Title: Lecture - Mathematical Physics II, PHYS 777

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Collection/Series: Mathematical Physics II (Elective), March 30 - May 1, 2026

Subject: Mathematical physics

Date: March 31, 2026 - 6:30 PM

URL: https://pirsa.org/26030053

Abstract:

LAST TIME

M is a 4 manifold with boundary

A is a gauge field

we saw

$$\int_M \mathrm{tr}\, F(A) \wedge F(A) = \int_{\partial M} 2CS(A)$$

A a gauge field on trivial

A a gauge field on trivial bundle

$\int_M \mathrm{tr}\, F\wedge F$ makes sense for any gauge field

And: If M is a closed 4 manifold

$\frac{1}{8\pi^2}\int_M \mathrm{tr}\, F\wedge F$ is an integer

Instanton number of a gauge field.

This is the 1st Pontryagin number

If N is a 3 manifold, $N = \partial M$

then we can define

$\int_N CS(A)$ to be $\int_M \mathrm{tr}\, F(\hat{A}) \wedge F(\tilde{A})$ where

$\tilde{A}$ is an extension of A to M

$\mathrm{tr}\, F(\tilde{A}) \wedge F(\hat{A})$ is obviously gauge invariant

Pontryagin number

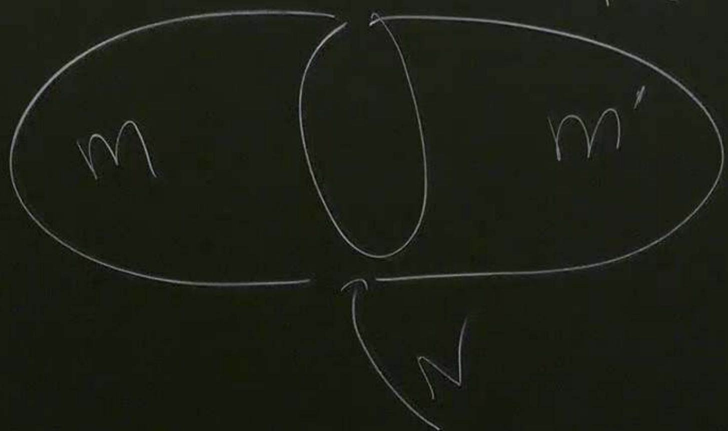
BUT

There may be some M' with $\partial M' = N$
and $\tilde{A}'$ gauge field on M' extends A.

But:

$$\int_M \mathrm{tr}\, F(\tilde{A}) \wedge F(\tilde{A}) - \int_{M'} \mathrm{tr}\left(F(\tilde{A}') \wedge F(\tilde{A}')\right) = 8\pi^2 (\text{an integer})$$

As, we can glue M, M' to be a closed manifold:



$\tilde{A}, \tilde{A}'$ glue to a gauge field on $M \cup_N M'$

$$\int_{M \cup_N M'} \mathrm{tr}\, F \wedge F = 8$$

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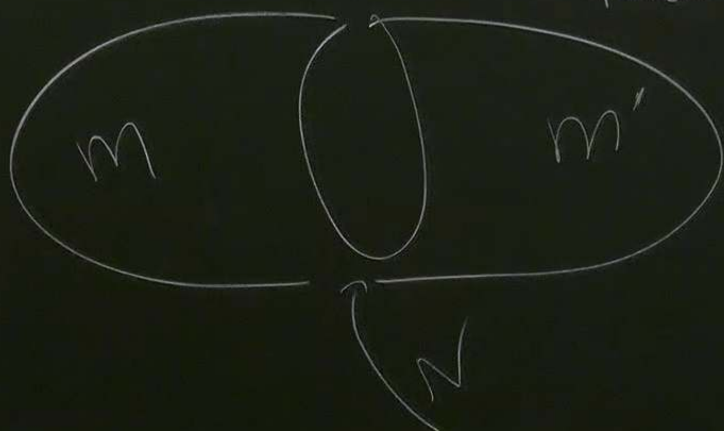

