Title: Lecture - Quantum Information I (Elective), 635

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Lecture 5 – Entropy and its properties

Alice

Bob

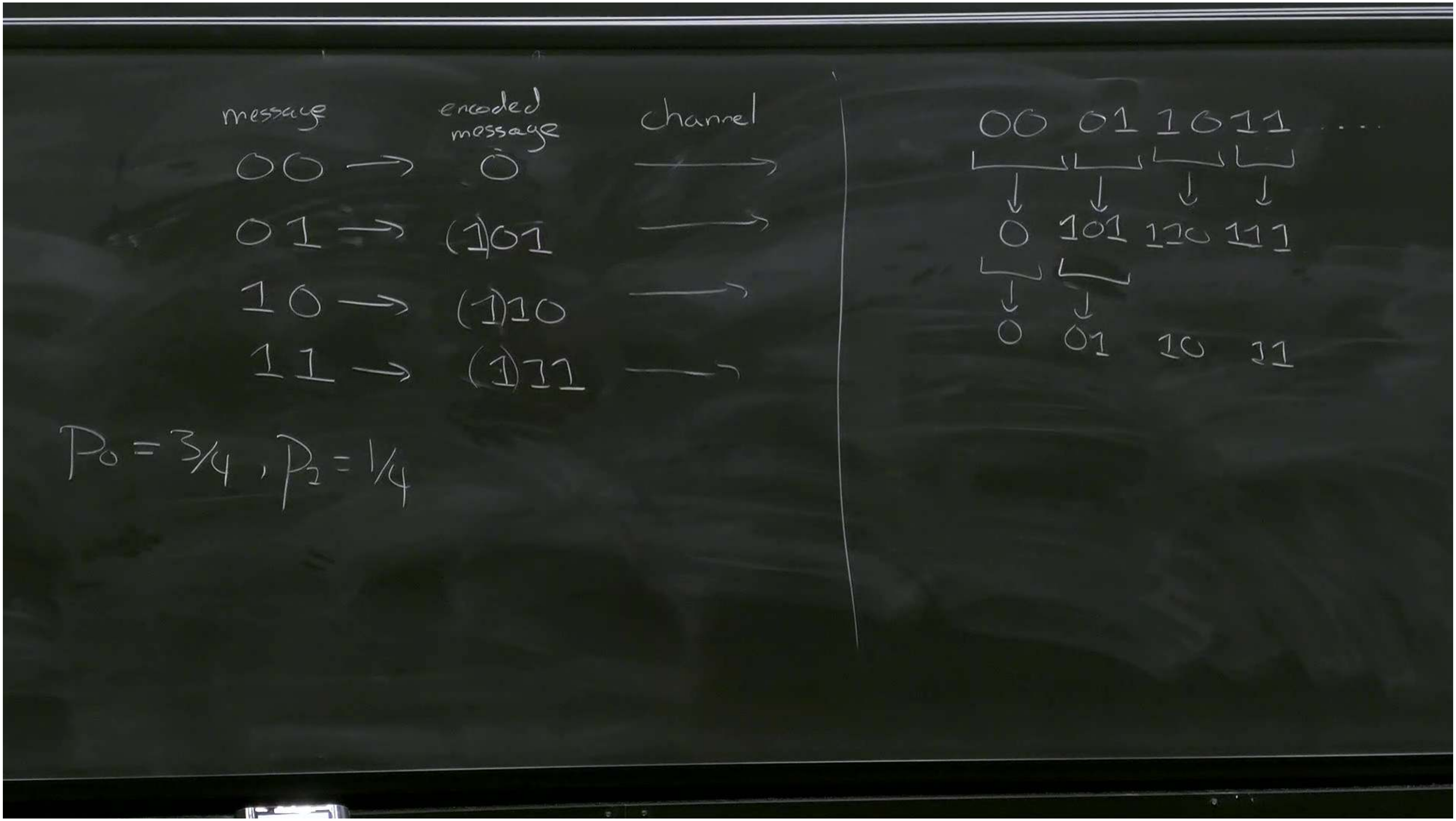
$\longrightarrow X_1, X_2 \ldots X_n \; \vec{y} \longrightarrow$ noiseless channel $\longrightarrow \vec{y}$ $\qquad X_1, \ldots X_n$

"classical source"

$X_i \in \{0,1\}$

$P_X = (p_0, p_1)$

$$\text{rate} = \lim_{n\to\infty} \left(\frac{\text{channel uses}}{\text{source uses}} \right)$$


message
encoded message
channel
00 → 0
01 → (1)01
10 → (1)10
11 → (1)11
P0 = 3/4 , P1 = 1/4
00 01 10 11 . . .
0 101 110 111
0 01 10 11

$$\frac{1}{2}\langle \text{channel uses}\rangle = \frac{1}{2}\left(p_{00}\cdot 1 + p_{01}\cdot 3 + p_{10}\cdot 3 + p_{11}\cdot 3\right)$$

$$= \sum_m p(m)\,\ell(m)$$

$$= \frac{1}{2}\left(\frac{3}{4}\times\frac{3}{4}\cdot 1 + \frac{3}{4}\cdot\frac{1}{4}\times 3 + \frac{1}{4}\cdot\frac{3}{4}\times 3 + \frac{1}{4}\frac{1}{4}\times 3\right)$$

$$= \frac{15}{16}$$

$r =$

$$r = \lim_{n\to\infty} \frac{\frac{n}{2} \times \left(\begin{array}{c}\text{channel use} \\ \text{to send 2 bits}\end{array}\right)}{n}$$

