Title: Lecture - Gravitational Physics (Elective), PHYS 636

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Subject: Cosmology, Strong Gravity

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LECTURE 8: MORE ON VACUUM* BLACK HOLES

Most astrophysical objects rotate, black holes are no exception → the Kerr metric

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right)dt^2 + \frac{4GMar}{\Sigma}\sin^2\theta\, dt\, d\varphi - \Sigma\, d\theta^2$$

$$- \frac{\Sigma\, dr^2}{\Delta} - \sin^2\theta\left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]d\varphi^2$$

$$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\varphi\right]^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma\, d\theta^2 - \frac{\sin^2\theta}{\Sigma}\left[a\, dt - (r^2+a^2 \ldots\right.$$

* & Λ.

HOLES

k holes

$-\Sigma\, d\theta^2$

$\theta] d\varphi^2$

$a\,dt - (r^2+a^2)d\varphi]^2$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 + a^2 - 2GMr$$

$$(a = J/M)$$

Geodesics

Consider a Killing vector

$$\frac{d}{d\tau}\left(k_\mu \dot{x}^\mu\right) = \dot{x}^\nu \nabla_\nu \left(k_\mu \dot{x}^\mu\right)$$

$$= \underbrace{\cancel{\dot{x}^\nu k_\mu \nabla_\nu \dot{x}^\mu}}_{\text{geod}} + \underbrace{\cancel{\dot{x}^\nu \dot{x}^\mu \nabla_\nu k_\mu}}_{\text{K.V.}}$$

i.e.

her

$\cancel{\dot X^\nu \dot X^\mu \nabla_\nu k_\mu}$

K.V.

i.e. $k_\mu \dot X^\mu$ conserved along geodesic.

$k = \frac{\partial}{\partial t}$: $\rightarrow$ $E = g_{t\mu}\dot X^\mu$

$$= \left(1 - \frac{2GMr}{\Sigma}\right)\dot t + \frac{2GMar\sin\theta}{\Sigma}\dot\phi$$

$k = \frac{\partial}{\partial \phi}$: $\rightarrow$ $h = -g_{\phi\mu}\dot X^\mu$

$$= \left(r^2 + a^2 + \frac{2GMr}{\Sigma}a^2\sin^2\theta\right)\dot\phi$$

$$- \frac{2GMar}{\Sigma}\sin^2\theta\,\dot t$$

Now consider $P_\alpha = (E, -h) = g_{\alpha\beta}\dot{X}^{\beta} \leftarrow t, \varphi$

$$P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta$$

$$= E^2 g^{tt} - 2Ehg^{t\varphi} + h^2 g^{\varphi\varphi}$$

$$= \frac{E^2}{\Delta\Sigma}\left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right] - \frac{4EhGMar}{\Delta\Sigma} - \frac{h^2}{\Delta\sin^2\theta}\left(1 - \frac{2GMr}{\Sigma}\right)$$

$$= \frac{1}{\Delta\Sigma}\left[(r^2+a^2)E - ah\right]^2 - \frac{\sin^2\theta}{\Sigma}\left[aE - \frac{h}{\sin^2\theta}\right]^2$$

$\frac{1}{\Sigma}$ $\quad \tilde{f}_h(r) \quad \tilde{g}(\theta)$

The geodesic satisfies

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \eta \to \begin{cases} +1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

$$\shortparallel$$

$$p^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2$$

$\theta = \pi/2$

$$\Rightarrow \quad \dot{r}^2 + V_{eff}(r) = 0$$

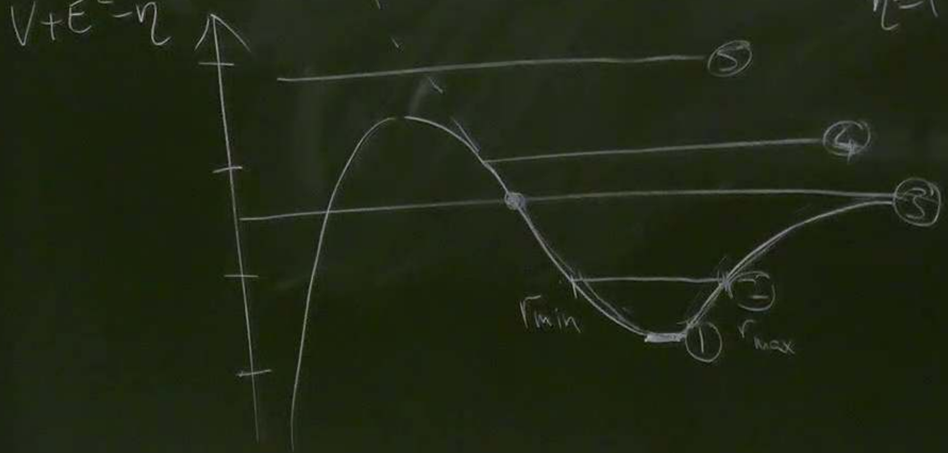
Where

$$V_{eff} = \frac{\eta\Delta}{r^2} - \frac{1}{r^4}\left\{ \begin{array}{l} [E(r^2+a^2) - ah]^2 \\ + \Delta[aE-h]^2 \end{array} \right\}$$