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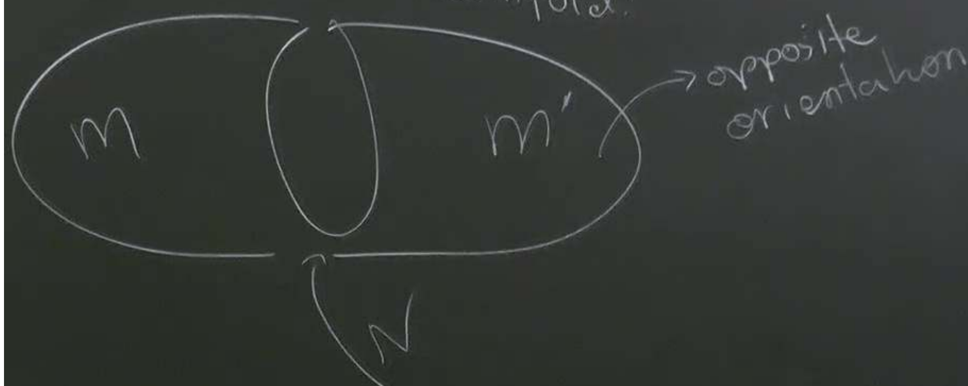


$$\int_{M \cup_N M'} \mathrm{tr}\, F \wedge F = 8\pi^2 \times \text{an integer}$$

$\tilde{A}, \tilde{A}'$ glue to a gauge field on $M \cup_N M'$

CS pat

$\int_A e$

Conclude

Although $CS(A)$ may depend on choices,
it's well-defined + gauge inv. up to $4\pi^2$ an integer

So $e^{\frac{i}{2\pi} k CS(A)}$ is well defined for $k \in \mathbb{Z}$

on choices.

o $4\pi^2$ an integer

or $K \in \mathbb{Z}$

CS path integral is

$$\int_A e^{\frac{i}{2\pi} K\, CS(A)}$$

$K \in \mathbb{Z}$, K called the level

In perturbation theory, CS(A) is gauge invariant + level does not need to be quantized.

