Title: Lecture - Gravitational Physics (Elective), PHYS 636

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Subject: Cosmology, Strong Gravity

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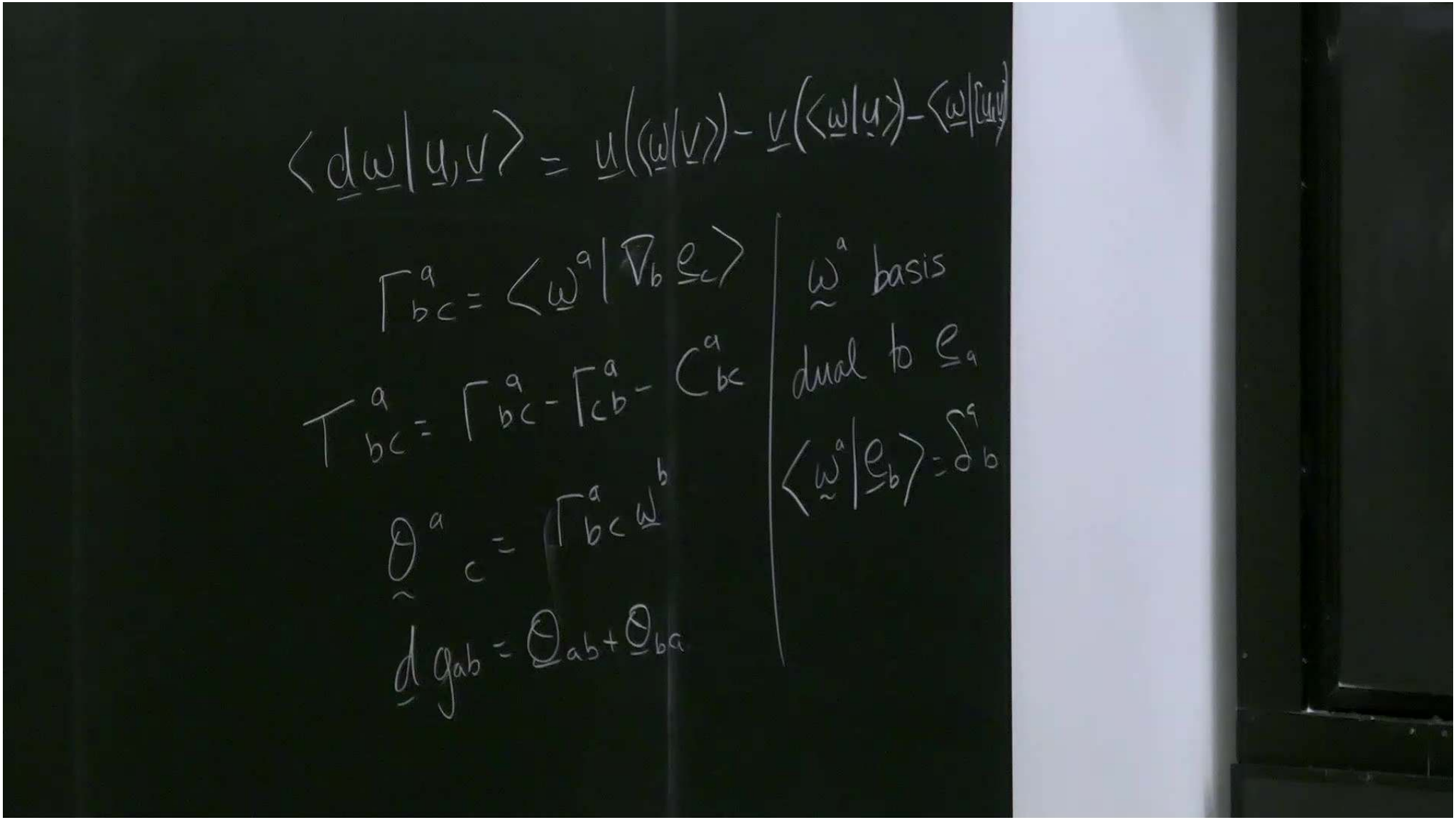

$\langle d\underline{\omega} | \underline{u}, \underline{v} \rangle = \underline{u}(\langle \underline{\omega} | \underline{v} \rangle) - \underline{v}(\langle \underline{\omega} | \underline{u} \rangle) - \langle \underline{\omega} | [\underline{u}, \underline{v}] \rangle$
$\Gamma^a_{bc} = \langle \underline{\omega}^a | \nabla_b \underline{e}_c \rangle$
$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$
$\underset{\sim}{\theta}{}^a{}_c = \Gamma^a_{bc} \underline{\omega}^b$
$\underline{d}\, g_{ab} = \underline{\theta}_{ab} + \underline{\theta}_{ba}$
$\underset{\sim}{\omega}^a$ basis
dual to $\underline{e}_a$
$\langle \underset{\sim}{\omega}^a | \underline{e}_b \rangle = \delta^a_b$

Lecture 4: C^3: Connections, Curvature & Cartan

$$\underline{d} g_{ab} = \underline{\Theta}_{ab} + \underline{\Theta}_{ba}$$

$g_{ab} = \langle \underline{g} | \underline{e}_a, \underline{e}_b \rangle$ is scalar

$$\underline{d}(g_{ab}) = \underline{\nabla}(g_{ab}) = \underline{\nabla}\left(\langle \underline{g} | \underline{e}_a, \underline{e}_b \rangle\right)$$

$$= \langle \nabla \underline{g} | \underline{e}_a, \underline{e}_b \rangle + \langle \underline{g} | \nabla \underline{e}_a, \underline{e}_b \rangle + \langle \underline{g} | \underline{e}_a, \nabla \underline{e}_b \rangle$$

metric ∇

$$= \underset{\sim}{\theta}^{c}{}_{a} \langle g | \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle + \underset{\sim}{\theta}^{c}{}_{b} \langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_c \rangle$$

$$= \underset{\sim}{\theta}^{c}{}_{a}\, g_{cb} + \underset{\sim}{\theta}^{c}{}_{b}\, g_{ac}$$

$$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$$

So if $\underset{\sim}{e}_a$ is an o/n basis, ie $g_{ab} = \eta_{ab}$, then

$\underset{\sim}{d} g_{ab} = 0 \Rightarrow \underset{\sim}{\theta}_{ab}$ is antisymmetric.