$$= \eta^2 - E^2 - \frac{2GM\eta}{r} + \frac{a^2}{r^2}(\eta - E^2) + \frac{h^2}{r^2} - \frac{2GM}{r^3}(aE-h)^2$$

SCH $V = \underbrace{\eta - E^2}_{\text{sets a baseline}} - \frac{2GM\eta}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}$

$\eta = 1$



1. $V = V' = 0$ Circular orbit
2. $r \in [r_{min}, r_{max}]$ Elliptical orbit
3. Parabolic scattering (marginal)
4. $\dot{r}^2 > 0$ – Scattering orbit
5. Capture

$V<0$ for $r\in\mathbb{R}$

At a turning pt

$$V' = \frac{2GM}{r^2} - \frac{2h^2}{r^3} + \frac{6GMh^2}{r^4} = 0$$

$$\Rightarrow r = \frac{h^2}{2GM} \pm \sqrt{\frac{h^4}{4G^2M^2} - 3h^2}$$

+ root is circular orbit - stable

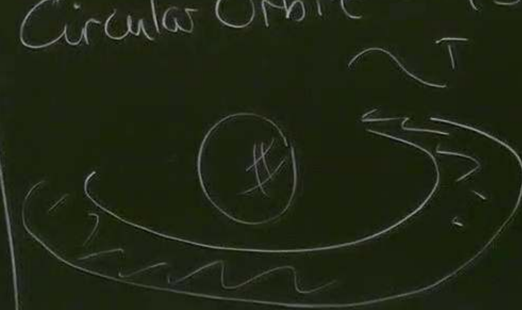
At $h^2 = 12G^2M^2$ roots coincide

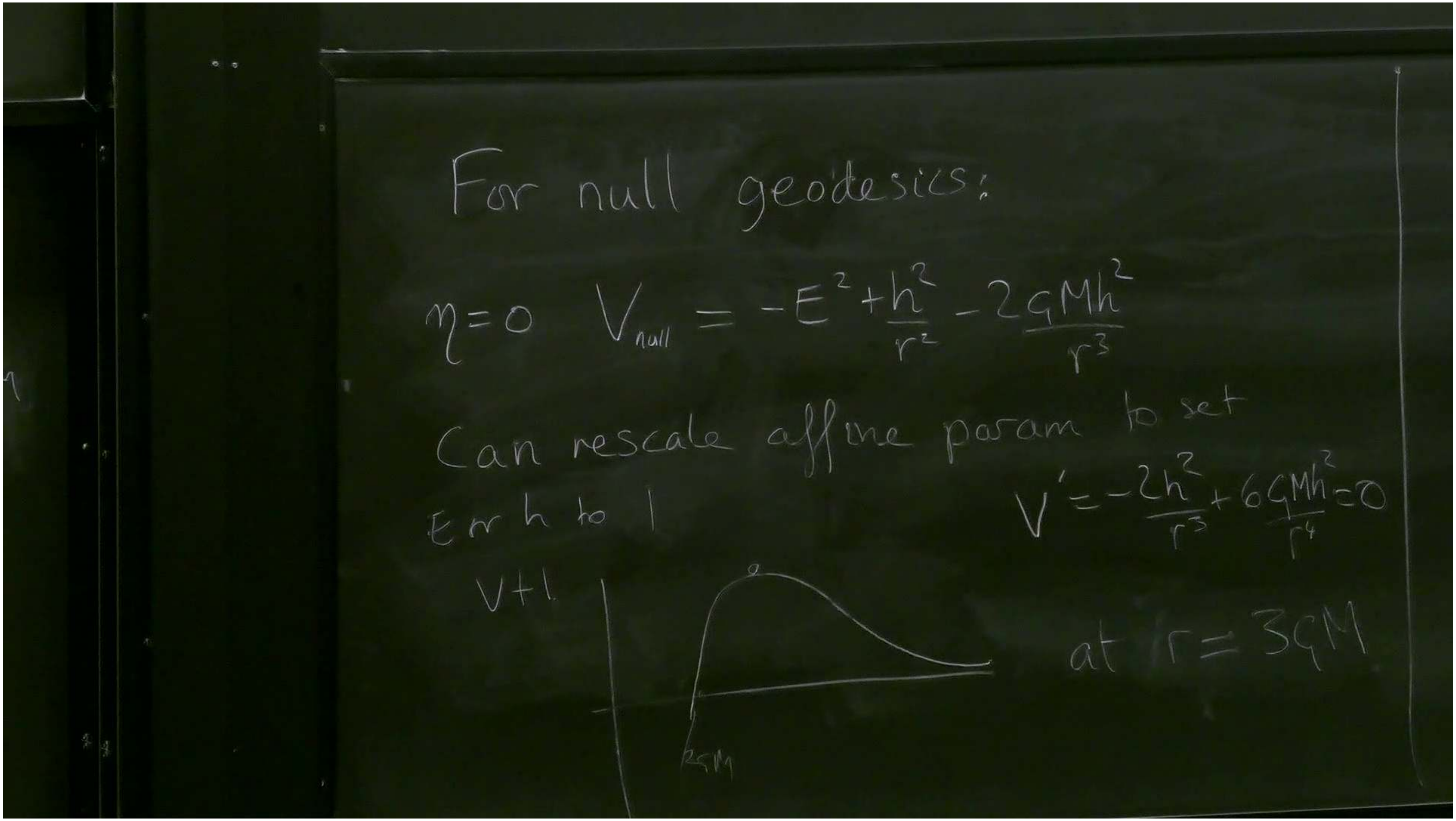
marginally stable

$$r = \frac{h^2}{2GM} = 6GM$$

stable bound orbits not possible $r<6GM$

This is the Innermost Stable Circular Orbit - ISCO




For null geodesics:
$\eta = 0 \quad V_{null} = -E^2 + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}$
Can rescale affine param to set
E or h to 1
$V' = -\frac{2h^2}{r^3} + \frac{6GMh^2}{r^4} = 0$
V+1
at r = 3GM
2GM

$\exists$ circular orbit for light,
but unstable – LIGHT RING

Finally set $u = 1/r$ $\quad \frac{du}{d\phi} = \frac{\dot{u}}{\dot{\phi}} = -\frac{\dot{r}}{h}$

$$\rightarrow \left(\frac{du}{d\phi}\right)^2 = \frac{\dot{r}^2}{h^2} = -\frac{(\eta - E^2)}{h^2} + 2GM\eta h^2 u - u^2 + 2GMu^3$$

$$u'^2 + u^2 + \frac{\eta - E^2}{h^2} = 2GM\eta h^2 u + 2GMu^3$$

$-2GMh^2 / r^3$

param to set

$$V' = -\frac{2h^2}{r^3} + \frac{6GMh^2}{r^4} = 0$$

at $r = 3GM$

with

Mu^3

without u^3 – get conic sections

For a scattering orbit

$bu = \cos\varphi$ (as $u \to 0$)

