Title: Lecture - Quantum Matter, PHYS 777

Speakers: Chong Wang

Collection/Series: Quantum Matter (Elective), PHYS 777, March 30 - May 1 2026

Subject: Condensed Matter

Date: March 30, 2026 - 3:30 PM

URL: https://pirsa.org/26030052

Quantum Matter

Ising model in 1+1 dimensions

$$H = -\sum_{i=1}^{L} Z_i Z_{i+1} - g \sum_i X_i$$

Typically we work with periodic b.c.

$i+L \sim i$

$i = 1, 2 \dots L$

Spin-$\frac{1}{2}$ (qubit) $\mathcal{H}_i \simeq \mathbb{C}^2$

$\mathcal{H} = \bigotimes_i \mathcal{H}_i$

Pauli operator X_i, Y_i, Z_i

Z - eigenstate : $|\uparrow\rangle, |\downarrow\rangle$

X - $\dots$: $|+\rangle, |-\rangle$

We are interested in low energy physics.

Ground states + low energy excitations. , thermodynamic limit $L \to \infty$

Today: two limits. $g \to \infty$

$g \to 0$

① $g \to \infty$ $\quad H_0 = -\sum_i X_i$

G.S. $|\Omega\rangle = \bigotimes_i |X_i = 1\rangle = |+++\ldots\rangle$

lowest excitation. $\quad x \in \{1, 2 \ldots L\}$

$|x\rangle = |+++\ldots\underset{x}{-}\ldots+++\rangle$ $\quad$ in total L of these

Excitation energy above g.s. $= 2$. $\xrightarrow[L\to\infty]{}$ finite $(0 < < \infty)$

(Energy gap)

Large but finite g. $H = -\sum_i X_i - \underbrace{\frac{1}{g}\sum_i Z_i Z_{i+1}}_{\delta H}$ treat in perturbation theory.

To first order. g.s. energy (non-degenerate pert.):

$$\langle \Omega | \delta H | \Omega \rangle = 0$$

g.s wavefunction does change, overall structure similar

First excited states: degenerate perturbation.

Large but finite g. $H = -\sum_i X_i - \underbrace{\frac{1}{g}\sum_i Z_i Z_{i+1}}_{\delta H}$ treat in perturbation theory.

To first order. g.s. energy (non-degenerate pert.):

$$\langle \Omega | \delta H | \Omega \rangle = 0$$

g.s wavefunction does change, overall structure similar

First excited states: degenerate perturbation.

Need: matrix elements of δH in $\mathcal{H}_{2p} = \mathrm{Span}\{|x\rangle, x = 1, 2, \ldots L\}$

$$\langle x | \delta H | y \rangle = -\frac{1}{g}(\delta_{x+1,y} + \delta_{x-1,y})$$

$$-\frac{1}{g}\begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & \ddots \\ 1 & & & \ddots & \ddots \end{pmatrix}$$

Trick to diagonalize $\langle x|\delta H|y\rangle$

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{ikx} |x\rangle .$$

k : "Lattice momentum" defined only mod 2π

$$\text{P. b.c.} \Rightarrow e^{ik\cdot L} = 1 \Rightarrow k = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}, \quad n \sim n+L$$

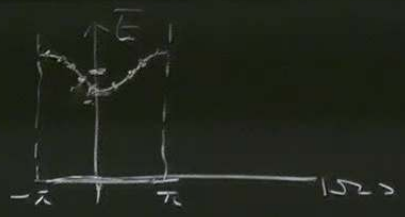
Plug in δH

$$\delta H |k\rangle = -\frac{1}{g} \cdot \frac{1}{\sqrt{L}} \sum_x e^{ikx} (|x+1\rangle + |x-1\rangle) = -\frac{1}{g} \frac{1}{\sqrt{L}} (e^{ik} + e^{-ik}) \sum_x e^{ikx} |x\rangle$$

$$= -\frac{2\cos k}{g} \cdot |k\rangle .$$

Excitation energy for $|k\rangle$

$$E_k = 2 - \frac{2\cos k}{g}$$



Lowest excitation $k = 0$

$\Delta = 2 - \frac{2}{g}$ still finite for small $1/g$.

$e^{ikx}|x\rangle$

Comments: ① Why do $|k\rangle$ work?

Translation symmetry $T_x : |x\rangle \to |x+1\rangle$

$[T_x, \delta H] = 0$

$\Rightarrow$ $T_x, \delta H$ can be simultaneously diag.

$T_x |k\rangle = e^{-ik} |k\rangle$.

More generally, can use $|k\rangle \sim \sum_x e^{ikx} \underbrace{|u_k\rangle_x}$ ↳ k-dependent wave function at x

↳ "Bloch state"

Bloch Thm: Bloch states work

② $\Delta: 2 \to 2 - O(\frac{1}{g})$ remains finite under small perturbation

In general, we believe this is true for local interaction. $H = \sum_i H_i$

Finite gap → if energy scale $\ll \Delta$ → The system basically dead

⇒ Insulator vs. metal

$g \to 0$. $H_0 = -\sum_i Z_i Z_{i+1}$

Two degenerate g.s $|\Omega_+\rangle = |\uparrow\uparrow\uparrow\ldots\rangle$

$|\Omega_-\rangle = |\downarrow\downarrow\downarrow\ldots\rangle$.

Lowest excitation: $|\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\ldots\uparrow\rangle$

domain wall

p.b.c: at least 2 domain walls

$\Delta = 4$

Small g ? $\delta H = -g \sum_i X_i$ $(\Delta = 4 + O(g))$

What happens to $|\Omega_{T,i}\rangle$?

Z_2 symmetry of Ising. $X := \bigotimes_i X_i$ $[H, X] = 0$ $(X_i Z_i = -Z_i X_i)$

Can organize eigenstates of H as X-eigen.

X-eigenstates: $|\Omega_+\rangle = |\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle$

$|\Omega_-\rangle = |\uparrow\uparrow\uparrow\ldots\rangle - |\downarrow\downarrow\downarrow\ldots\rangle$

→ "Bloch state"

Splitting $\delta \sim \langle \Omega_+ | \delta H^n | \Omega_+ \rangle - \langle \Omega_- | \delta H^n | \Omega_- \rangle = \begin{cases} 0 \text{ for } n \ll L \\ O(g^L) \quad n \sim O(L) \\ \sim e^{-L/\xi} \end{cases}$ } Robust degeneracy

n-th order perturbation

Key observation: $|\Omega_\pm\rangle$ are locally indistinguishable.

Reduced density matrix $\rho_A = tr_{\bar{A}} |\Omega_\pm\rangle\langle\Omega_\pm| = \frac{1}{2}\left(|\uparrow\uparrow\uparrow\ldots\rangle\langle\uparrow\uparrow\uparrow\ldots| + |\downarrow\downarrow\downarrow\ldots\rangle\langle\downarrow\downarrow\ldots|\right)$.

$\bar{A}$ A $\bar{A}$

$\Rightarrow \forall O_A$ acting on A, $\langle\Omega_+|O_A|\Omega_+\rangle = \langle\Omega_-|O_A|\Omega_-\rangle$.