Title: Lecture - Topological String Theory

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Collection/Series: Topological String Theory Mini-Course, Jan 15 - March 19, 2026

Date: February 19, 2026 - 7:00 PM

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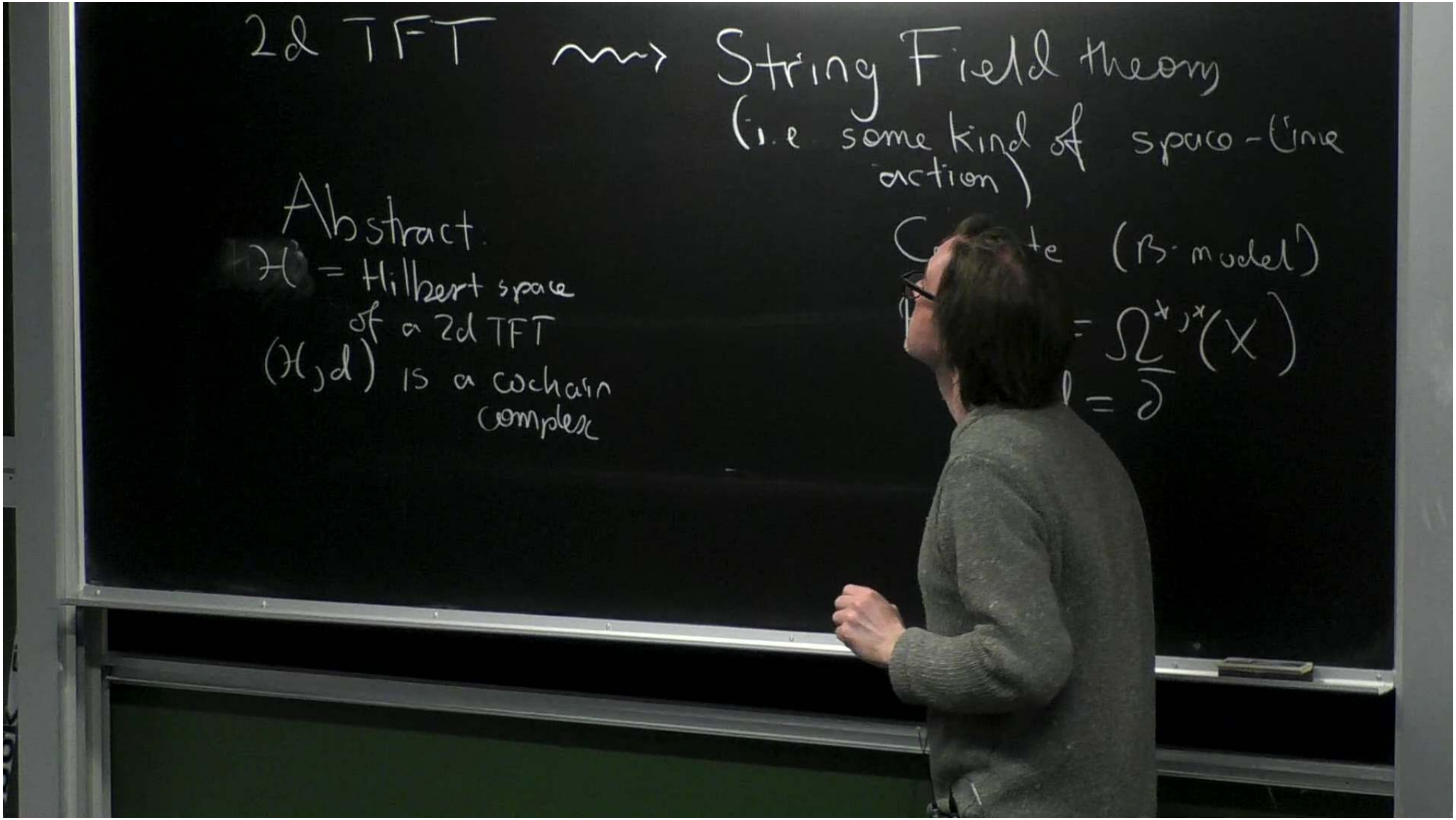

2d TFT ⇝ String Field theory
(i.e some kind of space-time action)
Abstract
$\mathcal{H}$ = Hilbert space of a 2d TFT
$(\mathcal{H}, d)$ is a cochain complex
(B-model)
$= \Omega^{*,*}(X)$
$= \bar{\partial}$

$\mathcal{H}$ has an action of H_*S^1
given by $D: \mathcal{H} \to \mathcal{H}$

$D^2 = 0 \quad [D, d] = 0$

$D = \partial$

Couple to gravity take HS^1 invariants

$$\mathcal{H}^{S^1} = \left(\mathcal{H}[[t]], d + tD\right) \qquad \left(\Omega^{\cdot,\cdot}(X)[[t]], \bar{\partial} + t\partial\right)$$

Fields of string field theory

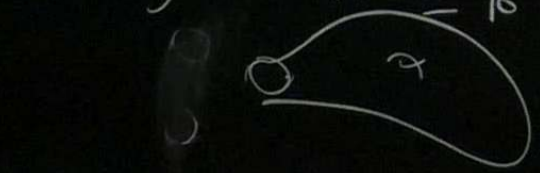
Before taking S^1 'invariants, we had maps

$\mathcal{H}^{\otimes n} \to \mathbb{C}$

associated to Riemann surface Σ, n boundary components

boundaries are framed

given an $\cong$ to a fixed S^1



After taking S^1 'invariants

$S^1 \simeq \mathrm{Diff}_+ S^1$

we don't need to

choose a coord.

boundary comp

If $\Sigma \in \mathcal{M}_{g,n}$ is a RS with n marked points, we get a map

$$(\mathcal{H}^{S^1})^{\otimes n} \longrightarrow \mathbb{C}$$

associated to Σ.

