Title: Lecture - Gravitational Physics (Elective), PHYS 636

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Subject: Cosmology, Strong Gravity

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Lecture 2: Lie Derivative

Recall vector is map $I: C^\infty(M) \to C^\infty(M)$

Therefore can act on a fn by more than 1 vector.

$$\underline{u}\,\underline{v}\,f \underset{\text{chart}}{\longleftrightarrow} u^\mu \frac{\partial}{\partial x^\mu}\left[v^\nu \frac{\partial}{\partial x^\nu} f\right]$$

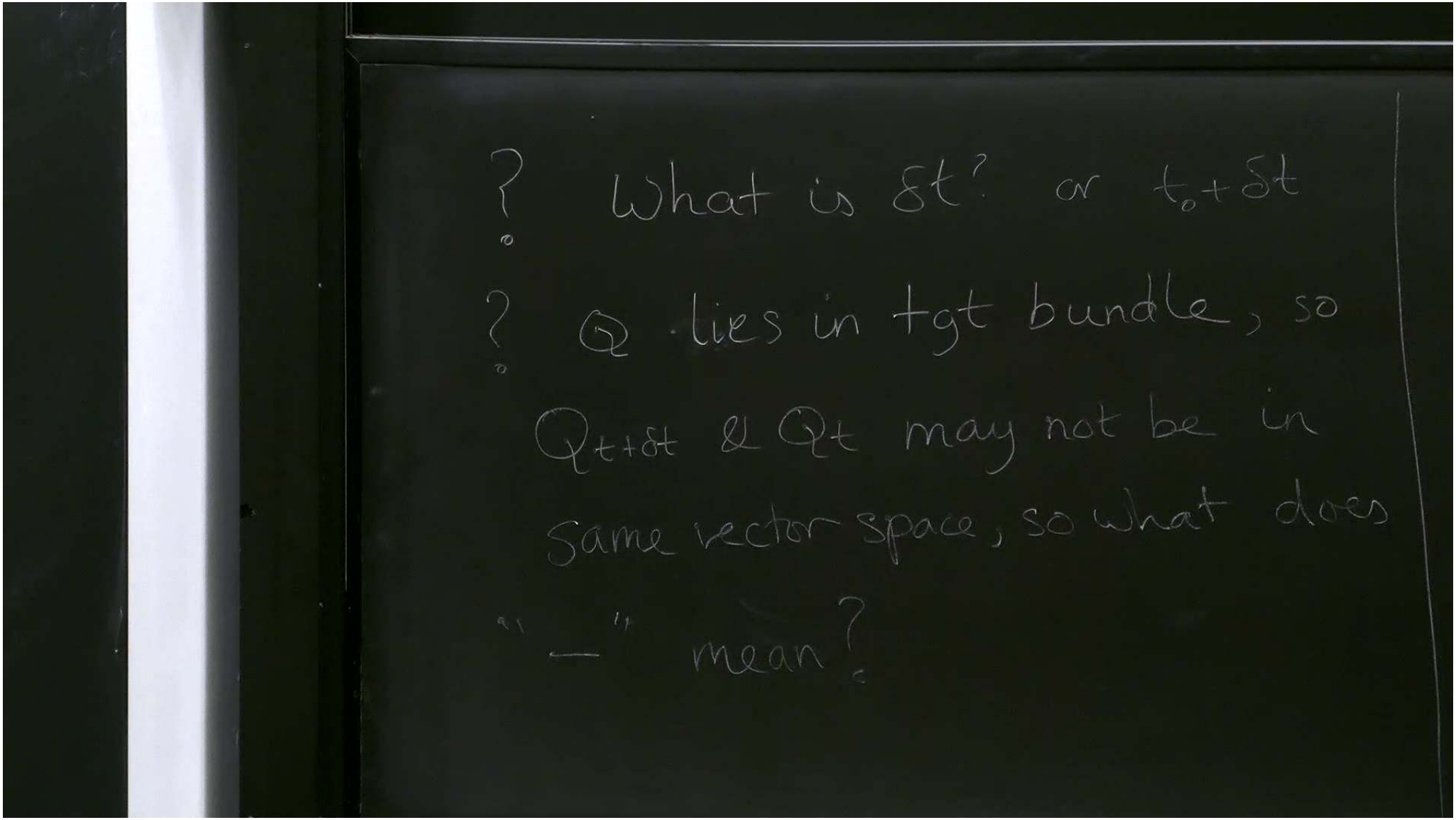
$$\left[u^\mu v^\nu{}_{,\mu} \frac{\partial}{\partial x^\nu}\right] f + u^\mu v^\nu \frac{\partial^2 f}{\partial x^\mu \partial x^\nu}$$