Consider $\underline{\Theta}^{a}{}_{c} \wedge \underline{\omega}^{c}$

$$= \Gamma^{a}_{[bc]}\, \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \frac{1}{2}\left(T^{a}_{bc} + C^{a}_{bc}\right) \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \underline{\underline{I}} + \frac{1}{2} \langle \underline{\omega}^{a} | [\underline{e}_{b}, \underline{e}_{c}] \rangle\, \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

But $\langle \underline{\omega}^a \,|\, [\underline{e}_b, \underline{e}_c] \rangle = \underline{e}_b \big(\langle \underline{\omega}^a \,|\, \underline{e}_c \rangle \big) - \underline{e}_c \big(\langle \underline{\omega}^a \,|\, \underline{e}_b \rangle \big) - \langle \underline{d\omega}^a \,|\, \underline{e}_b, \underline{e}_c \rangle$

($\langle \underline{\omega}^a \,|\, \underline{e}_c \rangle = \delta^a_c$, $\langle \underline{\omega}^a \,|\, \underline{e}_b \rangle = \delta^a_b$)

$$= - \langle \underline{d\omega}^a \,|\, \underline{e}_b, \underline{e}_c \rangle$$

Hence

$$\underline{\underline{T}}^a = \underline{\Theta}^a{}_c \wedge \underline{\omega}^c + \frac{1}{2} \langle \underline{d\omega}^a \,|\, \underline{e}_b, \underline{e}_c \rangle \, \underline{\omega}^b \wedge \underline{\omega}^c$$

$$= \underline{\Theta}^a{}_c \wedge \underline{\omega}^c + \underline{d\omega}^a$$ - Cartan's 1st Structural eqn.

Curvature

$$R(\underline{u},\underline{v})\,\underline{W} = \left(\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right)\underline{W}$$

$$T_p(M) \times T_p(M) \times T_p(M) \longrightarrow T_p(M)$$

– a commutator of derivatives.

$- \nabla_{[\mu,\nu]}) \underline{W}$

$\longrightarrow T_p(M)$

c f (non ab) gauge theories

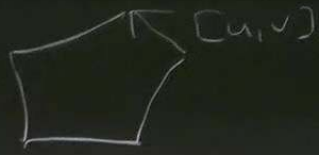
$$D_\mu \underline{\Phi} = (\partial_\mu + ie A_\mu) \underline{\Phi}$$

(c.f $\nabla_a V^b = \partial_a V^b + \Gamma_a{}^b{}_c V^c$)

Field strength is $\underline{F} = \underline{d}\underline{A}$ for e-m

$$[D_\mu, D_\nu] \underline{\Phi} = F_{\mu\nu} \underline{\Phi} \quad (em)$$

Curvature



$$R(\underline{u},\underline{v})\,\underline{W} = \left(\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right)\underline{W}$$

$$T_p(M) \times T_p(M) \times T_p(M) \longrightarrow T_p(M)$$

– a commutator of derivatives.

For non-abelian

$$[D_\mu, D_\nu] \to (\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu])$$

In components,

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$$

Defining the curvature 2-forms as

$$\underline{R}^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd}\, \underline{\omega}^{c} \wedge \omega^{d}$$

Cartan's 2nd structural eqn is

$$\underline{R}^{a}{}_{b} = \underline{d}\,\underline{\theta}^{a}{}_{b} + \underline{\theta}^{a}{}_{c} \wedge \underline{\theta}^{c}{}_{b}$$

Example: Sph. symm, static, 4D.

$SO(3)$ / Kv.

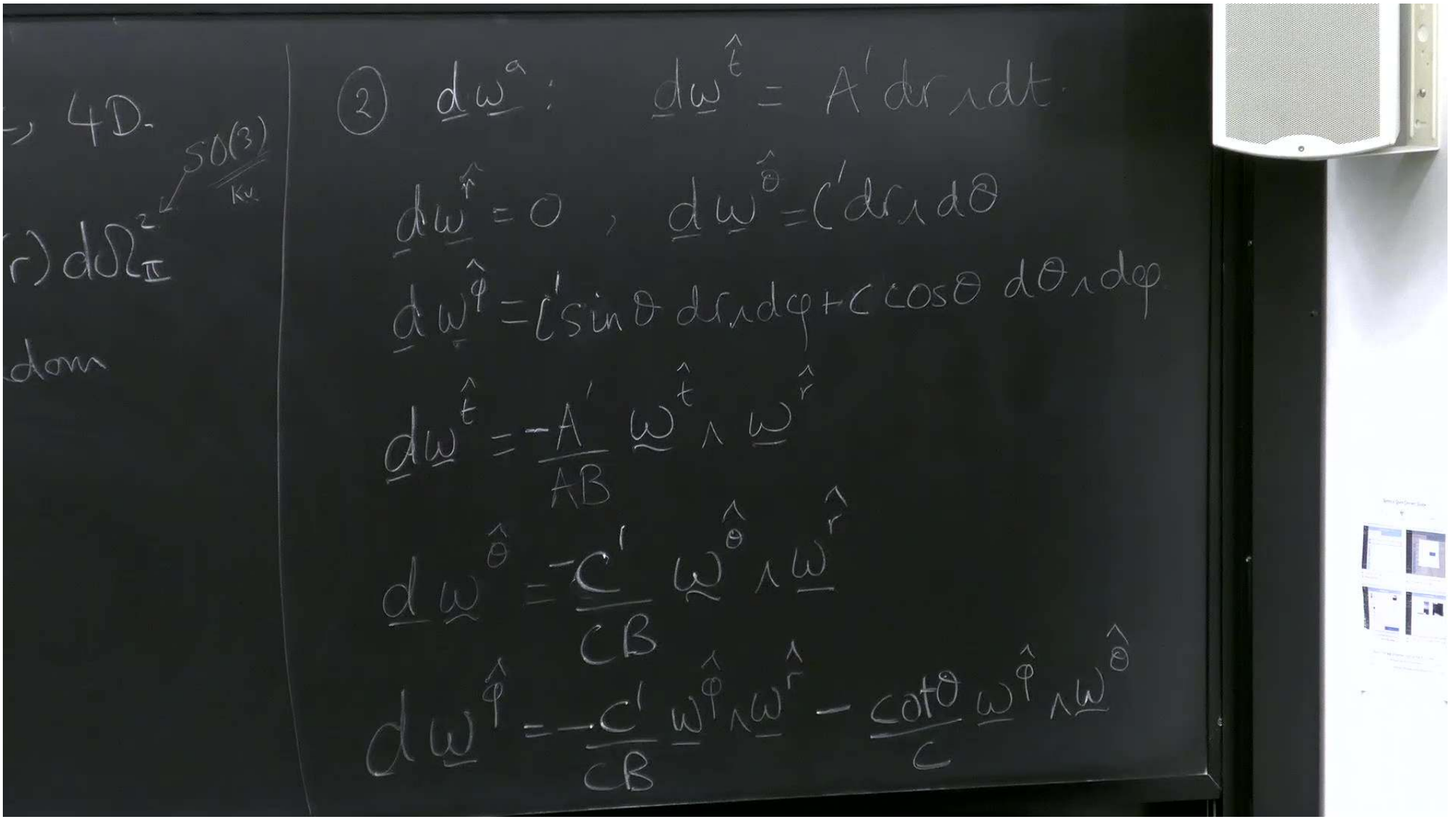
$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r)d\Omega_{II}^2$$