$C_* B\mathrm{Diff}(\Sigma)$

$\wr$

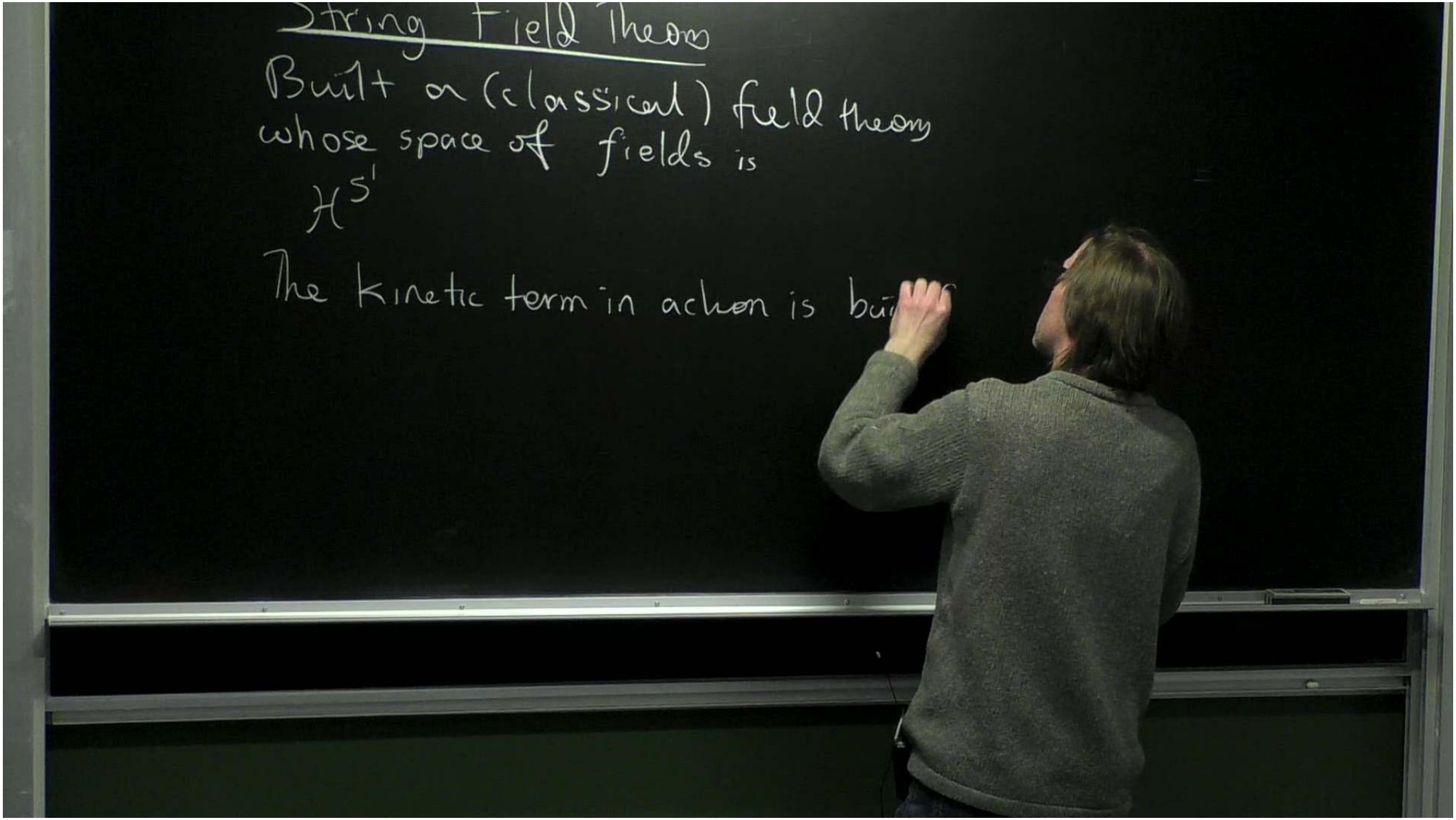
$\mathcal{M}_{g,n}$

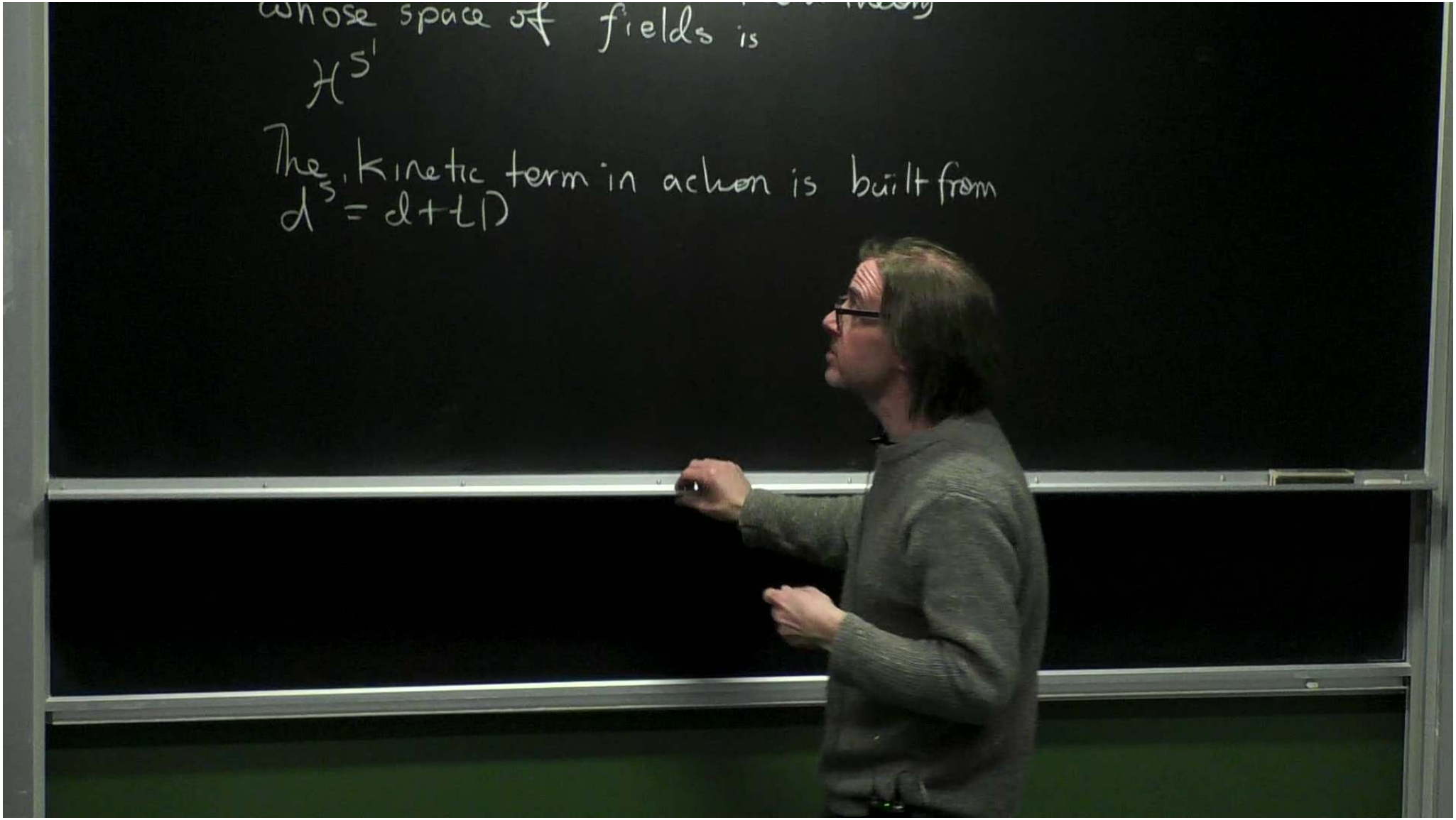
$3g-3+n>0$

This extends to a cochain map

$$C_*(\mathcal{M}_{g,n}) \otimes (\mathcal{H}^{S^1})^{\otimes n} \longrightarrow \mathbb{C}$$

(since depends on Σ up to coherent homotopy)


String Field Theory
Built on (classical) field theory
whose space of fields is
$\mathcal{H}^{S^1}$
The kinetic term in action is


whose space of fields is
$\mathcal{H}^{S^1}$
The kinetic term in action is built from
$d^S = d + t\iota_D$

