Title: Lecture - Gravitational Physics (Elective), PHYS 636

Speakers: Ruth Gregory

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LECTURE 6 The Gravitational Action

Almost always use an action principle to "back up" our eqns of motion

$$S \sim \int \underbrace{d^4x}_{\text{COORD VOLUME}} \mathcal{L}$$

$$d^4x' \to \left|\frac{\partial x'}{\partial x}\right| d^4x$$

use $\sqrt{-g}\,d^4x$ for covariance

– proper volume

Need to vary $\sqrt{-g}$ Use

$$\det M = \exp \operatorname{tr} \log M$$

$$\Rightarrow \delta \det g = (\det g)\, \delta(\operatorname{tr} \log g)$$

$$= (\det g)\, g^{ab}\, \delta g_{ab}$$

$$= -(\det g)\, g_{ab}\, \delta g^{-1\,ab}$$

ce

Use R to construct a grav^l Lagrangian.

$\log g)$

ab.

$g^{-1\,ab}$

Lagrangian.

$$\delta R = \delta(R_{\mu\nu}g^{\mu\nu})$$
$$= \delta R_{\mu\nu}g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}$$

To get $\delta R_{\mu\nu}$, use normal coords, a chart s.t. $\Gamma^{\mu}_{\nu\lambda} = 0$ at P (though $\Gamma^{\mu}_{\nu\lambda,\sigma} \neq 0$)

Since $g_{\mu\nu,\sigma} = 0$ at P, $\partial = \nabla$ AT P

Have $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \underset{NC}{\equiv} \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu}$ "

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu}$$

$$\rightarrow \nabla_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\lambda}_{\mu\lambda} \quad \text{AT P}$$

this is covariant ∴ true in all frames

(Palatini Lemma)

Use same method to derive

$$\delta \Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(\nabla_\lambda \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\lambda} - \nabla_\sigma \delta g_{\nu\lambda}\right)$$

$$= -\frac{1}{2}\left(\nabla_\lambda \delta g^{-1\,\mu}{}_{\nu} + \nabla_\nu \delta g^{-1\,\mu}{}_{\lambda} - \nabla^\mu \delta g^{\sigma\gamma} g_{\sigma\nu} g_{\gamma\lambda}\right)$$

$$\Rightarrow \quad g^{\mu\nu}\delta R_{\mu\nu} = \Box \delta g^{-1\,\lambda}{}_{\lambda} - \nabla_\mu \nabla_\nu \delta g^{-1\,\mu\nu}.$$

$$= \frac{1}{\sqrt{-g}}\, \partial_\mu \left\{ \sqrt{-g}\left(\nabla^\mu \delta g^{-1\,\lambda}{}_{\lambda} - \nabla_\nu \delta g^{\mu\nu}\right)\right\}$$

TOTAL DERIV.

Thus

$$\delta \int_{\mathcal{M}} d^4x \sqrt{-g}\, R = \int d^4x \sqrt{-g}\, \delta g^{ab} \left(R_{ab} - \frac{1}{2} R g_{ab} \right)$$

$$+ \int_{\partial \mathcal{M}} d^3x \sqrt{h}\, n_a \left(\nabla^a \delta g^{\ b}_{b} - \nabla_b \delta g^{ab} \right)$$

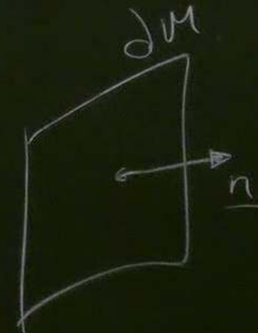
(n outward $\perp$ to $\mathcal{M}$)

Compare to SM $\mathcal{L}$ to get

Einstein Hilbert $\quad -\frac{1}{16\pi G} \int \sqrt{-g}\, R\, d^4x$

The boundary term?

To simplify analysis use Gaussian Normal gauge



$\nabla_n n = 0$

$n = \frac{\partial}{\partial z}$ $\quad n^z = 1, \; n_z = -1$

$ds^2 = h_{\mu\nu} dx^\mu dx^\nu - dz^2$

Chart always defined within radius of curvature of ∂M.

Thus $\delta g_{ab} \leftrightarrow \delta h_{\mu\nu}$

$\delta g_{az} = 0$ at ∂M

Now consider extrinsic curv.

$$K_{ab} = \nabla_a n_b = \Gamma^z_{ab} = g_{ab,z} \text{ or } h'_{ab}$$

$$\Rightarrow \quad \delta K_{ab} = \delta \Gamma^z_{ab} = -\frac{1}{2} n^c \left(\nabla_a \delta g_{bc} + \nabla_b \delta g_{ac} - \nabla_c \delta g_{ab} \right)$$

$$= \frac{1}{2} \left(\delta g_{bc} \nabla_a n^c + \delta g_{ac} \nabla_b n^c - \nabla_n \delta g_{ab} \right)$$

$$= \frac{1}{2} \delta g_{bc} K_a^c + \cdots$$

ge

0

efined

vature

$= - K$

n_{ab}

$+\nabla_b \delta g_{ac}$

$-\nabla_c \delta g_{ab})$

$\nabla_b n^c + \nabla_n \delta g_{ab})$

$$= -K_{c(a}\,\delta g^{-1c}{}_{b)} - \frac{1}{2}\nabla_n(\delta g^{-1})_{ab}$$

$$\Rightarrow h^{ab}\,\delta K_{ab} = -K_{ac}\,\delta g^{-1ac} - \frac{1}{2}\nabla_n \delta g^{-1c}{}_{c}$$

