Title: Lecture - Quantum Gravity, PHYS 644

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Collection/Series: Quantum Gravity (Elective), PHYS 644, March 30 - May 1 2026

Subject: Quantum Gravity

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Recap

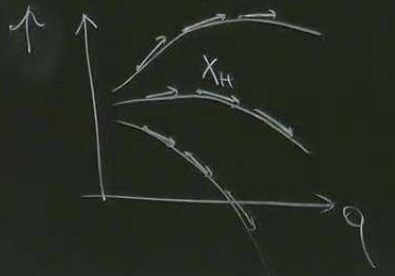
- Action principle $\delta S = \int_{t_0}^{t_1} (E.o.M)_i \, \delta q^i + \left[\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right]_{t_0}^{t_1}$, $q : \mathbb{R} \to Q$

  $p \, \delta q$

- From Euler Lagrange to Hamilton's EoM

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} =: X_H \quad \text{where } H(q,p) = p\dot{q} - L$$

- Geometric interpretation over phase space $P = T^*Q$



physical histories = integral curves of Hamiltonian vector field X_H

- (Canonical) Poisson bracket

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}$$

1. skew
2. bi-derivation (Leibniz in each entry)
3. Jacobi

$\Rightarrow X_H = \{H, \cdot\}$ Hamilt v.f. o

- Symplectic mfd (P, ω)
  - $\dim P = 2n$
  - $\omega \in \Omega^2(P)$: $\omega = \frac{1}{2} \omega_{IJ}$
  - closed, $d\omega = 0$
  - non-degenerate, $\omega^\flat : TP$

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$\rightarrow X_H = \{H, \cdot\}$ Hamilt v.f. of $H \in C^\infty(P)$

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  - $\dim P = 2n$
  - $\omega \in \Omega^2(P)$: $\omega = \frac{1}{2}\omega_{IJ}\, dz^I \wedge dz^J$
  - closed, $d\omega = 0$
  - non-degenerate, $\omega^\flat : TP \to T^*P$ injective

Rmk $n < \infty \Rightarrow \omega^\flat$ invertible

$\{\cdot,\cdot\} = \frac{1}{2}\pi^{IJ}\partial_I \wedge \partial_J$

with $\pi^{IJ} = -(\omega_{IJ})^{-1}$

is Poisson (1-3)

Thm (Darboux) locally $\exists$ coords $z = (q,p)$ s.t. $\omega = \sum dp_i \wedge dq^i$