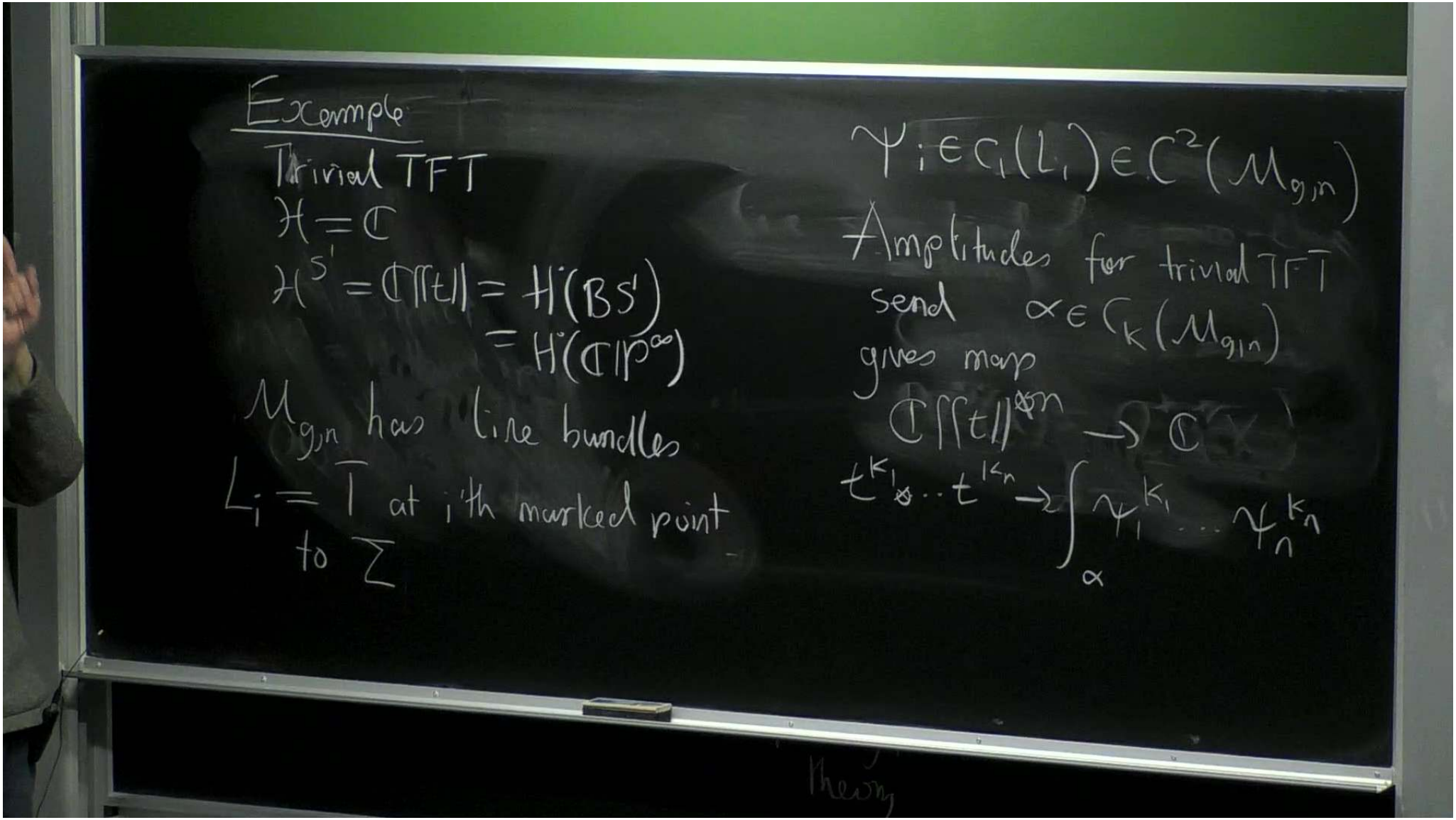
Interaction is a sum

$$I = \sum_{n \geq 3} I_n$$

$$I_n \in \mathrm{Sym}^n\left(\mathcal{H}(S^1)^\vee\right)$$

is given by $[M_{0,n}] : \mathcal{H}(S^1)^{\otimes n} \to \mathbb{C}$

$I_n = \int$ of amplitudes over $M_{0,n}$


Example
Trivial TFT
$\mathcal{H} = \mathbb{C}$
$\mathcal{H}(S^1) = \mathbb{C}[[t]] = H^*(BS^1)$
$= H^*(\mathbb{CP}^\infty)$
$\mathcal{M}_{g,n}$ has line bundles
$L_i = T$ at ith marked point
to Σ
$\psi_i \in C_1(L_i) \in C^2(\mathcal{M}_{g,n})$
Amplitudes for trivial TFT
send $\alpha \in C_k(\mathcal{M}_{g,n})$
gives map
$\mathbb{C}[[t]]^{\otimes n} \to \mathbb{C}$
$t^{k_1} \otimes \cdots t^{k_n} \to \int_\alpha \psi_1^{k_1} \cdots \psi_n^{k_n}$

What is $[\mathcal{M}_{0,n}]$ as $\mathcal{M}_{0,n}$ is not compact?

Consider $\bar{\mathcal{M}}_{0,n}$ the DM compactification. This is compact

Not htpy equiv to $\bar{\mathcal{M}}_{0,n}$.

Let $\tilde{\mathcal{M}}_{0,n}$ = real blow up of $\bar{\mathcal{M}}_{0,n}$ at boundary divisors

$\bar{\mathcal{M}}_{0,n}$ allow nodes

In $\widetilde{\mathcal{M}}_{0,n}$ have nodal surfaces

but the S^1 bundles of tangent r. $/\mathbb{R}^+$

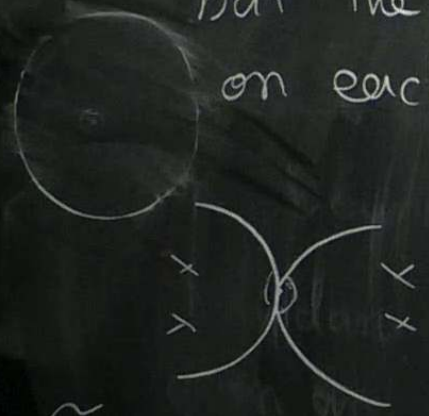
on each side of node are identified.

This allows us to infinitesimally smooth

$zw = 0$

$\Rightarrow zw = q$

extra datum $\equiv$ phase of q



$\widetilde{\mathcal{M}}_{0,n}$ is a real manifold with corners,

locus with k nodes is real codim k.