Note, still a gauge degree of freedom

① Identify o/n basis:

$\omega^{\hat{t}} = A\,dt$ $\quad \omega^{\hat{\theta}} = r\,d\theta$

$\omega^{\hat{r}} = B\,dr$ $\quad \omega^{\hat{\varphi}} = r\sin\theta\,d\varphi$



$\to$ 4D.

$SO(3)$ / Kv.

$r)\, d\Omega^2_{II}$

dom

② $\underline{d}\underline{\omega}^a$: $\quad \underline{d}\underline{\omega}^{\hat{t}} = A'\, dr \wedge dt$

$$\underline{d}\underline{\omega}^{\hat{r}} = 0 \,, \quad \underline{d}\underline{\omega}^{\hat{\theta}} = C'\, dr \wedge d\theta$$

$$\underline{d}\underline{\omega}^{\hat{\varphi}} = C' \sin\theta\, dr \wedge d\varphi + C \cos\theta\, d\theta \wedge d\varphi$$

$$\underline{d}\underline{\omega}^{\hat{t}} = -\frac{A'}{AB}\, \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$$

$$\underline{d}\underline{\omega}^{\hat{\theta}} = -\frac{C'}{CB}\, \underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{r}}$$

$$\underline{d}\underline{\omega}^{\hat{\varphi}} = -\frac{C'}{CB}\, \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{r}} - \frac{\cot\theta}{C}\, \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}$$

tc, 4D.

$SO(3)$ / Kv.

$(r)\, d\Omega_{II}^2$

eedom

② $\underline{d\omega}^a$: $\quad \underline{d\omega}^{\hat t} = A'\, dr \wedge dt$

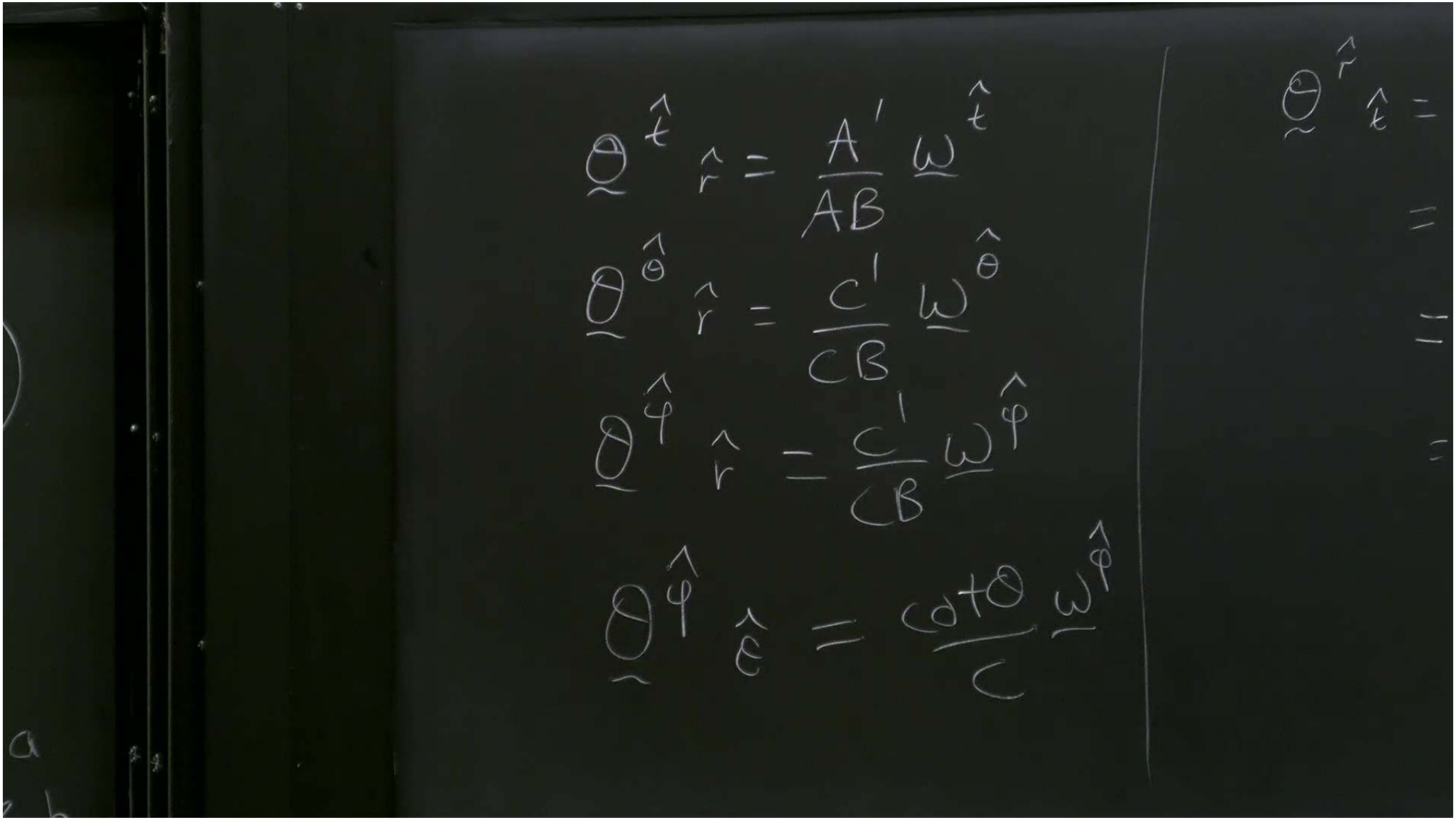
$\underline{d\omega}^{\hat r} = 0 \,, \quad \underline{d\omega}^{\hat\theta} = C'\, dr \wedge d\theta$