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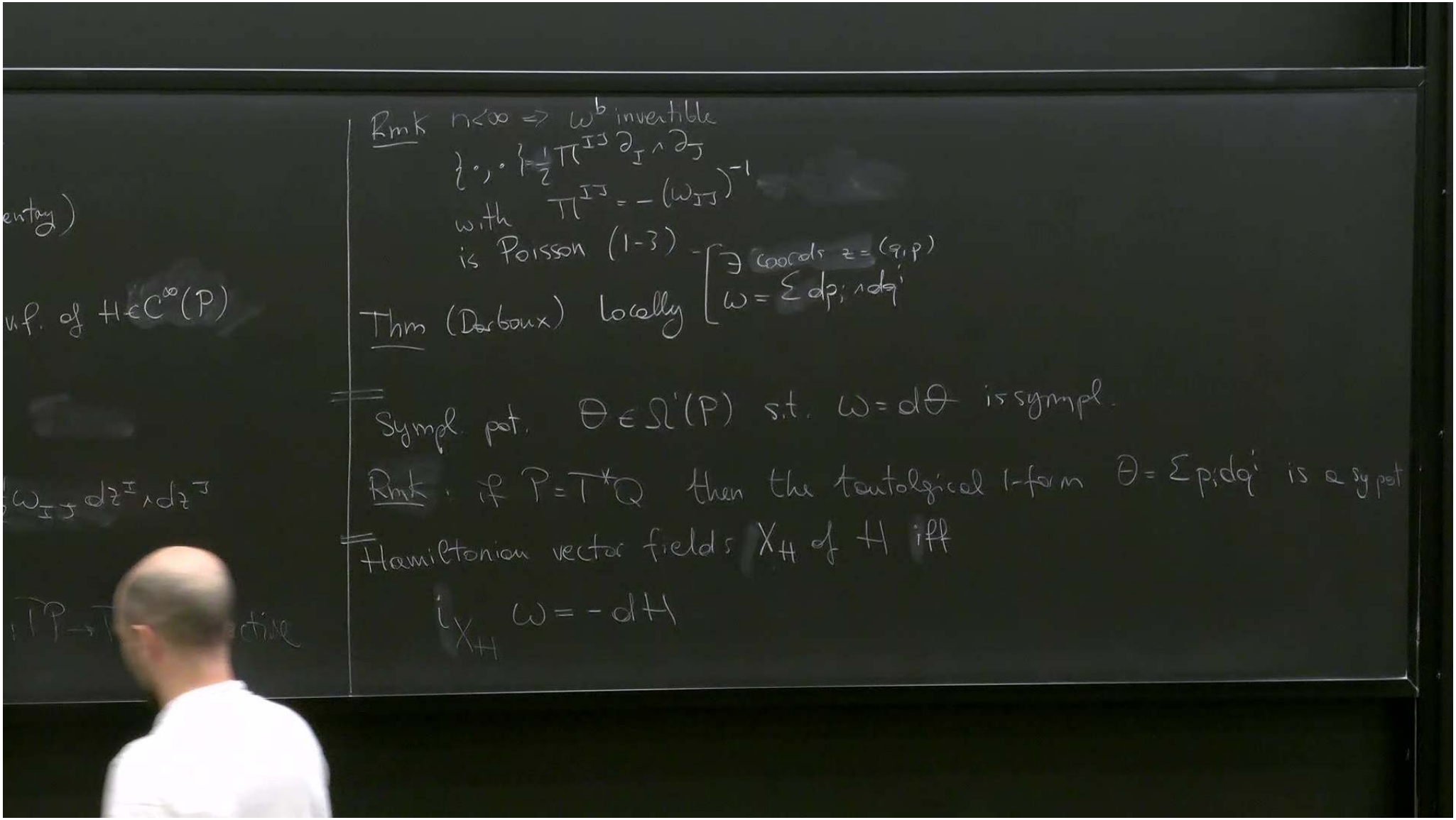
Thm (Darboux) locally $\begin{cases} \exists \text{ coords } z = (q,p) \\ \omega = \sum dp_i \wedge dq^i \end{cases}$

$\Leftrightarrow$ Sympl pot. $\Theta \in \Omega^1(P)$ s.t. $\omega = d\Theta$ is sympl.

Rmk: if $P = T^*Q$ then the tautological 1-form $\Theta = \sum p_i dq^i$ is a sy pot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Hamiltonian vector fields X_H of H iff
$i_{X_H}\omega = -dH$

$\omega_{IJ} dz^I \wedge dz^J$

$\Pi: P \to T^*P$ injective

Rmk. if $P = T^*Q$ then the tautological 1-form $\Theta = \sum p_i dq^i$ is a symplectic potential

Hamiltonian vector fields: X_H of H iff

$$i_{X_H} \omega = -dH \quad (!)$$

Pf

$$X_H^I \omega_{IJ} = -\partial_J H$$

recall $\Pi = -\omega^{-1}$, and multiply from the right.