Since $\tilde{\mathcal{M}}_{0,n}$ is a manifold w. corners,

$\tilde{\mathcal{M}}_{0,n}$ is htpy equiv to $\mathcal{M}_{0,n}$, its interior

So, we get operations

$$C_*(\tilde{\mathcal{M}}_{0,n}) \otimes (H^{S^1})^{\otimes n} \to \mathbb{C}$$

and, we have a fundamental chain

$$[\tilde{\mathcal{M}}_{0,n}] \in C_{2n-6}(\tilde{\mathcal{M}}_{0,n})$$

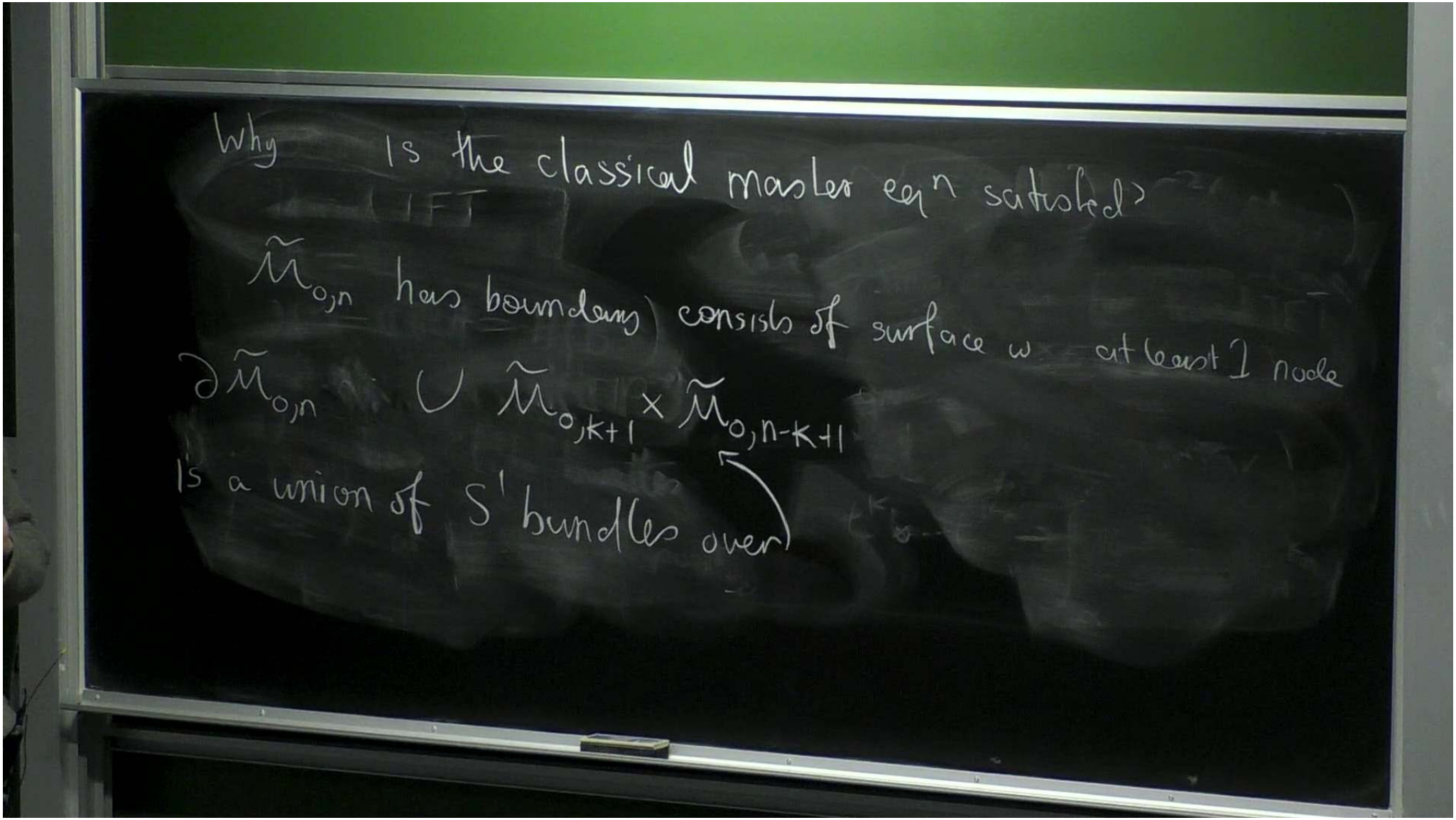
I_n = ∫ of amplitudes over $\overline{\mathcal{M}}_{0,n}$