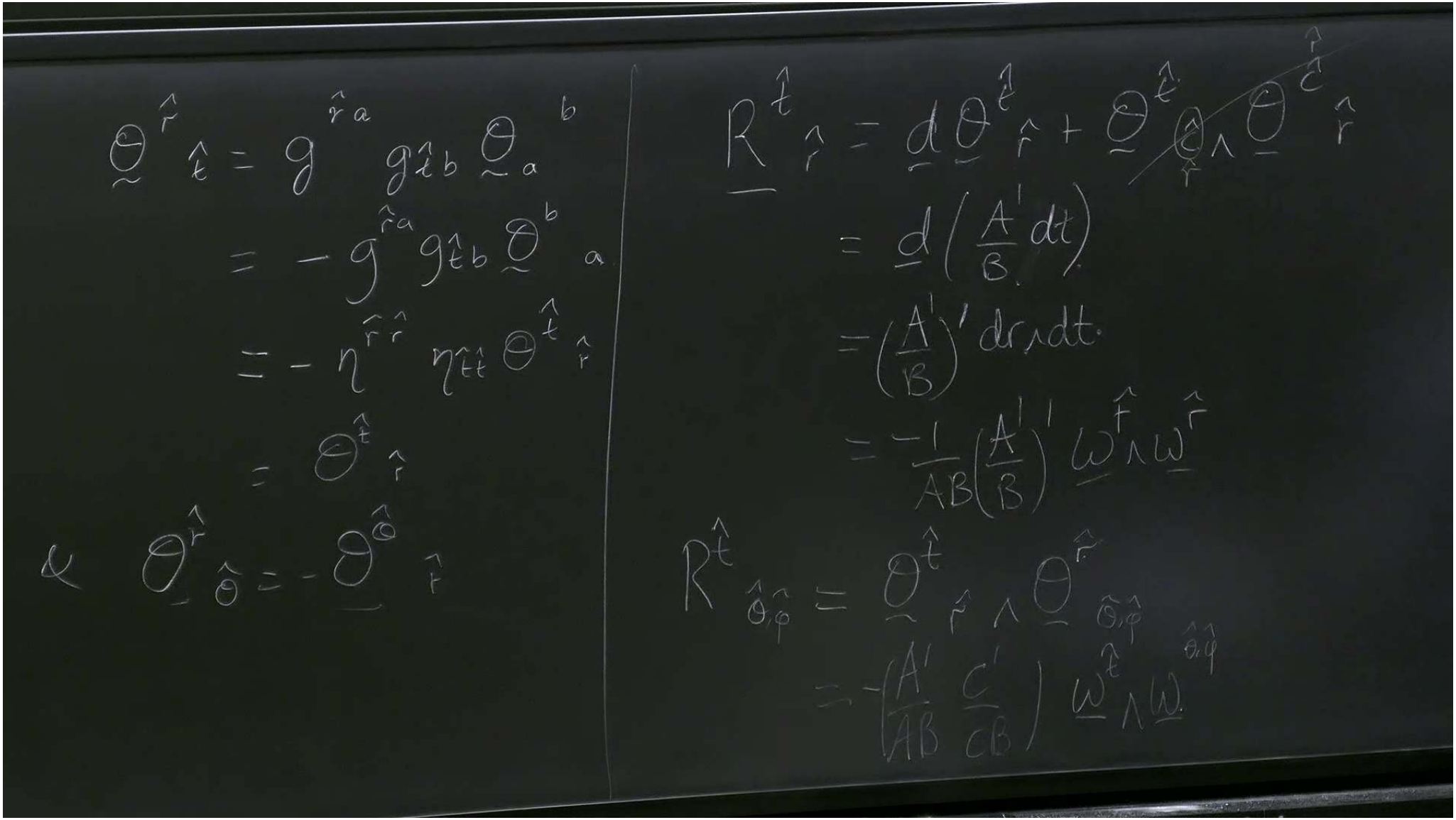
$\underline{d\omega}^{\hat\varphi} = C' \sin\theta\, dr \wedge d\varphi + C \cos\theta\, d\theta \wedge d\varphi$