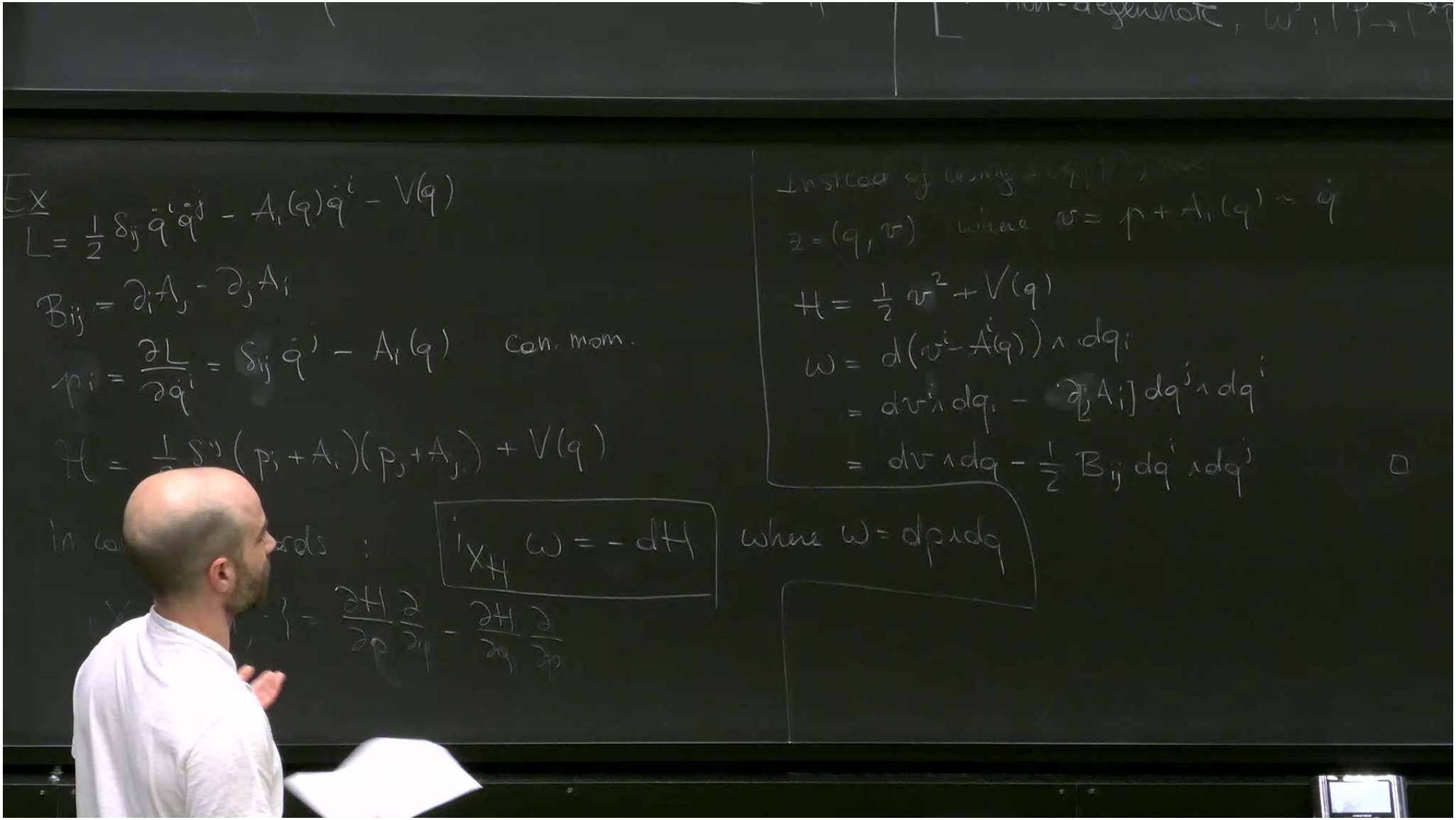
$$(X_H)^I \partial_I = \Pi^{IJ} \partial_J H \, \partial_I$$

