Title: Lecture - Topological String Theory

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Collection/Series: Topological String Theory Mini-Course, Jan 15 - March 19, 2026

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$S^{\text{kinetic}}(\eta) = \int \partial\eta \wedge \partial\eta$

Last Time:

Any TFT $\rightsquigarrow$ a sol'n to classical master eq'n on cochain complex

$$\mathcal{H}(S^1)^{S^1} = \left(\mathcal{H}(S^1)[[t]], \ell + tD\right)$$

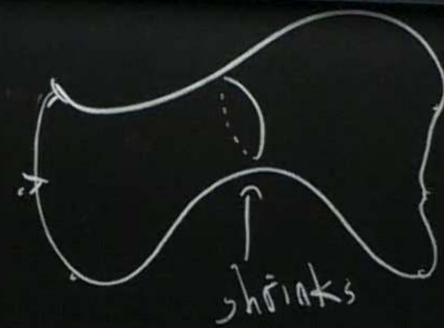
- S is given by $\int$ over $g=0$ worldsheet moduli
- BV bracket on $\mathcal{O}(\mathcal{H}(S^1)^{S^1})$ comes from

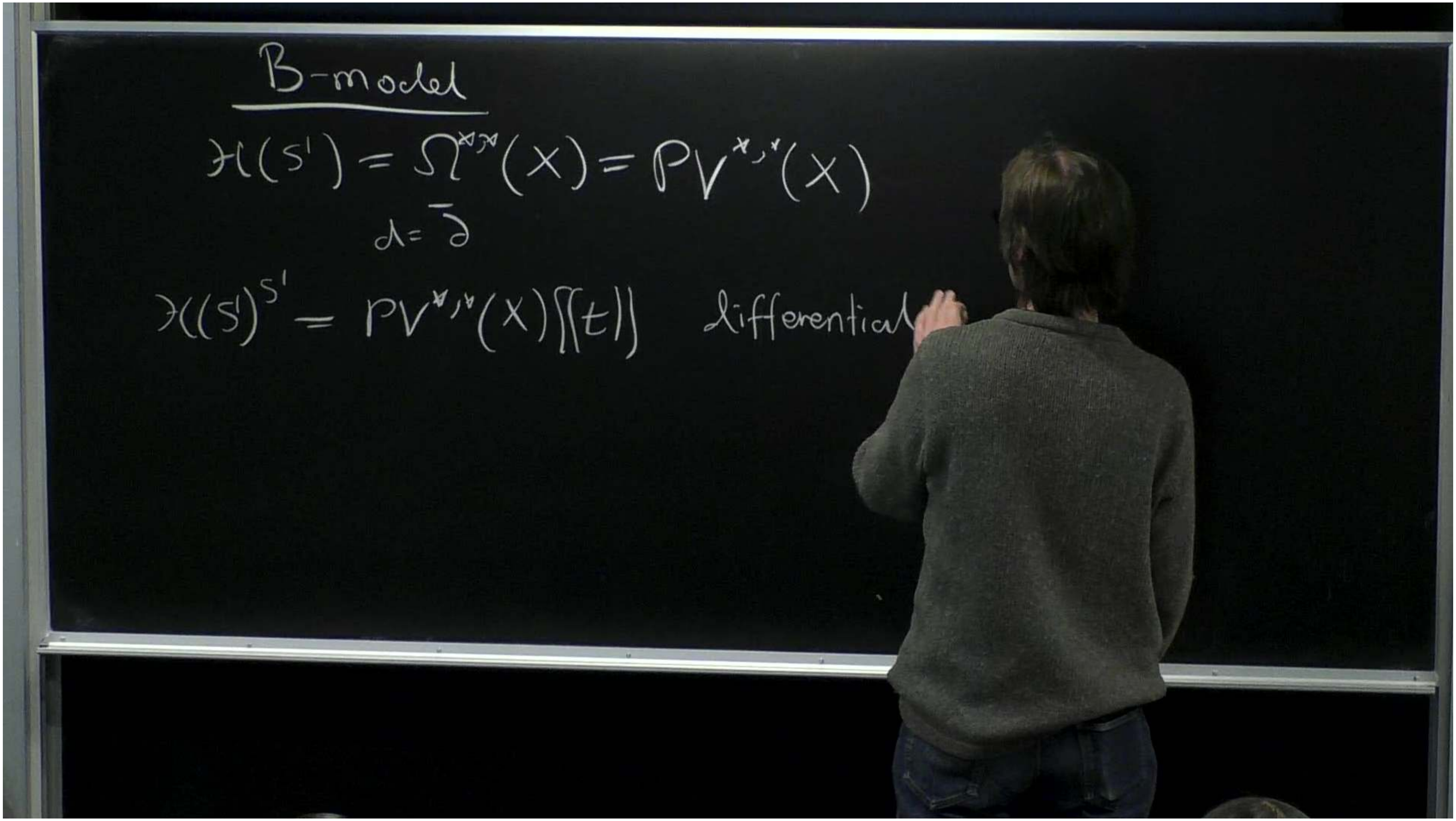
the tensor

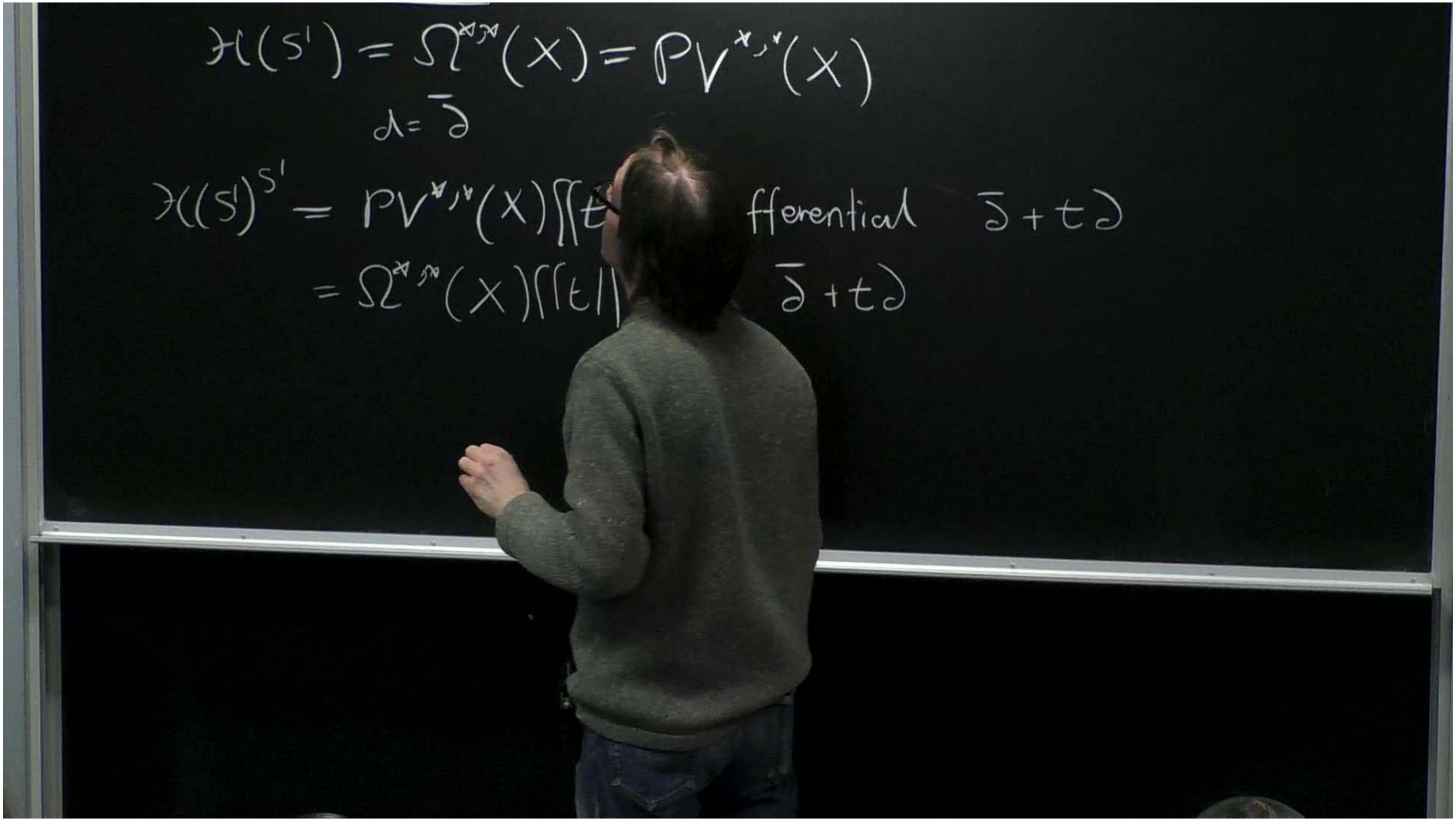
$$\mathcal{H}((S^1)^{S^1} \otimes \mathcal{H}((S^1)^{S^1}$$