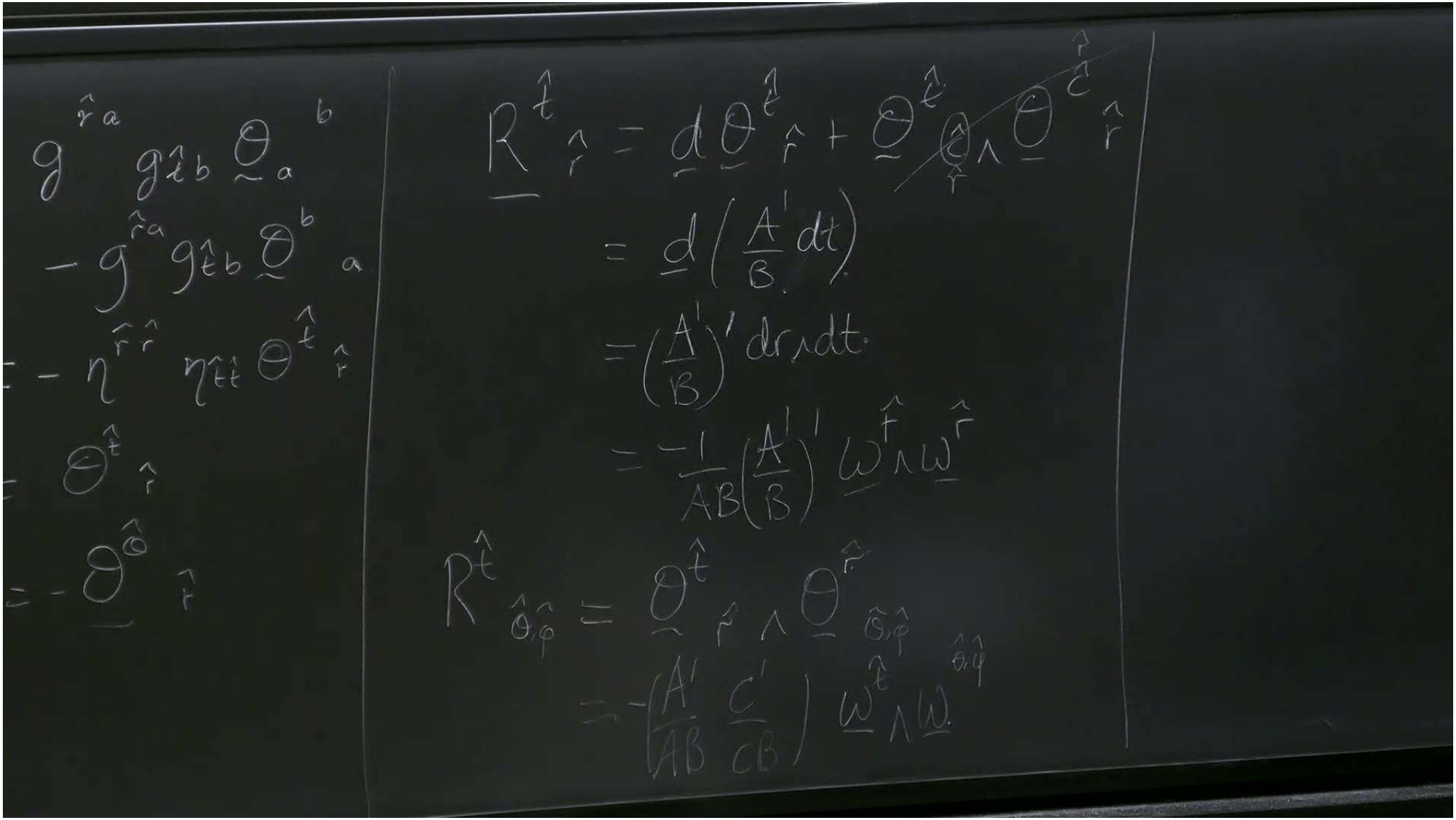
$$\underline{d\omega}^{\hat t} = -\frac{A'}{AB}\, \underline{\omega}^{\hat t} \wedge \underline{\omega}^{\hat r} = -\underline{\theta}^{\hat t}{}_{\hat r} \wedge \underline{\omega}^{\hat r}{}_{\hat t}$$