$$\Rightarrow \delta K = -\frac{1}{2}\nabla_n \delta g^{-1c}{}_{c}$$

Next, $n^c \delta g_{bc} = 0 \Rightarrow$

$$n_a \nabla_b \delta g^{-1ab} = -(\delta g^{-1ab})\,\nabla_b n_a$$

$$= -\delta g^{-1ab} K_{ab}$$

Hence $\nabla_n \delta g^{-1c}{}_c - n_a \nabla_b \delta g^{-1ab}$

$$= -2\delta(K\sqrt{-h}) - K h_{ab} \delta h^{ab}$$
$$+ K_{ab} \delta h^{ab}.$$

Therefore, if we add a boundary term

$$\frac{1}{8\pi G} \int d^3x \sqrt{h}\, K$$ to our action,

our variation is now appropriate.

INWARD

$$\delta\left[-\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R + \frac{1}{8\pi G}\int K\sqrt{h}\,d^3x\right]$$

EINSTEIN-HILBERT GIBBONS-HAWKING-YORK

$$= -\frac{1}{16\pi G}\int_{M} d^4x\sqrt{g}\,G_{ab}\,\delta g^{ab}$$

$$+\frac{1}{16\pi G}\int_{\partial M} d^3x\sqrt{h}\,(K_{ab}-K h_{ab})\,\delta h^{ab}$$

$\sqrt{h}\, d^3x \big]$

...KING–YORK

$(K h_{ab})\delta h^{ab}$

Application – Action of Schwarzschild

There are many vac solns in GR that are not distinguished by Einstein-Hilbert alone

Take SCH soln & analytically continue

$$ds_i^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_{II}^2$$

– GH : Euclidean metric should be nonsingular

As $r \to 2GM$ $\quad (\tau, r)$

$$ds^2 \sim \frac{(r-2GM)}{2GM} d\tau^2 + \frac{2GM\, dr^2}{r-2GM}$$

Try $\rho^2 = \lambda (r - 2GM)$

$$\to \quad 2\rho\, d\rho = \lambda\, dr.$$

$$\Rightarrow \frac{2GM\, dr^2}{r-2GM} = \frac{2GM}{\cancel{\rho^2}/\lambda} \frac{4\cancel{\rho^2}\, d\rho^2}{\lambda^{2}} = \frac{8GM}{\lambda} d\rho^2$$

Set $\lambda = 8$

$ds^2_{\tau r} \sim$

(τ, r)

$$\ldots M)\, d\tau^2 + \frac{2GM\, dr^2}{r-2GM}$$

$(r-2GM)$

$\lambda\, dr.$

$$= \frac{2GM}{\rho^2/\lambda} \frac{4\rho^2\, d\rho^2}{\lambda^2} = \frac{8GM}{\lambda} d\rho^2$$

Set $\lambda = 8GM$, then

$$ds^2_{\tau r} \sim d\rho^2 + \frac{\rho^2}{16G^2M^2} d\tau^2$$

This is the origin in polars

if $\theta = \frac{\tau}{4GM}$ i.e. τ is periodic

with periodicity $\Delta\tau = \beta = 8\pi GM$.

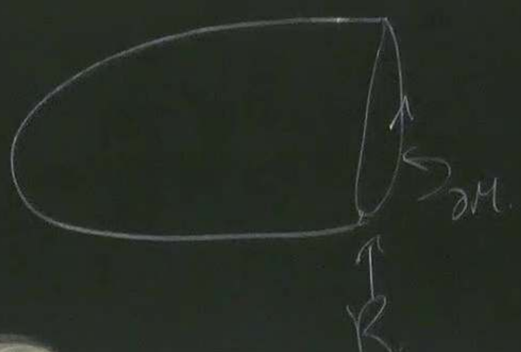
BLACK HOLE CIGAR

Characteristic of a field theory at finite temp

$$T = \frac{1}{\beta} = \frac{1}{8\pi G M} \longrightarrow \frac{\hbar c^3}{8\pi G M k_B}$$

USUAL WAY TO CALCULATE T_H

In computing action, only boundary term contributes



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$\mathbb{R}^3 \times S^1$

$$ds^2 = \left(1 - \frac{2GM}{R}\right) d\tau^2 + dr^2 + r^2 d\Omega_2^2$$

$R^2 d\Omega_{II}^2$

$\sqrt{1-\frac{2GM}{R}}\ \frac{\partial}{\partial r}$

n^r

$\sqrt{1-\frac{2GM}{R}}$

$$\int \sqrt{h}\, d\tau\, d\theta\, d\varphi\, K = -4\pi\beta\left[GM + 2(R-2GM)\right]$$

$$= 4\pi\beta\left[2R - 3GM\right]$$

Divergent!

For flat space, $K_0 = -\frac{2}{R}$,

$$\sqrt{h} = \left(1-\frac{2GM}{R}\right)^{1/2} R^2 \sin\theta .$$

$$\int K_0 \sqrt{h_0}\, d^3x = -4\pi\beta \cdot 2R\sqrt{1-\frac{2GM}{R}} = -4\pi\beta\left[2R - 2GM + O\left(\frac{1}{R}\right)\right]$$

Hence $I - I_0 = \frac{1}{8\pi G}\int \sqrt{h}\, d^3x\, (K - K_0)$

$$= -\frac{1}{8\pi G} \cdot 4\pi\beta\left[2R - 3GM - (2R - 2GM)\right]$$

$$= \frac{\beta M}{2} \rightarrow 4\pi G M^2$$