$$\uparrow$$

$$(D\otimes 1)\left(\begin{pmatrix} \circ \\ \varepsilon \\ \varepsilon \end{pmatrix}\right) \longrightarrow \mathcal{H}((S^1)^{\otimes 2}$$




B-model
$\mathcal{H}(S^1) = \Omega^{*,*}(X) = PV^{*,*}(X)$
$d = \bar{\partial}$
$\mathcal{H}(S^1)^{S^1} = PV^{*,*}(X)[[t]]$ differential


ℋ(S¹) = Ω*,*(X) = PV*,*(X)
d = ∂̄
ℋ(S¹)^S¹ = PV*,*(X)[[t
fferential ∂̄ + t∂
= Ω*,*(X)[[t]]
∂̄ + t∂

$$(\partial \otimes \underline{1}) \cdot S_{\Delta} \in \bar{\Omega}^{d+1, d}(X \times X)$$

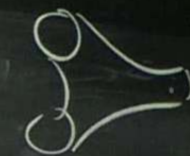
$d = \dim X$

$$= PV^{d-1, d}(X \times X)$$

<u>What is S?</u> First, cubic action

$PV^{\bullet,\bullet}(X)^{\otimes 3} \longrightarrow \mathbb{C}$. No worldsheet moduli integrals

Cubic action = TFT 3 point function.

TFT has a product comes from 

And an $\int$ map  : $PV^{*,*}(X) \to \mathbb{C}$

Cubic action is $\int \eta \wedge \eta \wedge \eta$

Product $= \wedge$ on $PV^{*,*}$

Fact:  is

$$\eta \in PV^{*,*}(X) \to \int_X \Omega_X \wedge (\eta \vee \Omega_X)$$

Only non-zero on $PV^{d,d}(X)$

Only non-zero on $\int_X PV^{d,d}(X)$

Full action: degree n in fields is a map

$$(PV^{*,*}(X)[[t]])^{\otimes n} \longrightarrow \mathbb{C} \quad \text{associated to}$$

$\int$ over $\mathcal{M}_{0,n}$

$$\eta_1 t^{k_1} \dots \eta_n t^{k_n} \to \left(\int (\eta_1 \wedge \dots \wedge \eta_n)\right) \int_{\bar{\mathcal{M}}_{0,n}} \psi_1^{k_1} \dots \psi_n^{k_n}$$

$\psi_i \in H^2(\bar{\mathcal{M}}_{0,n})$ are ψ classes

Only non-zero if $\sum k_i = n-3$

What is kinetic term?