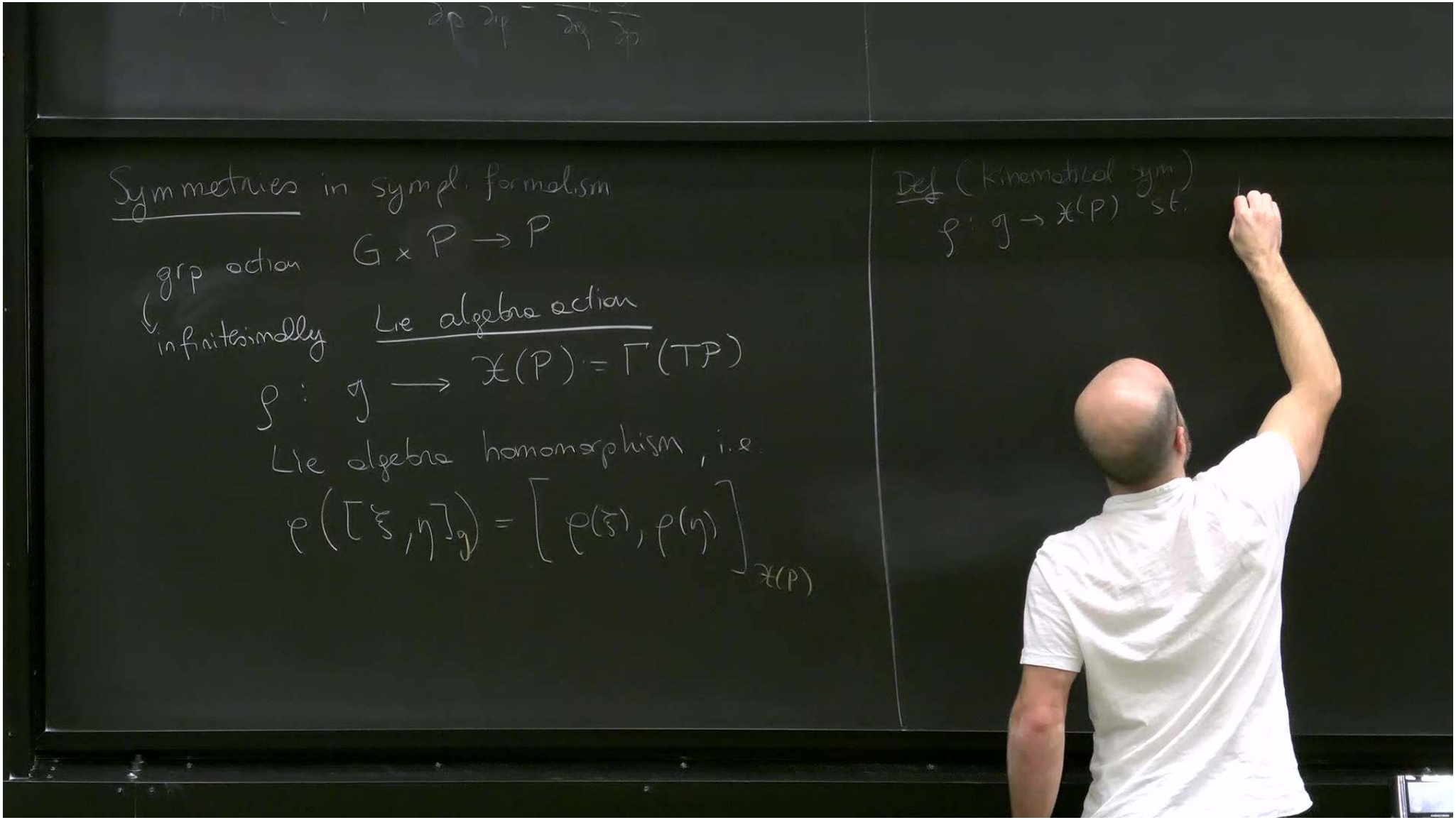
$$X_H = \{H, \cdot\} \quad \square$$

$\omega(X_H) = -dH$ $\qquad$ $i_Y i_X \omega = (\omega(X))(Y) \equiv \omega(X, Y)$

$X_H \lrcorner \omega = -dH$

$i_{X_H} \omega$


Ex
$L = \frac{1}{2}\delta_{ij}\dot{q}^i\dot{q}^j - A_i(q)\dot{q}^i - V(q)$
$B_{ij} = \partial_i A_j - \partial_j A_i$
$p_i = \frac{\partial L}{\partial \dot{q}^i} = \delta_{ij}\dot{q}^j - A_i(q)$ Can. mom.
$H = \frac{1}{2}\delta^{ij}(p_i + A_i)(p_j + A_j) + V(q)$
$i_{X_H}\omega = -dH$
$= \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$
$z = (q, v)$ where $v = p + A_i(q)\dot{q}$
$H = \frac{1}{2}v^2 + V(q)$
$\omega = d(v^i - A^i(q)) \wedge dq_i$
$= dv^i \wedge dq_i - [\partial_j A_i]\, dq^j \wedge dq^i$
$= dv \wedge dq - \frac{1}{2} B_{ij}\, dq^i \wedge dq^j$
where $\omega = dp \wedge dq$


Symmetries in sympl. formalism
grp action $G \times P \to P$
infinitesimally Lie algebra action
$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(P) := \Gamma(TP)$
Lie algebra homomorphism, i.e.
$\rho([\xi,\eta]_{\mathfrak{g}}) = [\rho(\xi), \rho(\eta)]_{\mathfrak{X}(P)}$
Def (kinematical sym)
$\rho : \mathfrak{g} \to \mathfrak{X}(P)$ s.t.