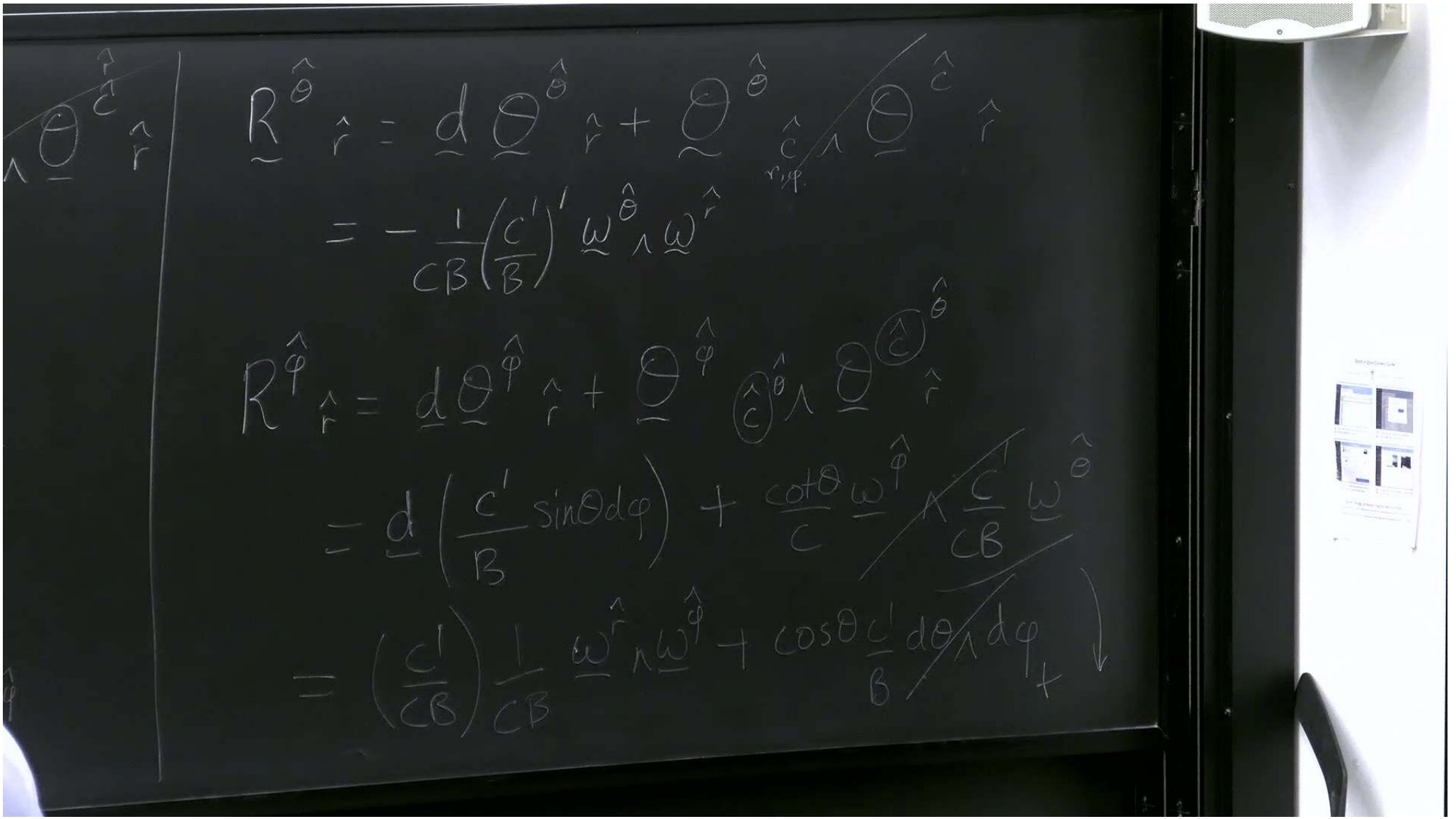
$$\underline{d\omega}^{\hat\theta} = -\frac{C'}{CB}\, \underline{\omega}^{\hat\theta} \wedge \underline{\omega}^{\hat r} = -\underline{\theta}^{\hat\theta}{}_{\hat r} \wedge \underline{\omega}^{\hat r}{}_{\hat\theta}$$

$$\underline{d\omega}^{\hat\varphi} = -\frac{C'}{CB}\, \underline{\omega}^{\hat\varphi} \wedge \underline{\omega}^{\hat r} - \frac{\cot\theta}{C}\, \underline{\omega}^{\hat\varphi} \wedge \underline{\omega}^{\hat\theta} = -\underline{\theta}^{\hat\varphi}{}_{\hat r} \wedge \underline{\omega}^{\hat r}{}_{\hat\theta}$$


$\Theta^{\hat{t}\,\hat{r}} = \frac{A'}{AB}\omega^{\hat{t}}$
$\Theta^{\hat{\theta}\,\hat{r}} = \frac{C'}{CB}\omega^{\hat{\theta}}$
$\Theta^{\hat{\varphi}\,\hat{r}} = \frac{C'}{CB}\omega^{\hat{\varphi}}$
$\Theta^{\hat{\varphi}\,\hat{\theta}} = \frac{\cot\theta}{C}\omega^{\hat{\varphi}}$
$\Theta^{\hat{r}\,\hat{t}} =$


$\underset{\sim}{\theta}^{\hat{r}}{}_{\hat{t}} = g^{\hat{r}a} g_{\hat{t}b} \underset{\sim}{\theta}^{b}{}_{a}$
$= -g^{\hat{r}a} g_{\hat{t}b} \underset{\sim}{\theta}^{b}{}_{a}$
$= -\eta^{\hat{r}\hat{r}} \eta_{\hat{t}\hat{t}} \theta^{\hat{t}}{}_{\hat{r}}$
$= \theta^{\hat{t}}{}_{\hat{r}}$
$\& \quad \underset{\sim}{\theta}^{\hat{r}}{}_{\hat{\theta}} = -\underset{\sim}{\theta}^{\hat{\theta}}{}_{\hat{r}}$
$\underline{R}^{\hat{t}}{}_{\hat{r}} = \underline{d}\underset{\sim}{\theta}^{\hat{t}}{}_{\hat{r}} + \underset{\sim}{\theta}^{\hat{t}}{}_{\hat{c}} \wedge \underset{\sim}{\theta}^{\hat{c}}{}_{\hat{r}}$
$= \underline{d}\left(\frac{A'}{B} dt\right)$
$= \left(\frac{A'}{B}\right)' dr \wedge dt$
$= \frac{-1}{AB}\left(\frac{A'}{B}\right)' \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$
$R^{\hat{t}}{}_{\hat{\theta},\hat{\varphi}} = \underset{\sim}{\theta}^{\hat{t}}{}_{\hat{r}} \wedge \underset{\sim}{\theta}^{\hat{r}}{}_{\hat{\theta},\hat{\varphi}}$
$= -\left(\frac{A'}{AB} \frac{C'}{CB}\right) \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta},\hat{\varphi}}$