Consider $(\underbrace{\underline{u}\underline{v} - \underline{v}\underline{u}}_{\text{VECTOR}} f) \longleftrightarrow (\underbrace{u^\mu v^\nu{}_{,\mu} - v^\mu u^\nu{}_{,\mu}}_{\text{cpts of vector}}) \underbrace{\frac{\partial}{\partial \bar{x}^\nu}}_{\text{basis vector}} f$

$[\underline{u}, \underline{v}]$ is the Lie bracket of $\underline{u}$ & $\underline{v}$.

Will now generalise to Lie derivative of arb tensor

Recall the derivative "$\left.\frac{dQ}{dt}\right|_{t_0} = \lim_{\delta t \to 0} \frac{Q_{t_0+\delta t} - Q_{t_0}}{\delta t}$.


? What is δt? or $t_0 + \delta t$
? Q lies in tgt bundle, so
$Q_{t+\delta t}$ & Q_t may not be in
same vector space, so what does
"−" mean?

$t_0 + \delta t$

ndle, so

t be in

what does

Expect δt to take Q from some pt P to a neighbouring pt $P + \delta P$. So $Q_{t_0 + \delta t}$ is not in same vector space as Q_{t_0}.

Challenge is to link nearby tgt spaces.

Integral curves

Just as $\underline{T}$ is tangent to γ can construct families of curves to which the field T is tgt.

In a local chart

$$X^\mu(t_0+\delta t) = \underbrace{X^\mu_0(t_0) + \delta t\, T^\mu + \frac{1}{2}\delta t^2 T^\nu T^\mu{}_{,\nu}}_{X'^\mu \leftarrow \text{new coords.}} + \dots$$

An alternate viewpoint is that moving δt along integral curves gives a coord transfm

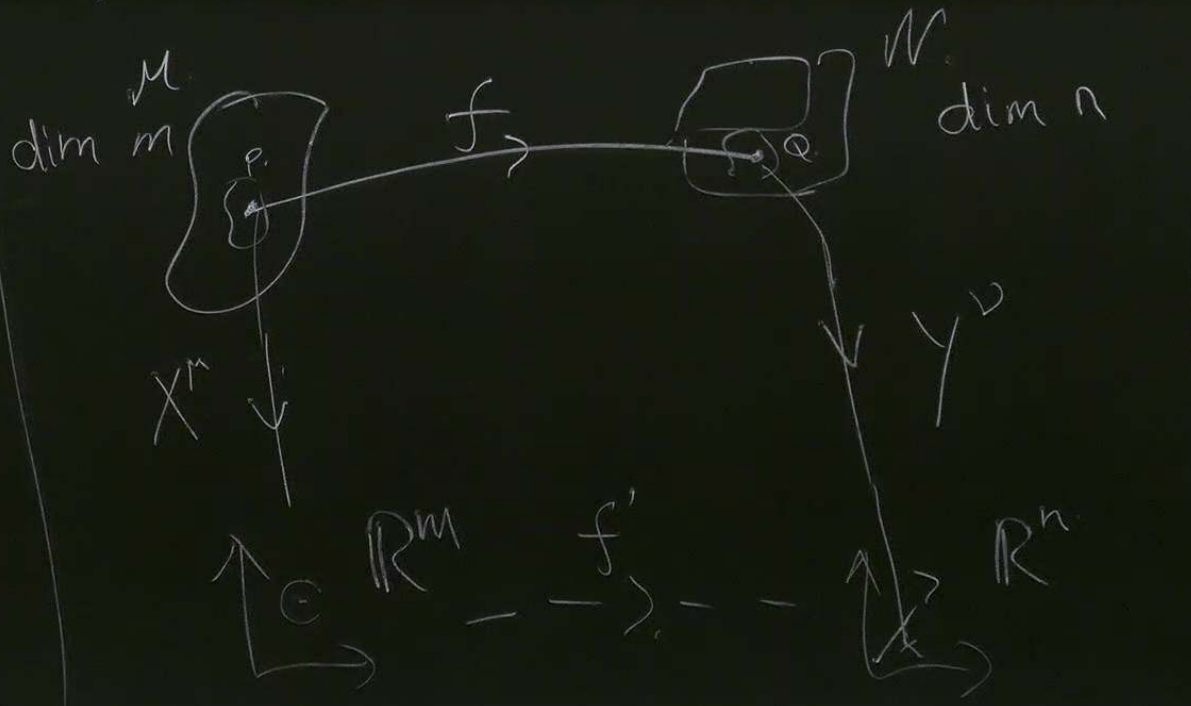
Integral curves give a way to link tangent spaces.