$i_{X_H}\omega$

Def (Kinematical sym)
$\rho : \mathfrak{g} \to \mathfrak{X}(P)$ s.t. $L_{\rho(\xi)}\omega = 0 \quad \forall \xi \in \mathfrak{g}$

$i_{\rho(\xi)}\omega = -df$

Rmk since $d\omega = 0$ & $L_X\omega = i_X d\omega + d i_X \omega$
(Cartan's magic formula)

$L_{\rho(\xi)}\omega = 0$ iff $d\, i_{\rho(\xi)}\omega = 0$

closed, when exact? (see notes)

If exact, then $\rho(\xi)$ is a Hamiltonian v.f.

and this is the case we are interested in.

$i_{X_4}\omega$

$= 0 \quad \forall \xi \in \mathfrak{g}$

$i_X d\omega + d i_X \omega$
magic formula)

0

, when exact ? (see notes)

Hamiltonian v.f.

interested in

$i_{\rho(\xi)}\omega = -df$

$\uparrow$
depends on ξ linearly $\Rightarrow \exists\ J : P \to \mathfrak{g}^* = \mathrm{Lin}(\mathfrak{g})$ s.t.

$$i_{\rho(\xi)}\omega(z) = -d\langle J(z), \xi\rangle = -\langle dJ, \xi\rangle$$