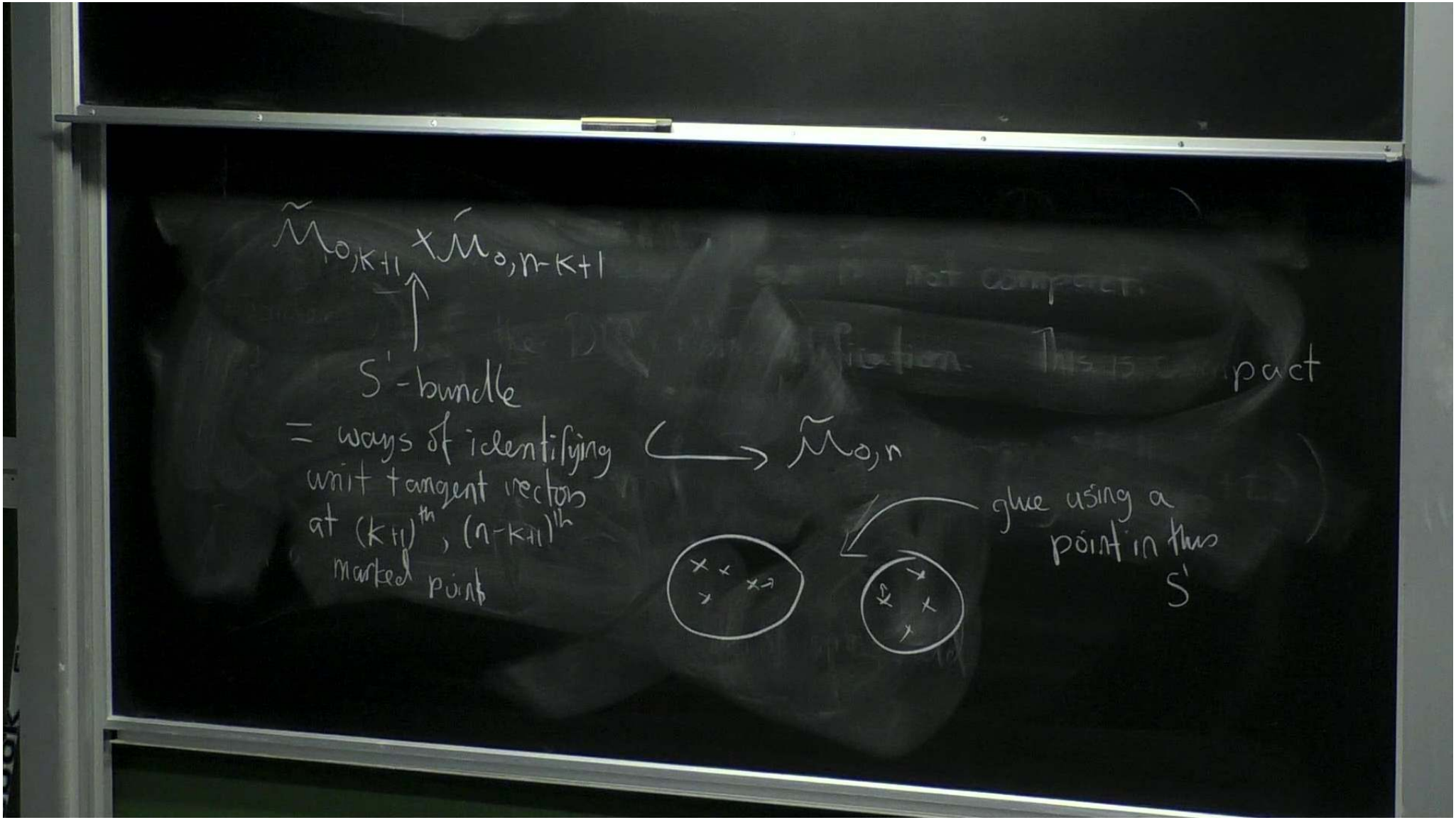
$[\tilde{\overline{\mathcal{M}}}_{0,n}]$ allows us to define $I_n : (\mathcal{H}^{S^1})^{\otimes n} \to \mathbb{C}$

and so $I = \sum_{n \geq 3} I_n$

Claim 1) I always satisfies a classical master eqn $d^{S^1} I + \frac{1}{2}\{I, I\} = 0$

2) If we include higher genus curves, it satisfies QME


Why is the classical master eqn satisfied?
IFT
$\tilde{\mathcal{M}}_{0,n}$ has boundary consists of surface w at least 1 node
$\partial\tilde{\mathcal{M}}_{0,n} \supset \bigcup \tilde{\mathcal{M}}_{0,k+1} \times \tilde{\mathcal{M}}_{0,n-k+1}$
is a union of S^1 bundles over


$\tilde{M}_{0,k+1} \times \tilde{M}_{0,n-k+1}$
S^1-bundle
= ways of identifying unit tangent vectors at $(k+1)^{th}$, $(n-k+1)^{th}$ marked point
$\hookrightarrow \tilde{M}_{0,n}$
glue using a point in this S^1

and, we have a fundamental chain

$$[\tilde{\mu}_{0,n}] \in C_{2n-6}(\tilde{\mu}_{0,n})$$

This eqn

$$d[\tilde{\mu}_{0,n}] \sim \sum [\hat{\tilde{\mu}}_{0,K+1}] \hat{\wedge} [\hat{\tilde{\mu}}_{0,n-K+1}] \hat{\wedge} [S']$$

{ fun class of extra S'

will become master eqn

$$d\bar{I}_n = \sum \{\bar{I}_{K+1}, \bar{I}_{n-K+1}\}$$

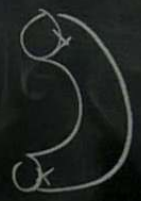
We need $\{\ \}$ to be related to extra S'