nt is that

ral curves gives

way to link

Maps & Manifolds

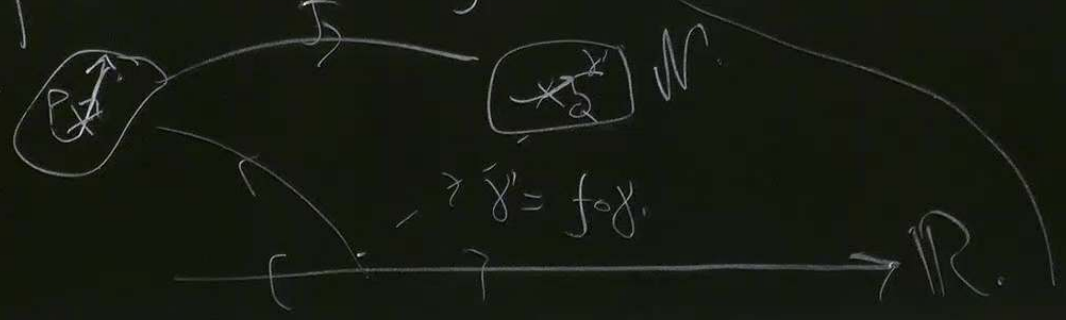


If n

If $m=n$, f 1-1 & onto, C^∞. then f is a diffeomorphism and M, N are the same

Associated to f are maps between tgt spaces

Push forward f_* $T_p(M) \to T_q(N)$



Using this, define the Pull-Back

$$f^*: T_q^*(N) \longrightarrow T_p^*(M)$$

s.t. $\langle f^*(\omega) | I \rangle = \langle \omega | f_*(I) \rangle$