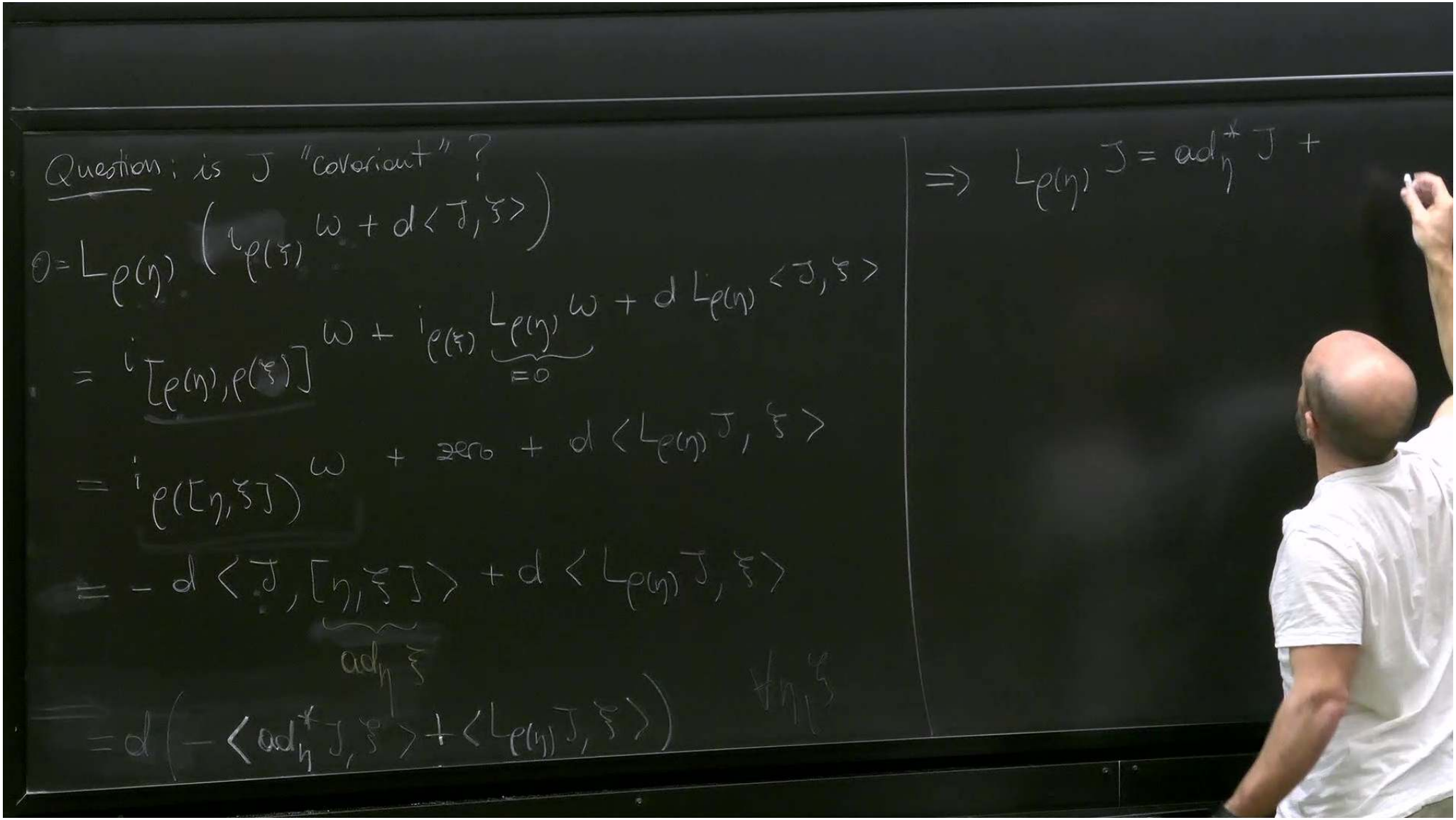
momentum map

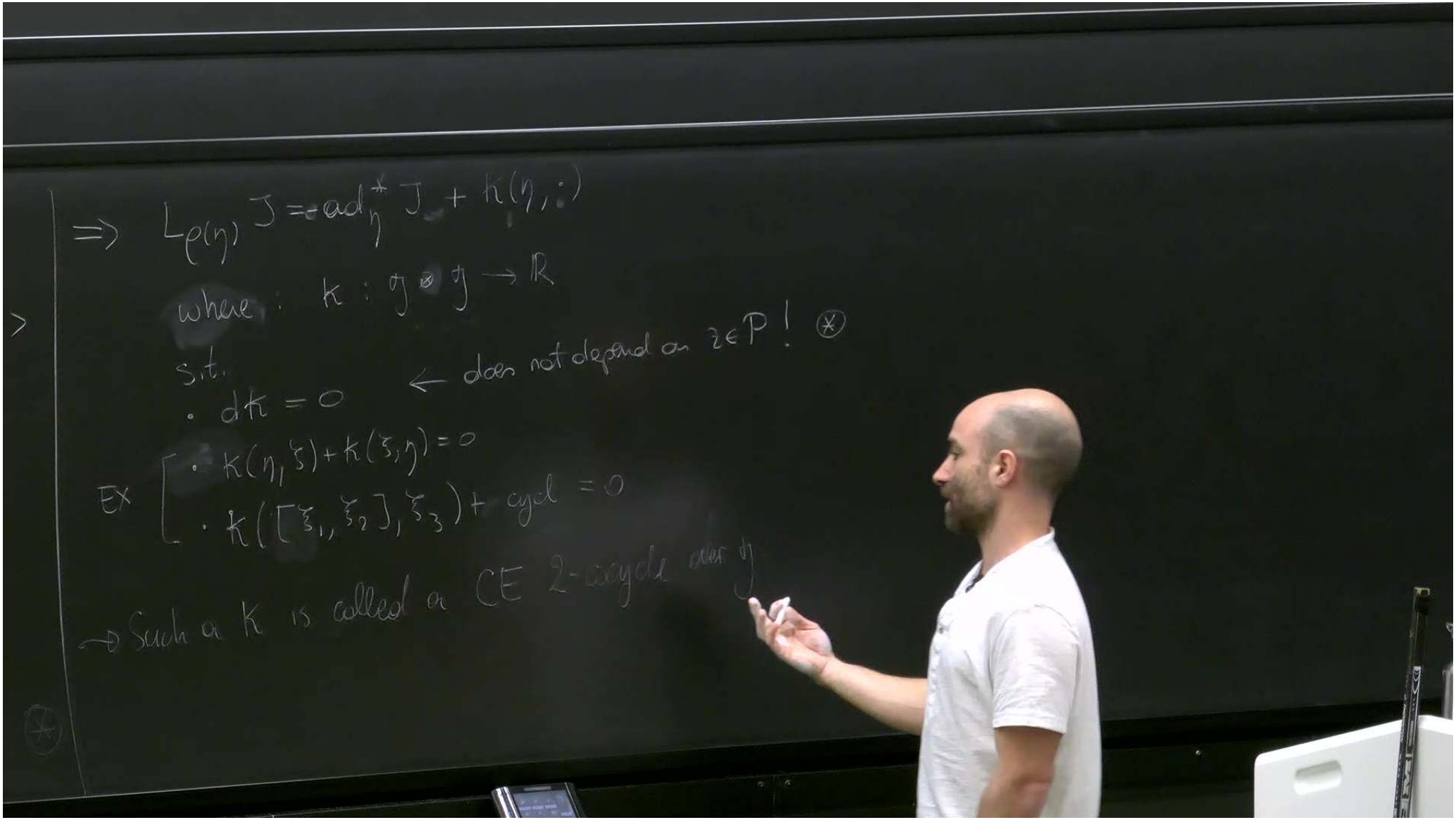
Ex: rotations $\mathfrak{g} = \mathfrak{so}(3)$, $\omega = d\vec{p} \wedge d\vec{x}$, $P = T^*\mathbb{R}^3$