$g^{\hat{r}a}\, g_{\hat{t}b}\, \underset{\sim}{\theta}_a{}^b$
$- g^{\hat{r}a}\, g_{\hat{t}b}\, \underset{\sim}{\theta}^b{}_a$
$- \eta^{\hat{r}\hat{r}}\, \eta_{\hat{t}\hat{t}}\, \theta^{\hat{t}}{}_{\hat{r}}$
$\theta^{\hat{t}}{}_{\hat{r}}$
$-\theta^{\hat{\theta}}{}_{\hat{r}}$
$\underset{\sim}{R}{}^{\hat{t}}{}_{\hat{r}} = \underset{\sim}{d}\,\underset{\sim}{\theta}^{\hat{t}}{}_{\hat{r}} + \underset{\sim}{\theta}^{\hat{t}}{}_{\hat{c}} \wedge \underset{\sim}{\theta}^{\hat{c}}{}_{\hat{r}}$
$= \underset{\sim}{d}\left(\frac{A'}{B}\, dt\right)$
$= \left(\frac{A'}{B}\right)' dr \wedge dt$
$= \frac{-1}{AB}\left(\frac{A'}{B}\right)' \underset{\sim}{\omega}^{\hat{t}} \wedge \underset{\sim}{\omega}^{\hat{r}}$
$R^{\hat{t}}{}_{\hat{\theta}\hat{\varphi}} = \underset{\sim}{\theta}^{\hat{t}}{}_{\hat{r}} \wedge \underset{\sim}{\theta}^{\hat{r}}{}_{\hat{\theta}\hat{\varphi}}$
$= -\left(\frac{A'}{AB}\,\frac{C'}{CB}\right) \underset{\sim}{\omega}^{\hat{t}} \wedge \underset{\sim}{\omega}^{\hat{\theta}\hat{\varphi}}$



$$\wedge \underline{\Theta}^{\hat{c}}{}_{\hat{r}}$$

$$\underset{\sim}{R}^{\hat{\theta}}{}_{\hat{r}} = \underset{\sim}{d}\,\underset{\sim}{\Theta}^{\hat{\theta}}{}_{\hat{r}} + \underset{\sim}{\Theta}^{\hat{\theta}}{}_{\hat{c}} \wedge \underline{\Theta}^{\hat{c}}{}_{\hat{r}}$$

r, φ

$$= -\frac{1}{CB}\left(\frac{C'}{B}\right)' \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{r}}$$

$$R^{\hat{\varphi}}{}_{\hat{r}} = \underline{d}\,\underset{\sim}{\Theta}^{\hat{\varphi}}{}_{\hat{r}} + \underline{\Theta}^{\hat{\varphi}}{}_{\hat{c}}{}^{\hat{\theta}} \wedge \underline{\Theta}^{\hat{c}}{}_{\hat{r}}$$

$$= \underline{d}\left(\frac{C'}{B}\sin\theta\, d\varphi\right) + \frac{\cot\theta}{C}\underline{\omega}^{\hat{\varphi}} \wedge \frac{C'}{CB}\underline{\omega}^{\hat{\theta}}$$

$$= \left(\frac{C'}{CB}\right)\frac{1}{CB}\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\varphi}} + \cos\theta\,\frac{C'}{B}\, d\theta \wedge d\varphi$$