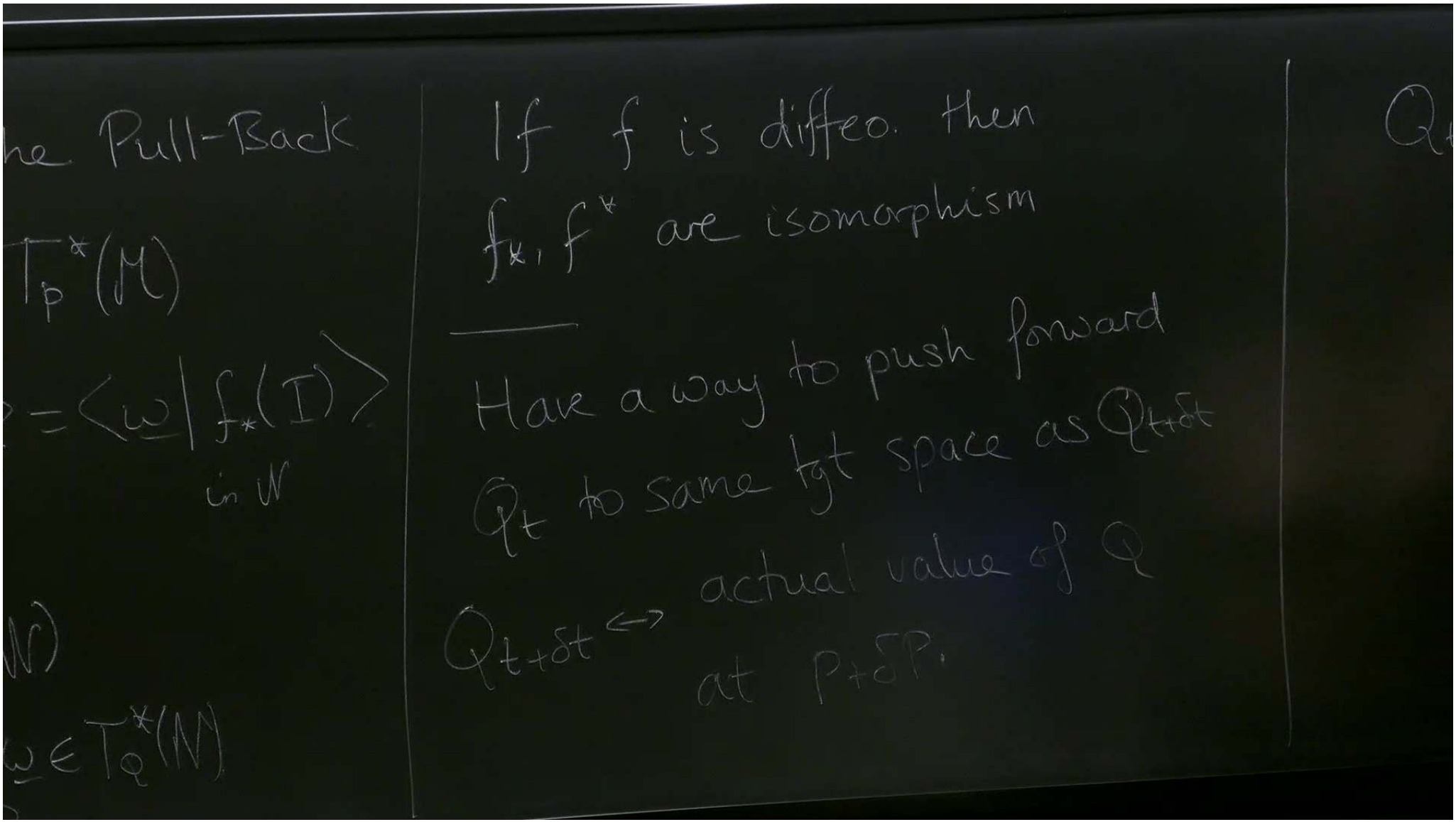
in M in N

$$T_p(M) \xrightarrow{f_*} T_q(N)$$

$f_*(I)$

$\forall \omega \in T_q^*(N)$

R


he Pull-Back
$T_p^*(M)$
$= \langle \omega | f_*(I) \rangle$
in N
$N)$
$\omega \in T_Q^*(N)$
If f is diffeo. then
f_*, f^* are isomorphism
Have a way to push forward
Q_t to same tgt space as $Q_{t+\delta t}$
$Q_{t+\delta t} \leftrightarrow$ actual value of Q
at $P+\delta P$.