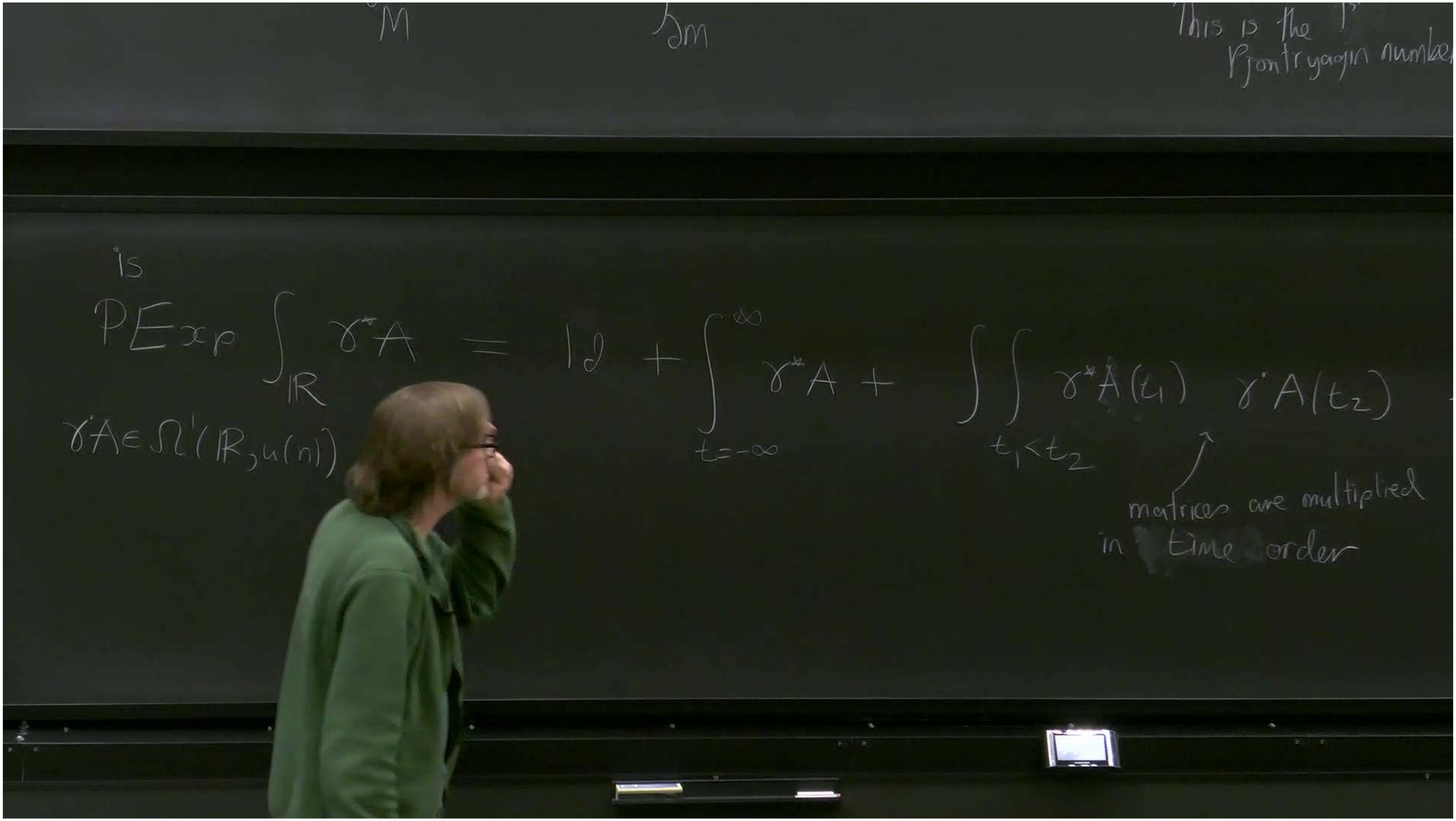
W

WILSON LINES

Let $A \in \Omega^1(M, u(n))$ be a gauge field

$\gamma : \mathbb{R} \longrightarrow M.$

The Wilson line on γ is


M
∂M
This is the
Pontryagin number
is
$PExp \int_{\mathbb{R}} \gamma^* A = Id + \int_{t=-\infty}^{\infty} \gamma^* A + \int\int_{t_1<t_2} \gamma^* A(t_1) \; \gamma^* A(t_2)$
$\gamma^* A \in \Omega^1(\mathbb{R}, u(n))$
matrices are multiplied
in time order

in time order

For $u(1)$, order of matrix mult. doesn't matter

$$\iint_{t_1<t_2} \gamma^* A(t_1)\, \gamma^* A(t_2) = \frac{1}{2}\int_{t_1,t_2} \gamma^* A(t_1)\, \gamma^* A(t_2) = \left(\int \gamma^* A\right)^2$$

$$\iiint_{t_1<t_2<t_3} \gamma^* A(t_1)\, \gamma^* A(t_2)\, \gamma^* A(t_3) = \frac{1}{3!}\left(\int \gamma^* A\right)^3$$