- Badly defined as BV bracket involves ∂

  The theory is shifted Poisson, not symplectic

Fields have a differential $\bar{\partial} + t\partial$

Usually, $\{ S^{\text{kinetic}}, - \}$ gives $\bar{\partial} + t\partial$

$\{ \alpha, \beta \}$ involves $\int \partial \alpha \, \beta$ we can say, if $\eta \in PV^{*,*}(X)$

$$S^{\text{kinetic}}(\eta) = \int \partial^{-1} \eta \, \bar{\partial} \eta$$

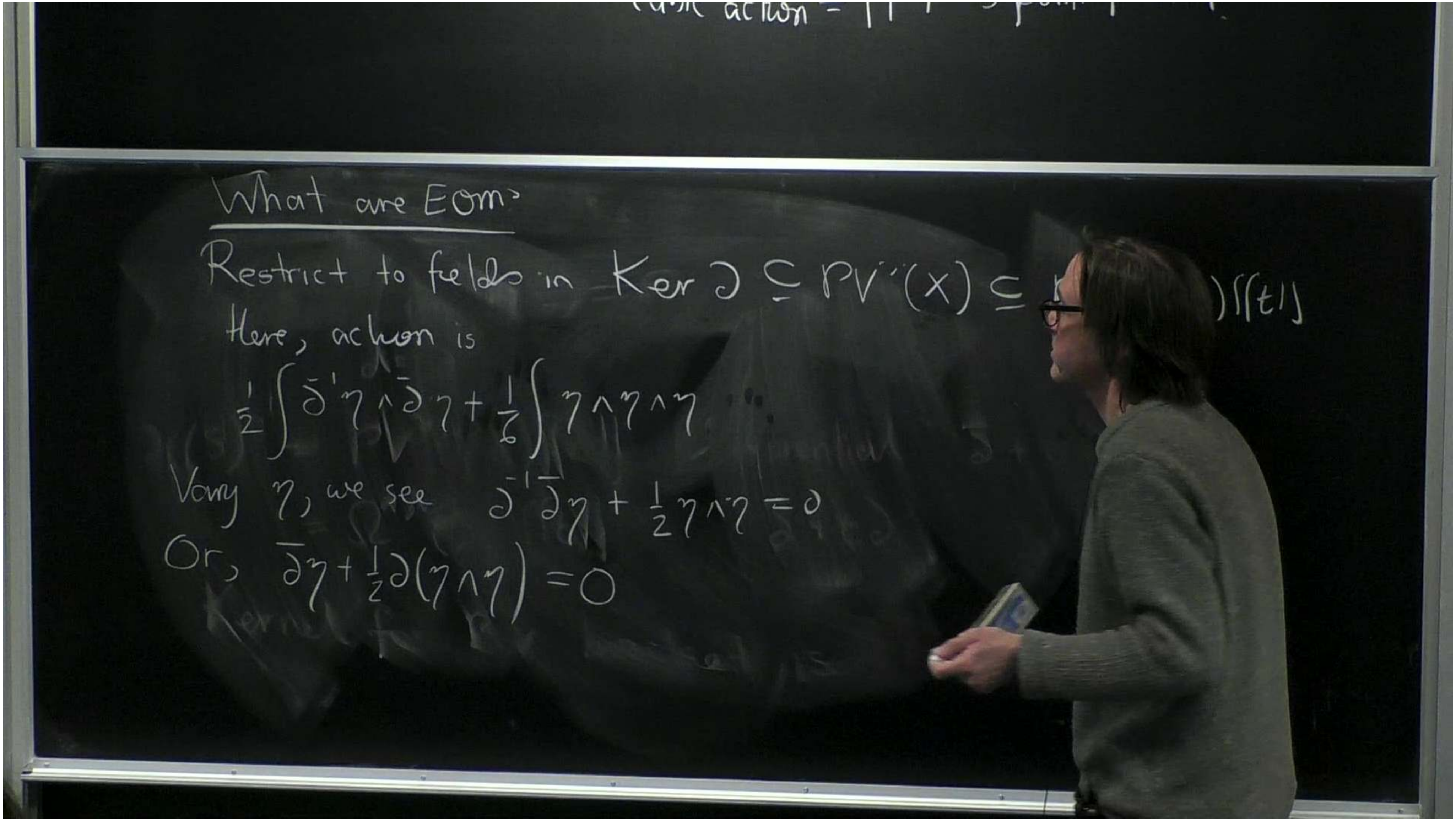
Note

B-model on X comes from $\mathrm{II}B$ on $X \times \mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon}$

$$\downarrow$$

$$X \times \mathbb{R}_{\geq 0}$$

Really, B-model comes from a 7d theory which is free, w. an interesting boundary condition.