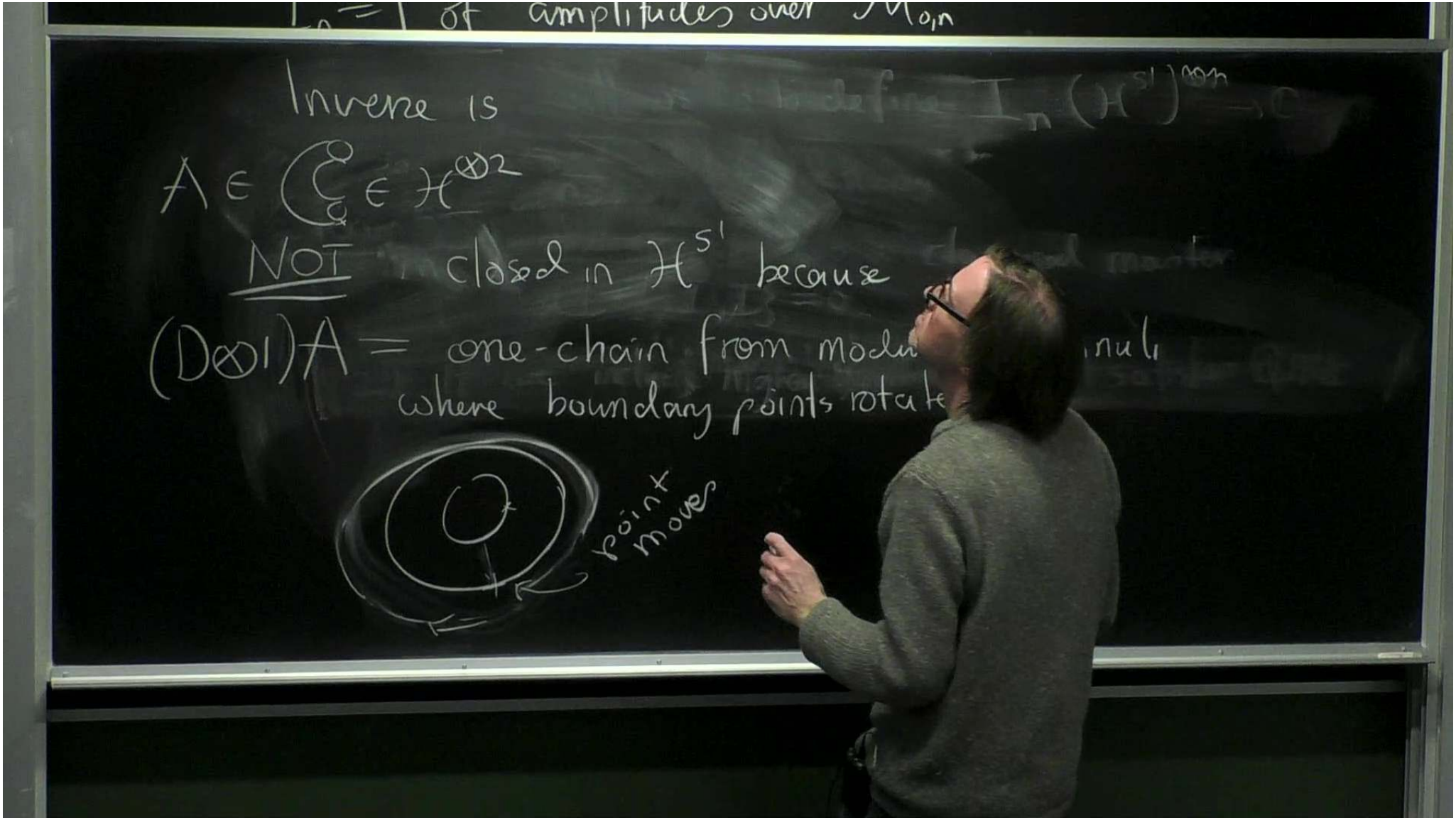
What is $\{\ ,\ \}$?

Fields are $\mathcal{H}(S^1)$

BV Poisson bracket lives in $\mathcal{H}(S^1) \otimes \mathcal{H}(S^1)$

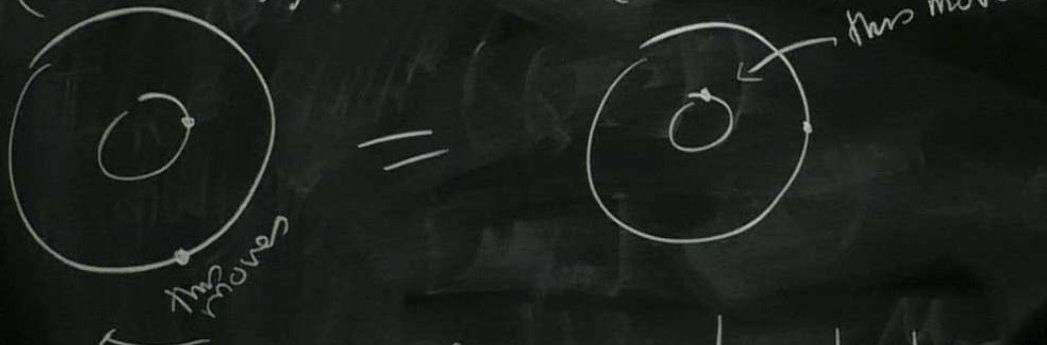
as it is a bi-vector w. constant coefficients;