For $u(1)$,

$P\mathrm{Exp}\int \gamma^*$

$A)^2$

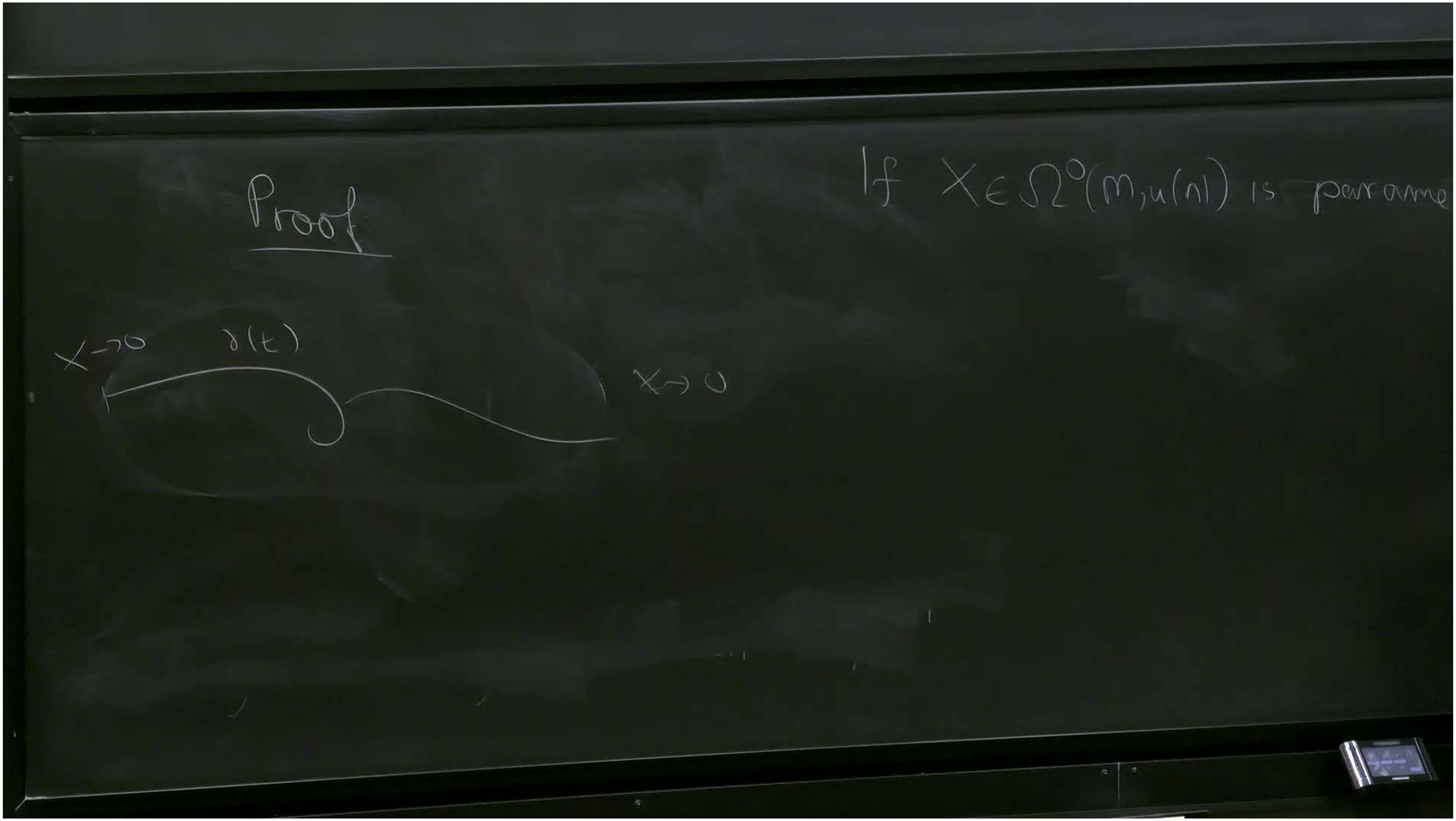
For $u(1)$,