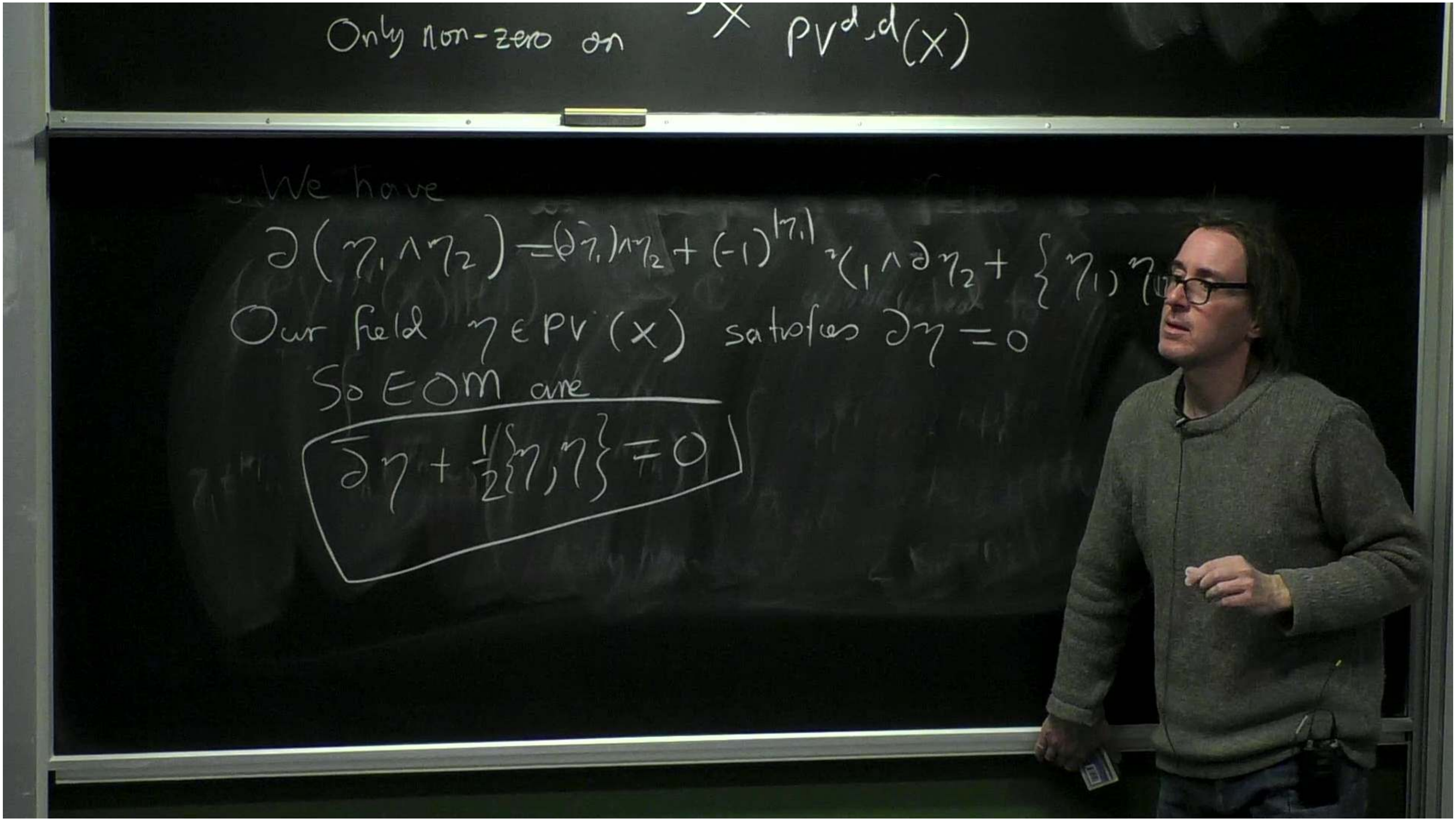
What are EOM?
Restrict to fields in $\operatorname{Ker} \partial \subseteq PV^{\bullet,\bullet}(X) \subseteq$ ... $)[[t]]$
Here, action is
$\frac{1}{2}\int \bar{\partial}^{-1}\eta \wedge \bar{\partial}\eta + \frac{1}{6}\int \eta\wedge\eta\wedge\eta$
Vary η, we see $\partial^{-1}\bar{\partial}\eta + \frac{1}{2}\eta\wedge\eta = 0$
Or, $\bar{\partial}\eta + \frac{1}{2}\partial(\eta\wedge\eta) = 0$