There is a non-degenerate pairing on $\mathcal{H}$ coming from annulus  $\mathcal{H}^{\otimes 2} \to \mathbb{C}$


of amplitudes over $\mathcal{M}_{0,n}$
Inverse is
$A \in \mathcal{C} \in \mathcal{H}^{\otimes 2}$
NOT closed in $\mathcal{H}^{S^1}$ because
$(D \otimes 1)A =$ one-chain from modu
where boundary points rotate
point moves

However,

$(D \otimes 1)\tilde{A} = \pm (1 \otimes D) A$



$(D \otimes 1)\tilde{A} \in \mathcal{H}(\otimes)\mathcal{H}$ is BV bracket

In kernel of d_1 and $D \otimes 1$ and $1 \otimes D$, since $D^2 = 0$

So

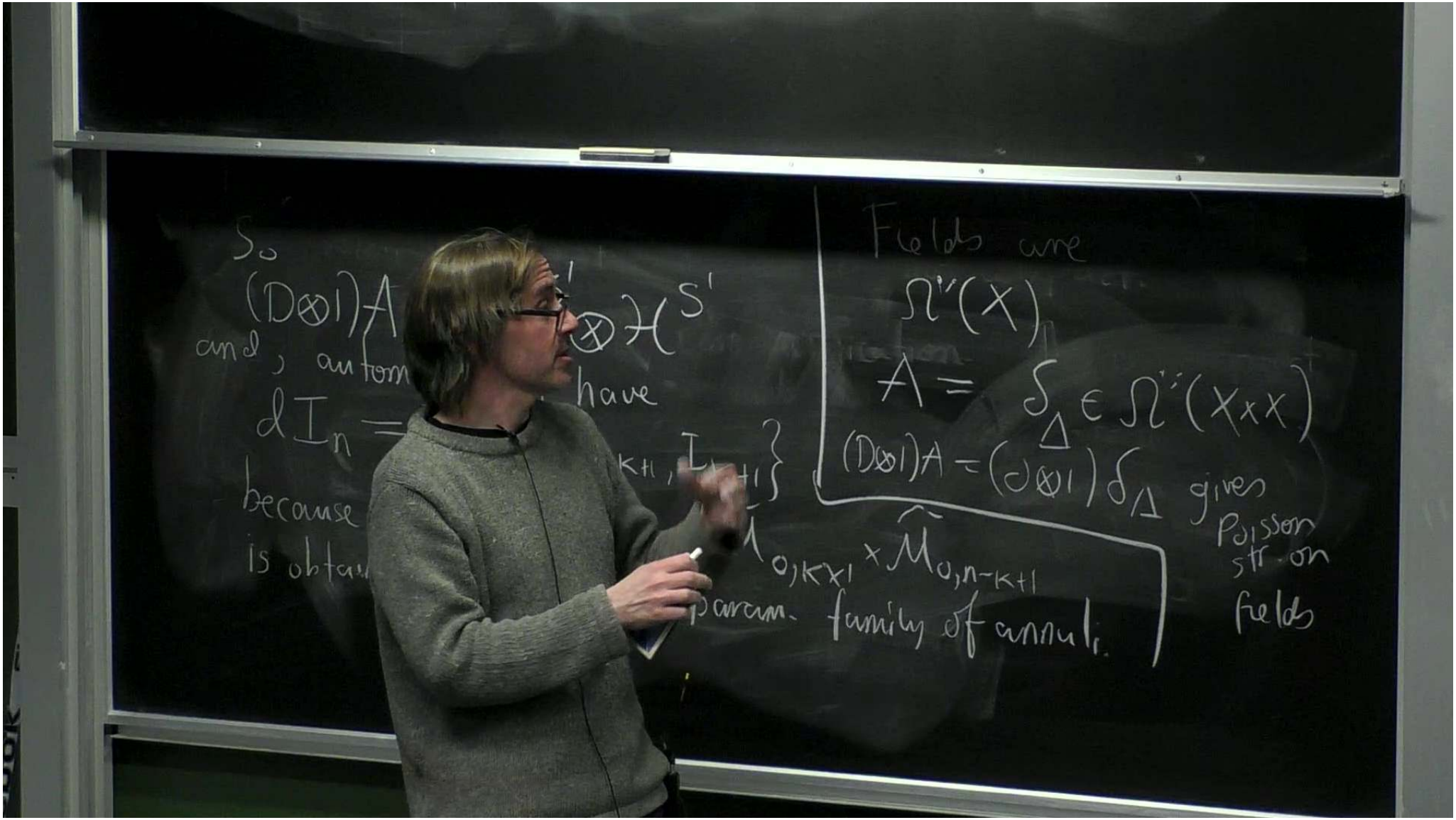
$(D \otimes 1) A \in \mathcal{H}(S^1) \otimes \mathcal{H}(S^1)$

and, automatically have

$$dI_n = \sum \{ I_{n-k+1}, I_{k+1} \}$$

because S^1-bundle over $\widetilde{\mathcal{M}}_{0,k+1} \times \widehat{\mathcal{M}}_{0,n-k+1}$ is obtained by gluing in 1 param. family of annuli.

pact


Fields are
$\Omega^{\cdot,\cdot}(X)$
$A = \delta_\Delta \in \Omega^{\cdot,\cdot}(X \times X)$
$(D \otimes 1)A = (\partial \otimes 1)\delta_\Delta$ gives Poisson str. on fields
$\overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,n-k+1}$
param. family of annuli.