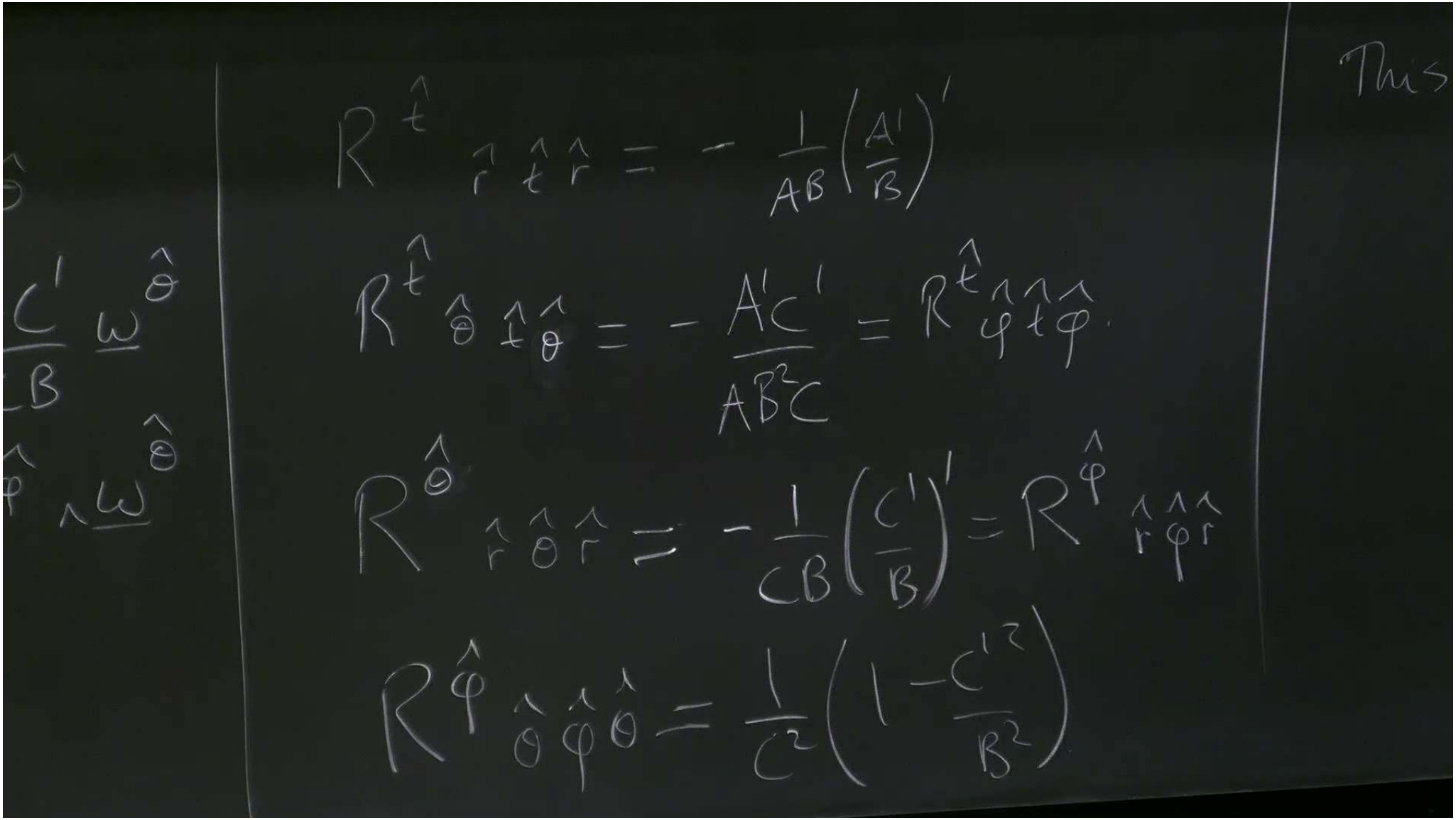
$+$

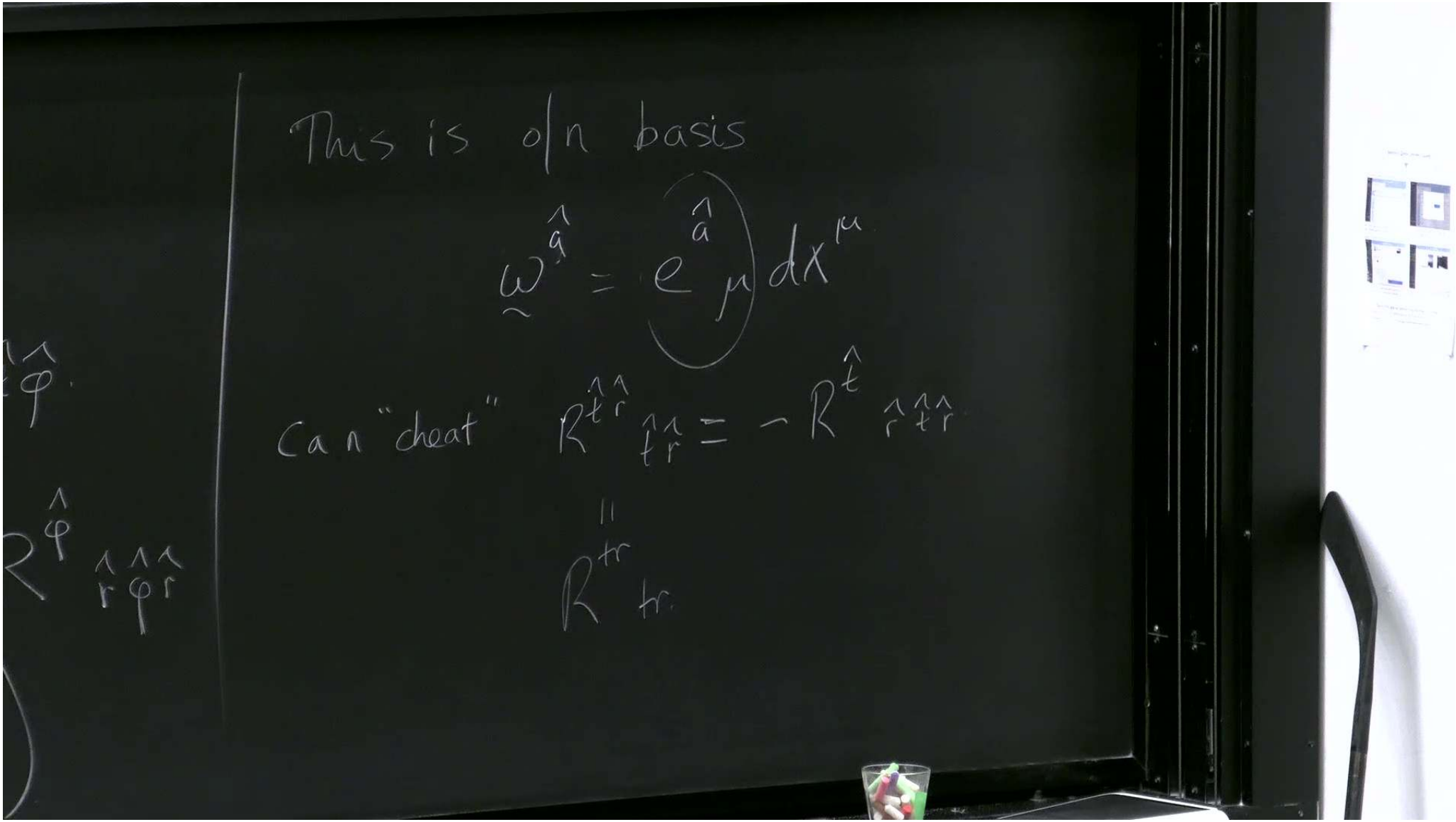
$$\underline{R}^{\hat{\varphi}}{}_{\hat{\theta}} = \underline{d}\,\underline{\Theta}^{\hat{\varphi}}{}_{\hat{\theta}} + \underline{\Theta}^{\hat{\varphi}}{}_{\hat{r}} \wedge \Theta^{\hat{r}}{}_{\hat{\theta}}$$

$$= \underline{d}\left(\frac{\cot\theta}{c}\underline{\omega}^{\varphi}\right) + \frac{c'}{cB}\underline{\omega}^{\hat{\varphi}} \wedge \frac{-c'}{cB}\underline{\omega}^{\hat{\theta}}$$

$$= -\sin\theta\, d\theta \wedge d\varphi - \frac{c'^2}{c^2B^2}\underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}$$

$$= \frac{1}{c^2}\left(1 - \frac{c'^2}{B^2}\right)\underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}$$


$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -\frac{1}{AB}\left(\frac{A'}{B}\right)'$
$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}}$
$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -\frac{1}{CB}\left(\frac{C'}{B}\right)' = R^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}\hat{r}}$
$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{C^2}\left(1-\frac{C'^2}{B^2}\right)$
This


This is o/n basis
$\omega^{\hat{a}} = e^{\hat{a}}{}_{\mu} dx^{\mu}$
Can "cheat" $R^{\hat{t}\hat{r}}{}_{\hat{t}\hat{r}} = -R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}$
$R^{tr}{}_{tr}$