$\frac{1}{4} \times 3 \,)$

Thm The optimal (noiseless) rate is.

$$\boxed{H(X) = -\sum_x p_x \log_2 p_x}$$

"Shannon entropy"

Quantum (von Neumann) entropy

Alice

$|\psi_x\rangle, p_x$

"quantum source" $\longrightarrow |\psi_0\rangle, |\psi_1\rangle \ldots |\psi_2\rangle \ldots \longrightarrow$ Noiseless quantum channel $\longrightarrow |\psi_0\rangle, |\psi_1\rangle \ldots$

Example:

source $|0\rangle, |1\rangle, |+\rangle, |-\rangle$

$1/4, 1/4, 1/4, 1/4$

$$H(X) = -\sum_{x=1}^{4} \frac{1}{4} \log\left(\frac{1}{4}\right)$$

$$= -\log(1/4) = 2$$

Thm The optimal quantum rate for source $\{(p_x, |\psi_x\rangle)\}$ is

$$\rho_A = \sum_x p_x |\psi_x\rangle\langle\psi_x|$$

$$S(A)_\rho = -\mathrm{tr}(\rho \log \rho)$$

"Von Neumann entropy"

Example:

source: $|0\rangle, \ |1\rangle, \ |+\rangle, \ |-\rangle$

$1/4, \ 1/4, \ 1/4, \ 1/4$

$$H(X) = -\sum_{x=1}^{4} \frac{1}{4} \log\left(\frac{1}{4}\right)$$

$$= -\log(1/4) = 2$$

$\sim n p_0$

$n p_1$

$$\rho = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$$

$$= \frac{1}{2}\Big(\underbrace{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|}_{I/2}\Big) + \frac{1}{2}\Big(\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

$$= I/2$$

Thm The optimal quantum rate for source $\{(p_x, |\psi_x\rangle)\}$ is

$$\rho_A = \sum_x p_x |\psi_x\rangle\langle\psi_x|$$

$$S(A)_\rho \equiv -\mathrm{tr}(\rho \log \rho)$$

"Von Neumann entropy"

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$S(A)_\rho = -\sum_i \lambda_i \log \lambda_i$$

$S(A)_\rho = -$

$= -$

$=$

Properties

1) $S(A)_\rho \geq 0$ $\qquad \sum_i \lambda_i (-\log \lambda_i) \quad , \quad \sum_i \lambda_i = 1$

2) $S(A)_\rho \leq \log d_A$

3) $S(AB)_{\rho_A \otimes \rho_B} = S(A)_{\rho_A} + S(B)_{\rho_B}$ $\qquad \log(\rho_A \otimes \rho_B) = (\log \rho_A) \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \log \rho_B$

4) $S(A)_{\text{[illegible]}}$

4) $S(A)_{|\psi\rangle\langle\psi|} = 0$

$$= -\mathrm{tr}\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \log \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \right)$$

$$= 0$$

$$\log\left(\sum \lambda_i |i\rangle\langle i| \right)$$

$$= \sum \log \lambda_i \, |i\rangle\langle i|$$

Quantum (von Neumann) entropy

Alice

$|\psi_x\rangle_A, p_x$

$\rightarrow |\psi_0\rangle, |\psi_1\rangle \ldots |\psi_2\rangle \ldots \rightarrow$ Noiseless quantum channel $\rightarrow |\psi_0\rangle, |\psi_1\rangle \ldots$

"quantum source"

$\log_2 d_A$

$$\rho = \sum p_x |\psi_x\rangle\langle\psi_x|$$

Example:

sour

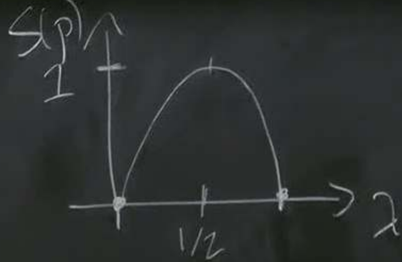
$H(X)$

$\sim n p_0$

$n p_1$

4) $S(A)_{|\psi\rangle\langle\psi|} = 0$

5) $\rho = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$



6) Strong subadditivity:

$$S(AB)_\rho + S(BC)_\rho \geq S(B)_\rho + S(ABC)_\rho$$

Relative entropy.

$$D(\rho\|\sigma) = \begin{cases} \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma) & \ker\sigma \subseteq \ker\rho \\ \infty & \text{else} \end{cases}$$

1) $D(\rho\|\sigma) \geq 0$

2) $D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$