the Pull-Back

$\rightarrow T_p^*(M)$

$I\rangle = \langle \omega | f_*(I) \rangle$

in W

$T_Q(W)$

$\forall \omega \in T_Q^*(M)$

If f is diffeo. then

f_*, f^* are isomorphism

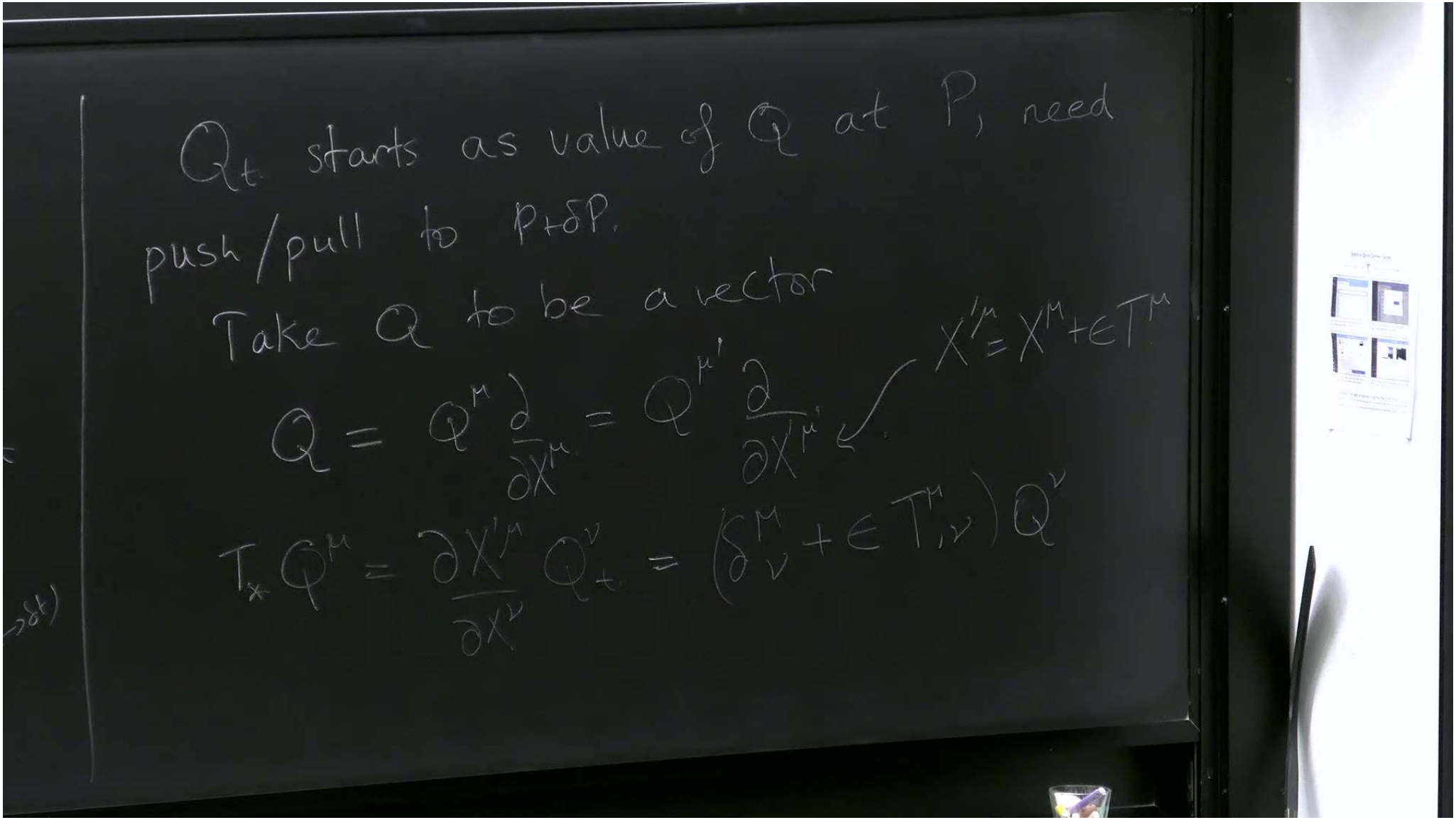
Have a way to push forward

Q_t to same tgt space as $Q_{t+\delta t}$

$Q_{t+\delta t} \leftrightarrow$ actual value of Q ($\epsilon \leftrightarrow \delta t$)

at $P_{t+\delta t}$

$= Q_t + \epsilon T^\mu \partial_\mu Q + \dots$


Q_t starts as value of Q at P, need push/pull to $P+\delta P$.
Take Q to be a vector
$Q = Q^\mu \frac{\partial}{\partial x^\mu} = Q^{\mu'} \frac{\partial}{\partial x'^{\mu'}}$
$x'^\mu = x^\mu + \epsilon T^\mu$
$T_* Q^\mu = \frac{\partial x'^\mu}{\partial x^\nu} Q^\nu_t = (\delta^\mu_\nu + \epsilon T^\mu_{,\nu}) Q^\nu$

Thus $(Q_{t+\delta t} - Q_t)^\mu = Q_t^\mu + \epsilon T^\nu \partial_\nu Q^\mu$

$- (Q_t^\mu + \epsilon T^\mu{}_{,\nu} Q_t^\nu)$

$= \epsilon (T^\nu Q^\mu{}_{,\nu} - T^\mu{}_{,\nu} Q^\nu)$

Lim as $\epsilon \to 0$ is Lie Deriv.

$(\mathcal{L}_T Q)^\mu = T^\nu Q^\mu{}_{,\nu} - T^\mu{}_{,\nu} Q^\nu$.