On $PV^{\bullet,\bullet}(X)$ there is the Schouten bracket which gives it a shifted Poisson alg. str

In words, $PV^{\bullet,\bullet}(X)$ is $\Omega^{0,\bullet}(\mathbb{C}^n)[\theta^1, \dots, \theta^n]$ $\quad \theta^i \leftrightarrow \frac{\partial}{\partial z_i}$

$$\{\eta_1, \eta_2\} = \frac{\partial \eta_1}{\partial z_i} \frac{\partial \eta_2}{\partial \theta^i} + (\leftrightarrow)$$

In words, $\partial = \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial z_i}$



Only non-zero on $\int_X PV^{d,d}(X)$
We have
$\partial(\eta_1 \wedge \eta_2) = (\partial \eta_1)\wedge\eta_2 + (-1)^{|\eta_1|}\eta_1 \wedge \partial\eta_2 + \{\eta_1, \eta_2\}$
Our field $\eta \in PV(X)$ satisfies $\partial\eta = 0$
So EOM are
$\bar{\partial}\eta + \frac{1}{2}\{\eta,\eta\} = 0$

- If $\eta \in PV^{1,1}(X)$ then it is a def of complex structure. Locally, $\bar\partial$ operator is deformed to

$$\bar\partial + \eta = \bar\partial + \eta_i^j \, d\bar z_i \frac{\partial}{\partial z_j}$$

New hol. fns are those in kernel of $\bar\partial + \eta$

$$(\bar\partial + \eta)^2 = 0 \iff \bar\partial \eta + \frac{1}{2}\{\eta, \eta\} = 0$$