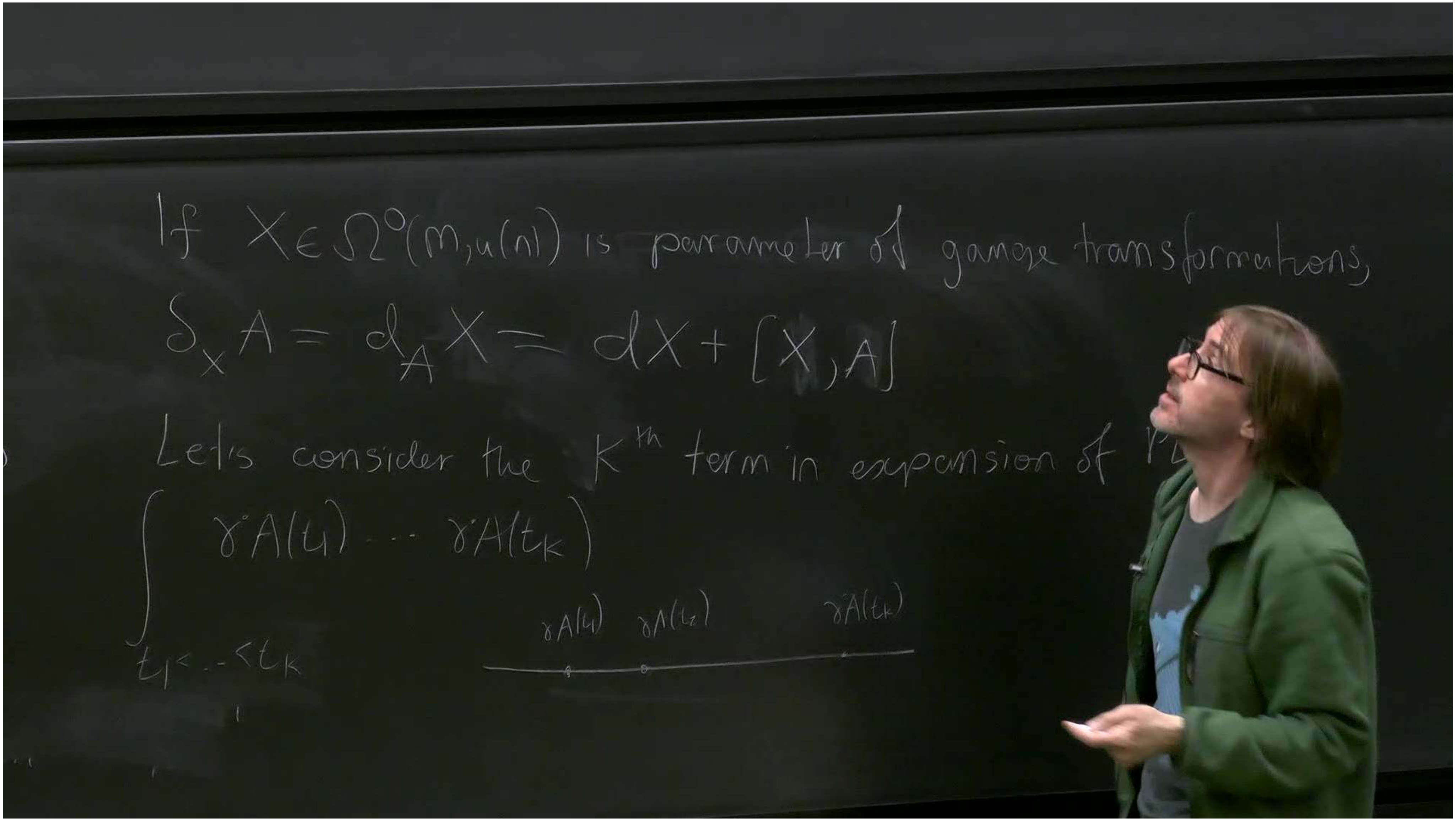
$$P\mathrm{Exp}\int \gamma^* A = \exp \int \gamma^* A.$$