$$\rho(\vec{a})\begin{vmatrix}\vec{x}\\ \vec{p}\end{vmatrix} = \begin{vmatrix}\vec{a}\times\vec{x}\\ \vec{a}\times\vec{p}\end{vmatrix}$$

$\mathfrak{g}^* \cong \mathbb{R}^3$

$$i_{\rho(\vec{a})}\omega = (\vec{a}\times\vec{p})\cdot d\vec{x} - (\vec{a}\times\vec{x})\cdot d\vec{p}$$
$$= d\left(\vec{a}\cdot(\vec{p}\times\vec{x})\right)$$
$$= d\langle \vec{a}, \vec{L}\rangle$$


Question: is J "covariant"?
$0 = L_{\rho(\eta)}\left(i_{\rho(\xi)}\omega + d\langle J, \xi\rangle\right)$
$= i_{[\rho(\eta),\rho(\xi)]}\omega + i_{\rho(\xi)}\underbrace{L_{\rho(\eta)}\omega}_{=0} + d L_{\rho(\eta)}\langle J, \xi\rangle$
$= i_{\rho([\eta,\xi])}\omega + \text{zero} + d\langle L_{\rho(\eta)}J, \xi\rangle$
$= -d\langle J, \underbrace{[\eta,\xi]}_{ad_\eta \xi}\rangle + d\langle L_{\rho(\eta)}J, \xi\rangle$
$= d\left(-\langle ad^*_\eta J, \xi\rangle + \langle L_{\rho(\eta)}J, \xi\rangle\right)$
$\forall \eta, \xi$
$\Rightarrow L_{\rho(\eta)}J = ad^*_\eta J +$


$\Rightarrow L_{\rho(\eta)} J = -\mathrm{ad}^*_\eta J + k(\eta, \cdot)$
where: $k : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$
s.t.
• $dk = 0$ ← does not depend on $z \in P$! ⊛
• $k(\eta, \xi) + k(\xi, \eta) = 0$
• $k([\xi_1, \xi_2], \xi_3) + \text{cycl} = 0$
Ex
→ Such a k is called a CE 2-cocycle over $\mathfrak{g}$

$= d\left(\tfrac{1}{2} \langle \mathrm{ad}^*_\eta J, \xi \rangle + \langle L_{\rho(\eta)} J, \xi \rangle \right)$

If $k = 0$ —

Rmk

(♡) fixes J only up to constants!

$$J(z) \longmapsto J'(z) = J(z) + \lambda \qquad \lambda \in \mathfrak{g}^*$$

However if $J(z)$ were equivariant, J' would not! (provided $\mathfrak{g}$ is non Abelian)

$$L_{\rho(\eta)} J' = L_{\rho(\eta)}(J + \lambda) = -\mathrm{ad}^*_\eta J + \text{zero} = -\mathrm{ad}^*_\eta J' + \underbrace{\mathrm{ad}^*_\eta \lambda}_{K(\eta,\cdot) = \langle \lambda, [\eta, \cdot] \rangle \leftarrow \text{trivial CE 2-cocycle.}}$$