$\iff$ we have an integral def. of complex structure

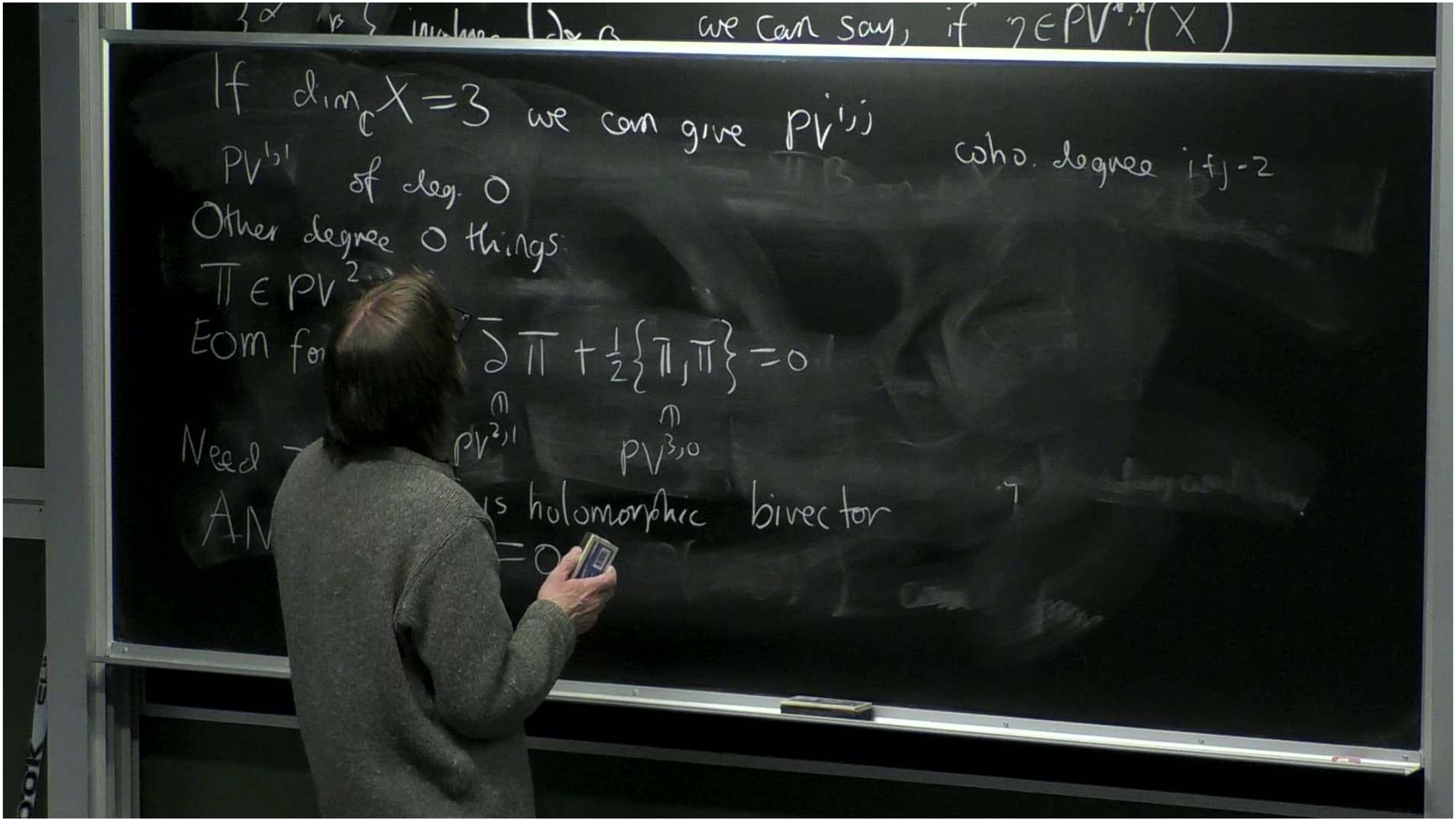
If $\dim_{\mathbb{C}} X = 3$ we can give $PV^{i,j}$ coho. degree $i+j-2$

$PV^{1,1}$ of deg. 0

Other degree 0 things:

$\Pi \in PV^{2,0}$

EOM for Π: $\underbrace{\bar\partial \Pi}_{\in PV^{2,1}} + \frac{1}{2}\underbrace{\{\Pi,\Pi\}}_{\in PV^{3,0}} = 0$


we can say, if $\eta \in PV^{*,*}(X)$
If $\dim_{\mathbb{C}} X = 3$ we can give $PV^{i,j}$
coho. degree $i+j-2$
$PV^{1,1}$ of deg. 0
Other degree 0 things:
$\Pi \in PV^{2}$
EOM for
$\bar{\partial} \Pi + \frac{1}{2}\{\Pi, \Pi\} = 0$
$PV^{2,1}$
$PV^{3,0}$
Need
is holomorphic bivector
$=0$

Π gives a Poisson bracket on sheaf of fns on X

$\{\Pi,\Pi\}=0 \iff$ this satisfies Jacobi.

$\gamma \in PV^{0,2}$. EOM are $\bar\partial\gamma=0$ $\gamma\in H^2(X,\mathcal{O}_X)$

Together, space of deg. 0 sol^s to EOM

= def. of X, as a complex manifold + hol Poisson tensor

+ a gerbe

Full field content

$\eta \in PV^{\bullet}(X)[[t]]$

EOM (after a transformation)

are $\bar{\partial}\eta + t\partial\eta + \frac{1}{2}\{\eta,\eta\} = 0$