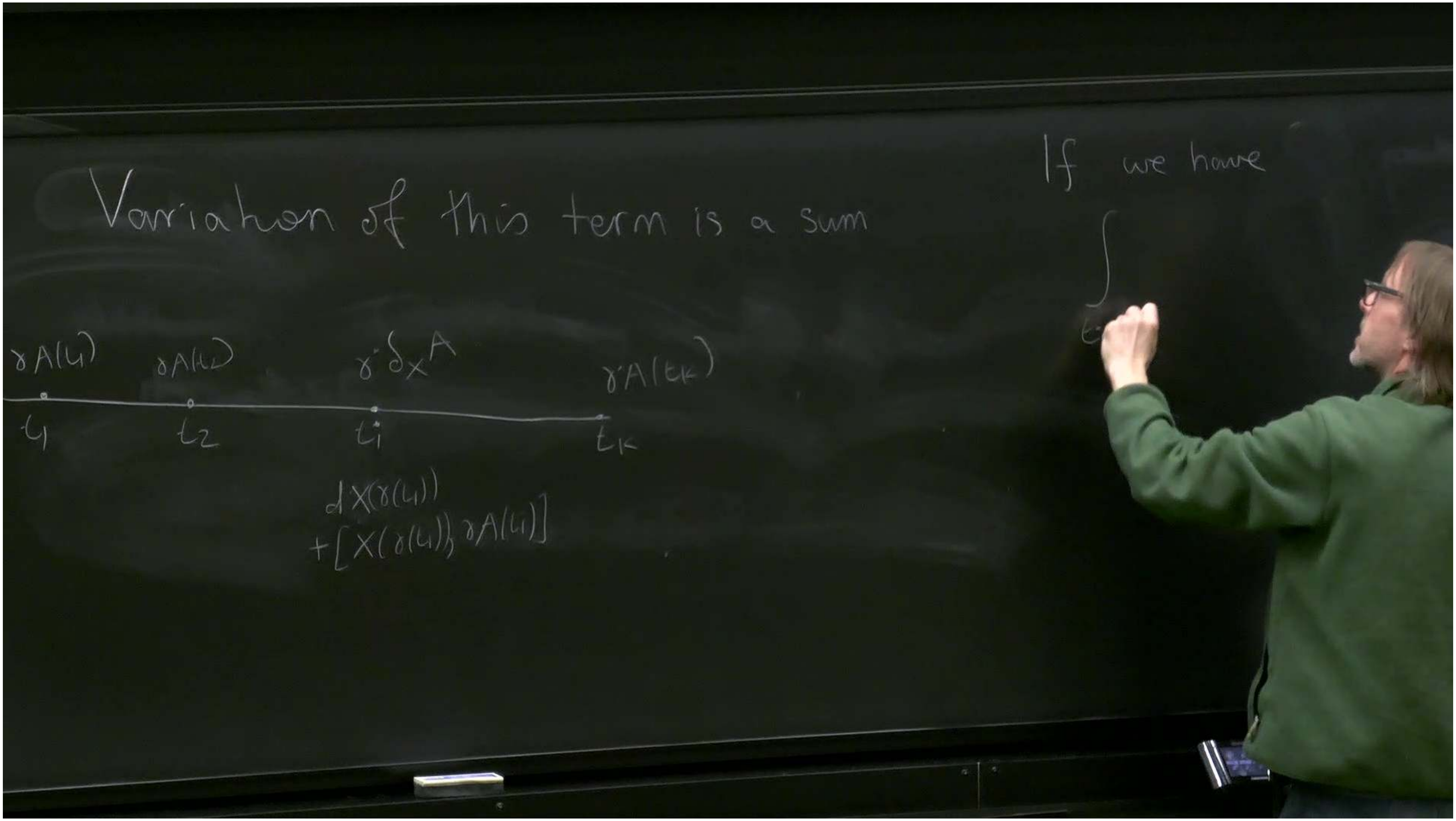
LEMMA $P\mathrm{Exp}\int \gamma^* A \in n \times n$ matrices

is invariant under gauge transformations X which satisfy $X(\gamma(t)) = 0$ at $t = \pm\infty$

$(A \to 0$ at $\gamma(\pm\infty))$


Proof
If $X \in \Omega^0(M, u(n))$ is parame
$X \to 0$
$\gamma(t)$
$X \to 0$


If $X \in \Omega^0(M, u(n))$ is parameter of gauge transformations,
$\delta_X A = d_A X = dX + [X, A]$
Let's consider the K^{th} term in expansion of
$\int_{t_1 < \dots < t_k} \gamma^* A(t_1) \cdots \gamma^* A(t_k)$
$\gamma^* A(t_1)$
$\gamma^* A(t_2)$
$\gamma^* A(t_k)$


