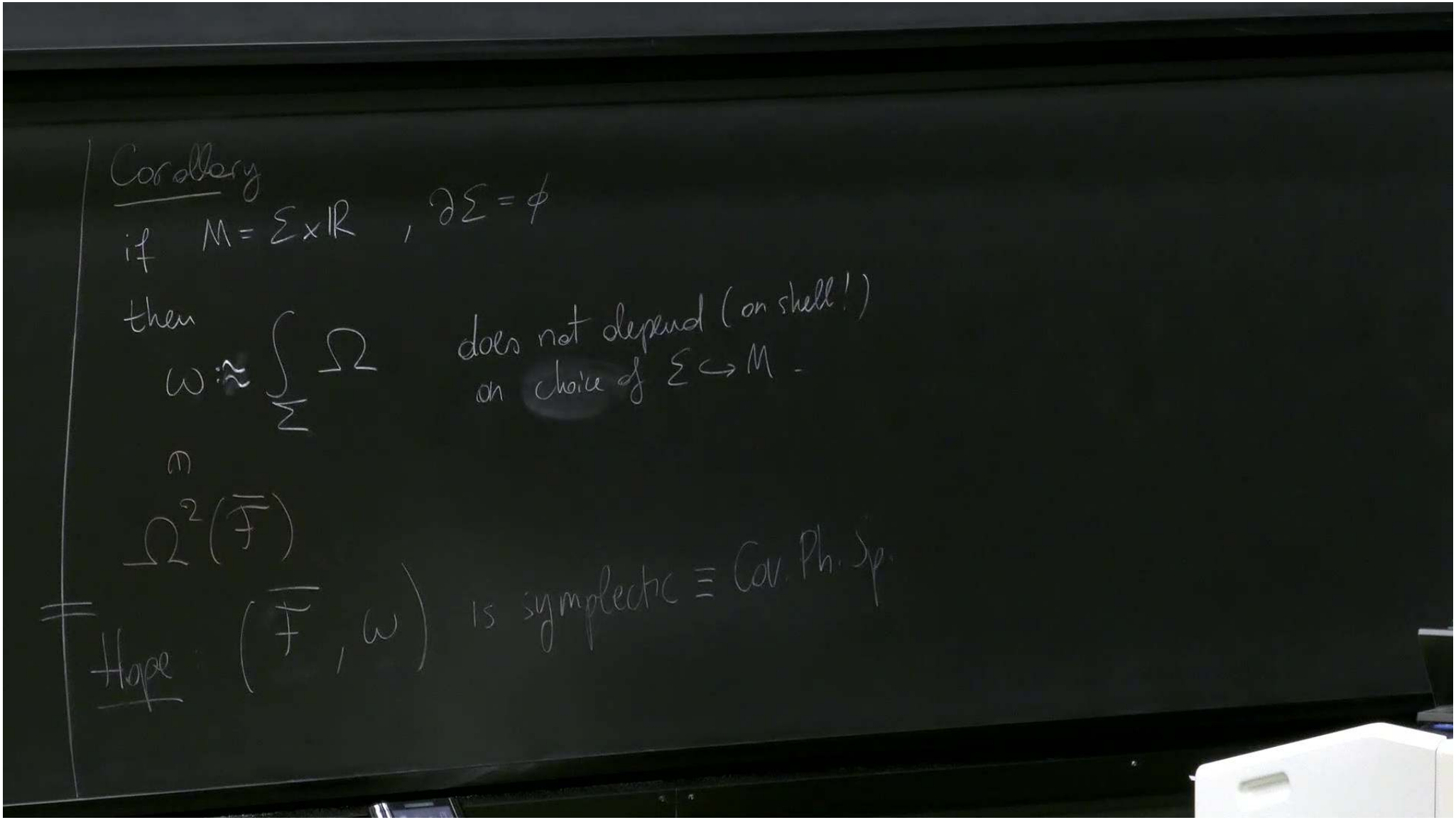
$$d\Omega \approx 0$$

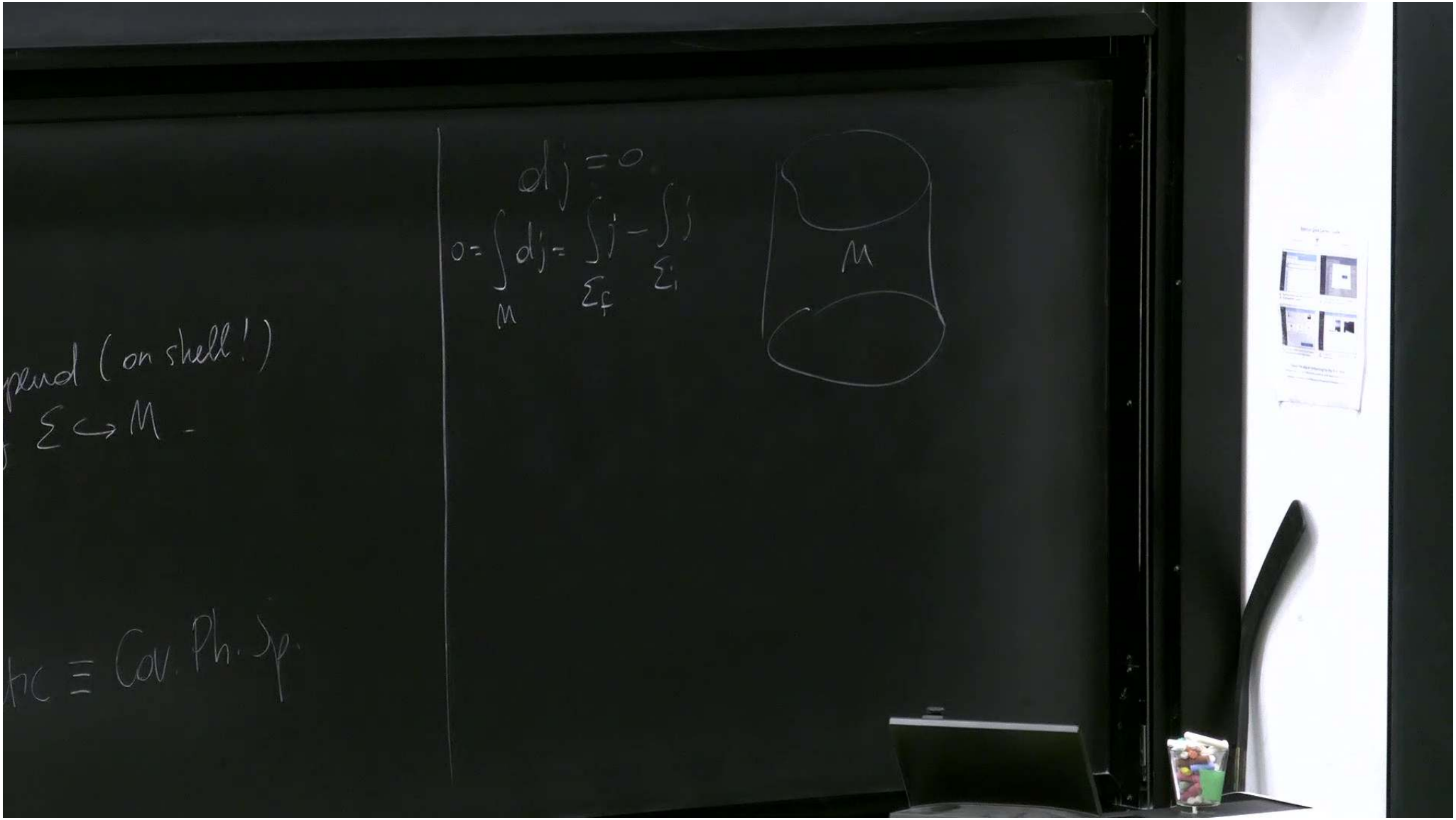
Pf: $\mathbb{d}L = E - d\Theta$

$$0 \equiv \mathbb{d}^2 L = \mathbb{d}E + d\mathbb{d}\Theta = \mathbb{d}E + d\Omega$$

Since $\overline{\mathcal{F}} = \{E(\varphi) = 0\}$, $\overset{*}{\underset{\mathcal{F}}{=}} d\Omega = 0$ ∎

Rmk on shell Ω "eats" vectors tangent to $\overline{\mathcal{F}}$ = sols to lin. e.o.m


Corollary
if $M = \Sigma \times \mathbb{R}$, $\partial\Sigma = \phi$
then
$\omega := \int_\Sigma \Omega$ does not depend (on shell!) on choice of $\Sigma \hookrightarrow M$.
$\Omega^2(\bar{\mathcal{F}})$
Hope $(\bar{\mathcal{F}}, \omega)$ is symplectic ≡ Cov. Ph. Sp.


dj = 0
0 = ∫_M dj = ∫_{Σ_f} j − ∫_{Σ_i} j
M
pund (on shell!)
Σ ↪ M
tic ≡ Cov. Ph. Sp.