$u' = -\sin\varphi / b.$

As $u \to 0$ $\quad u'^2 = -\frac{(\eta - E^2)}{h^2} = \frac{1}{b^2}$

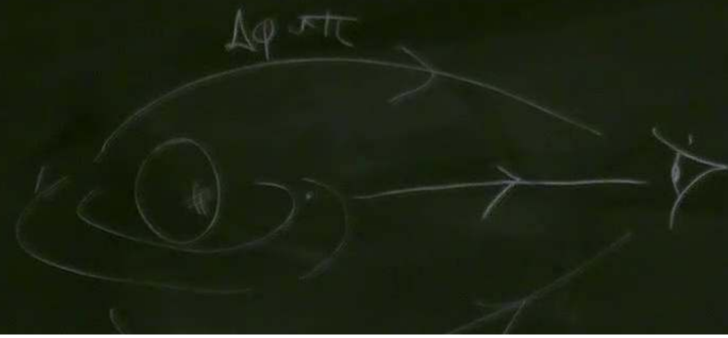
$x = r\cos\varphi$

For light $\quad b = h \quad (E = 1)$

$$\Rightarrow \quad u'^2 = \frac{1}{b^2} - u^2 + 2GMu^3 = g(u)$$

deflection angle eqn

$$\int_{u_0}^{u_1} \frac{du}{\sqrt{g(u)}} = \Delta\varphi \Big|_{u_0}^{u_1}$$



N

Note at $r=3GM$, $\frac{du}{d\varphi}=0$

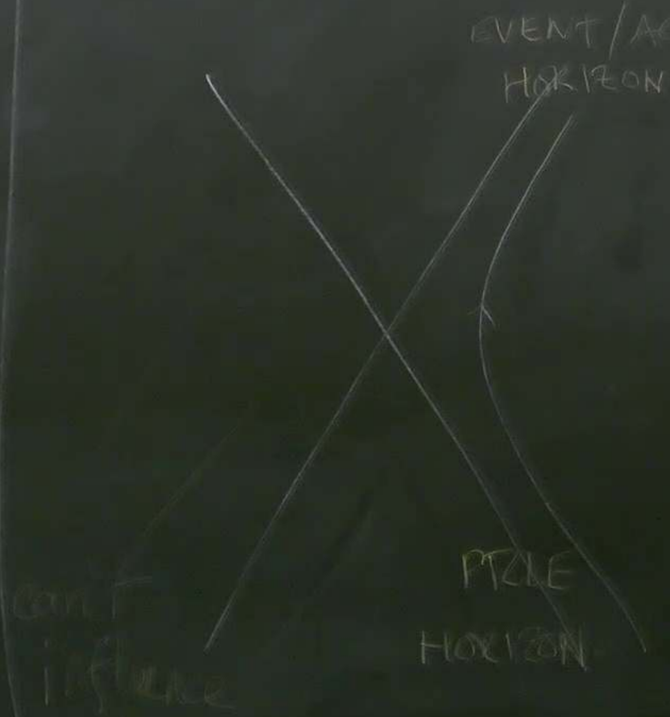
$$\Leftrightarrow \frac{1}{b^2} - u^2 + 2GMu^3 = 0$$

$$b^2 = 27G^2M^2$$

If $b < 3\sqrt{3}GM$, light is captured.

Acceleration in Geometry

An accelerating observer is represented as a hyperbola



$$X^\mu = \left(\frac{1}{A}\sinh A\tau, \frac{1}{A}\cosh A\tau, \underline{0}\right)$$

$$\dot{X}^\mu = \left(\cosh A\tau, \sinh A\tau, \underline{0}\right)$$

$$\ddot{X}^\mu = A^2 X^\mu \qquad \|\ddot{X}^\mu\| = A$$

$$\|\dot{X}^\mu\| = 1 \rightarrow \tau \text{ is proper time}$$

Observer only sees $x < T$ &

influences $x > -T$

Let $$T = \frac{\sqrt{1 - A^2\rho^2}}{A(1 + A\rho\cos\theta)} \sinh A\tau$$

$$X = \frac{\sqrt{1 - A^2\rho^2}}{A(1 + A\rho\cos\theta)} \cosh A\tau$$

$$Y + iZ = \frac{\rho \sin\theta}{(1 + A\rho\cos\theta)} e^{i\varphi}$$

$\rho = 0$ is worldline of acc obs. "DS STATIC"

$$\text{EX:} \quad ds^2 = \frac{1}{(1 + A\rho\cos\theta)^2}\left[(1 - A^2\rho^2)\,d\tau^2 - \frac{d\rho^2}{(1 - A^2\rho^2)} - \rho^2 d\Omega_{II}^2\right]$$

$\uparrow \Omega$

RINDLER

Can be generalised to include M & Λ

$$ds^2 = \frac{1}{\Omega}\left[f\,d\tau^2 - \frac{d\rho^2}{f} - \rho^2\left(\frac{d\theta^2}{g} + g\sin^2\theta\,\frac{d\varphi^2}{K^2}\right)\right]$$

$$f = (1 - A^2 r^2)\left(1 - \frac{2GM}{\rho}\right) - \frac{\Lambda\rho^2}{3} \qquad g = 1 + 2GMA\cos\theta$$

$A\tau$

$\cosh A\tau$

φ

b.h horizon near 2GM.

& acc horizon near $1/A$ or $\left(A^2 + \frac{\Lambda}{3}\right)^{-1/2}$

Can $\exists$ no acc horizon in AdS "SLOW ACC"

$$\theta = \delta\theta \qquad ds^2_{\theta,\varphi} \propto d\theta^2 + (1+2GMA)^2 \frac{\delta\theta^2}{K^2} d\varphi^2$$

$$\theta = \pi - \delta\theta \qquad ds^2_{\theta\varphi} \propto d\theta^2 + (1-2GMA)^2 \frac{\delta\theta^2}{K^2} d\varphi^2$$

$]$

Λ

$\theta \frac{d\varphi^2}{K^2} \Big) \Big]$

$-2GMA\cos\theta.$

Cannot make both poles regular if $MA \neq 0$.

If $K = 1 + 2qMA \rightarrow$ NP regular

SP. $d\theta^2 + \delta\theta^2 \Big(\underbrace{d\varphi \frac{(1-2qMA)}{1+2qMA}}_{\psi}\Big)^2$

ψ has periodicity $\frac{1-2qMA}{1+2qMA} \cdot 2\pi < 2\pi$ CONICAL DEFICIT