Variation of this term is a sum
If we have
γA(t1)
γA(t2)
γ·δX A
γA(tK)
t1
t2
ti
tK
dX(γ(t1))
+[X(γ(t1)), γA(t1)]

rm is a sum

$\gamma^* A(t_K)$

K

If we have

$$\int_{t_{i-1}<t_i<t_{i+1}} dX(\gamma(t_i))$$

can be replaced by boundary term

$$\dots \gamma A(t_{i+1}) \Big| \; X(t_i)\,\gamma^* A(t_i) \dots$$

$$- \; \gamma^* A(t_{i-1})\, X(t_{i-1})\, \gamma A(t_i) \dots$$

Contributions to the variation which come from inserting dX somewhere, give us

$$\sum \iint \gamma^* A(t_1) \cdots \left[X(\gamma(t_j)), \gamma^* A(t_j)\right] \cdots \gamma^* A(t_{k-1})$$

This

This exactly cancels the other term in variation of $\dot{A}$,

$$\delta_X A = dX + [A, X]$$

PHYSICAL MEANING OF PExp

Consider

$$\frac{d}{dT}\, \mathrm{PExp}\int_{-\infty}^{T} \gamma^{*}A = \left(\mathrm{PExp}\int_{-\infty}^{T} \gamma^{*}A\right) \gamma^{*}A(T)$$

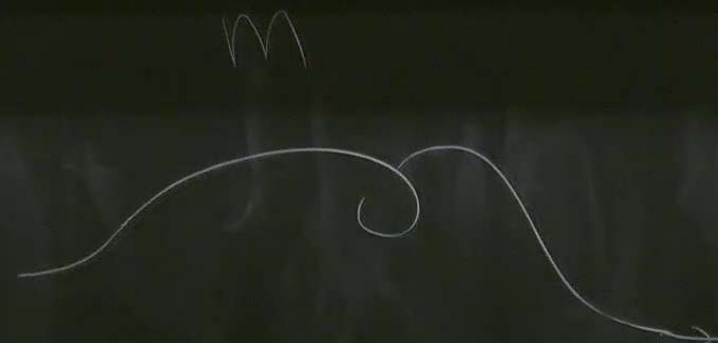
i.e. if we have a 1st matrix differential eq^n

$$(T) \qquad \frac{d}{dT} M(T) = M(T)\, A(T)$$

it is solved by $M(T) = P\mathrm{Exp} \int_{-\infty}^{T} A$

$\mathrm{PExp}_{-\infty}^{T} \int \gamma^* A$ solves the Schrödinger eq^n (ignore factor of i) for a time dependent Hamiltonian on our path which is $\gamma^* A$

$$H(T) = (\gamma^* A)(T)$$



A Wilson line for a gauge theory on M is a QM system on the path γ, with Hilbert space $\mathbb{C}^n$ and coupled to the gauge theory by declaring

$$H(T) = \dot\gamma \cdot A(T)$$

$A(T)$

i.e. if we have a 1^{st} matrix differential eq^n $\quad V(T) \in \mathbb{C}^n$

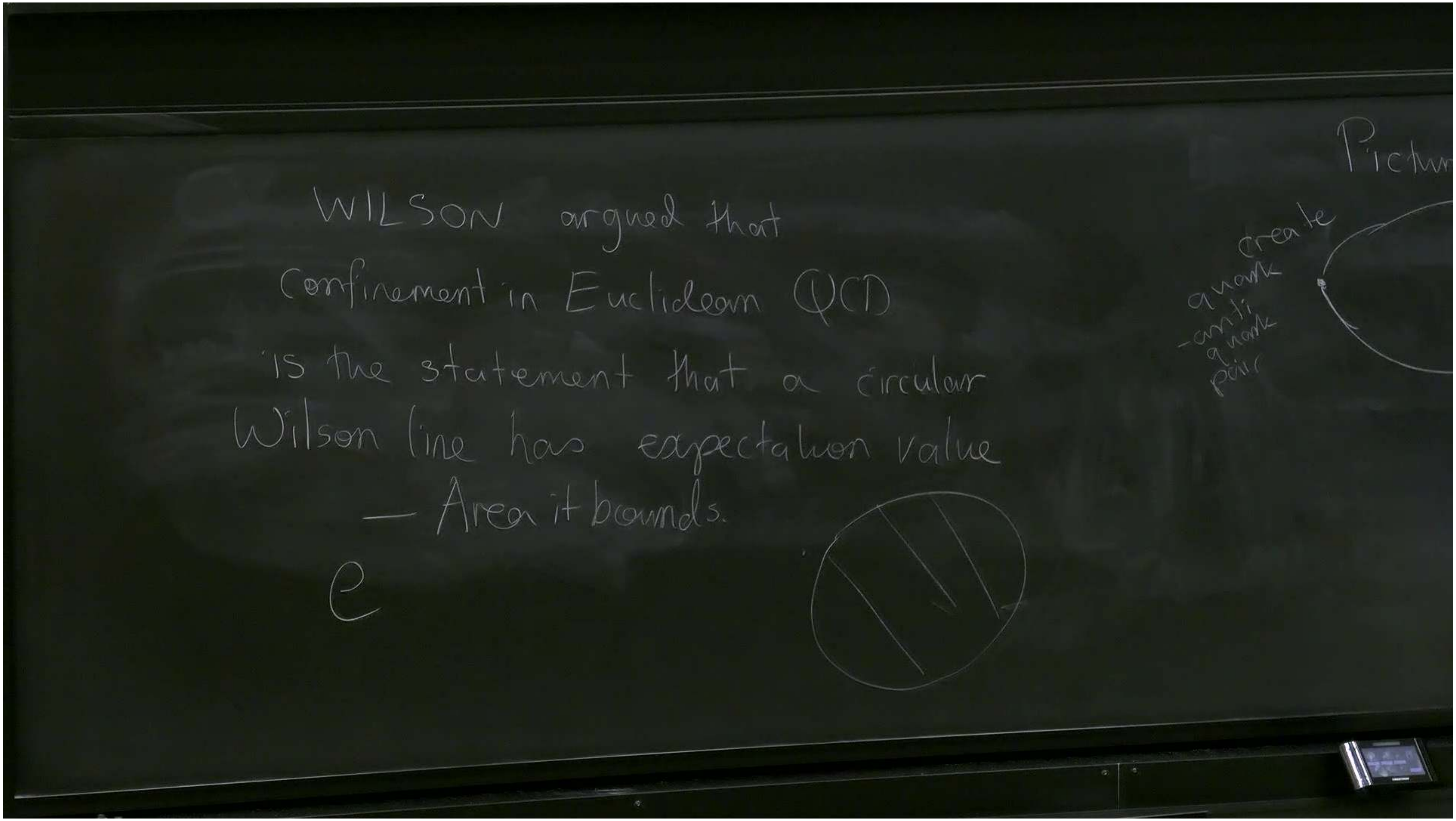
$$\frac{d}{dT} V(T) = A(T)\, V(T)$$

it is solved by $V(T) = \left(P\,\mathrm{Exp} \int_{-\infty}^{T} A \right) V(-\infty)$

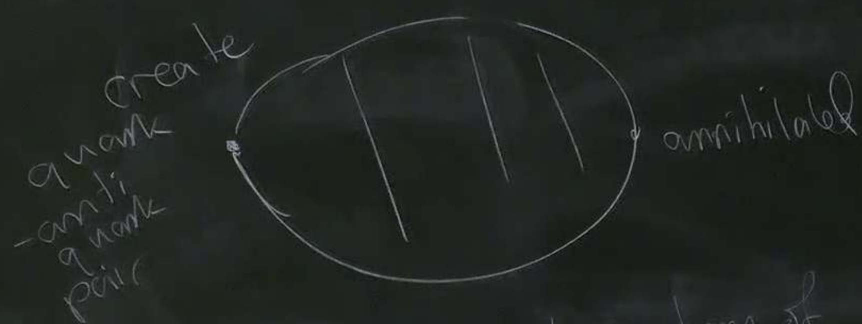
system on the path

γ, with Hilbert space $\mathbb{C}^n$

and coupled to the gauge theory by


WILSON argued that
confinement in Euclidean QCD
is the statement that a circular
Wilson line has expectation value
e
— Area it bounds.
Pictur
create
quark
-anti
quark
pair

Picture



Wilson line = trajectory of the quark and antiquark