EX

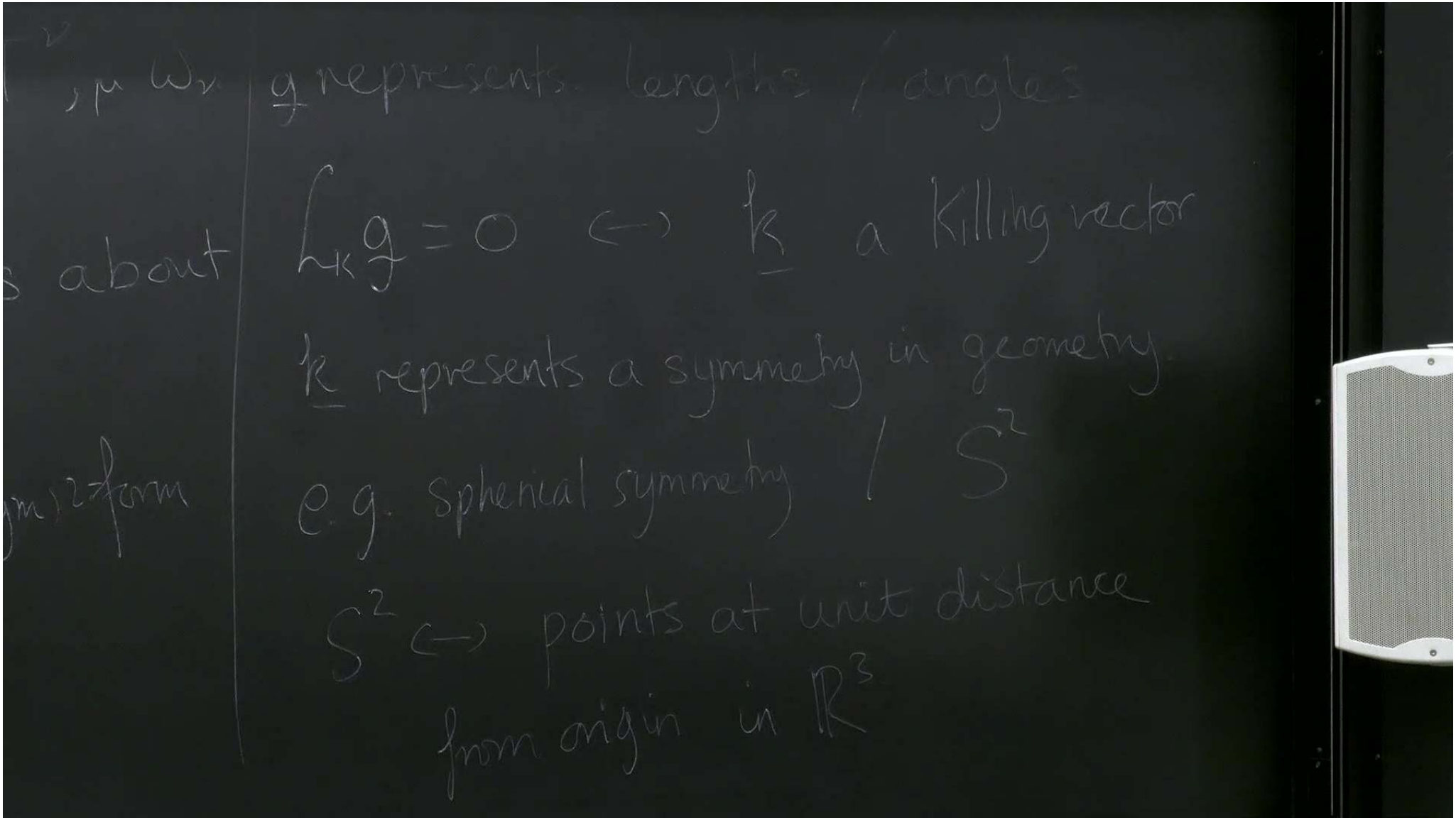
EX: $(\mathcal{L}_T \underline{\omega})_\mu = T^\nu \omega_{\mu,\nu} + T^\nu{}_{,\mu}\, \omega_\nu$

Lie derivatives can tell us about symmetry

The metric is a bilinear (Sym) 2-form

$g \in T^*_p(M) \otimes T^*_p(M)$

g represents

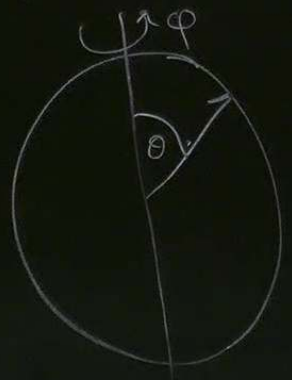

, μ W
s about
m 2-form
g represents lengths / angles
$\mathcal{L}_k g = 0 \leftrightarrow k$ a Killing vector
k represents a symmetry in geometry
e.g. spherical symmetry / S^2
$S^2 \leftrightarrow$ points at unit distance
from origin in $\mathbb{R}^3$

S^2 inherits a metric from $\mathbb{R}^3$.

most naturally expressed in polars

$$\underline{x} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\rightarrow ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 = d\Omega_{II}^2$$



$\underline{k}_q =$

$(\mathcal{L}_K$

$\mathbb{R}^3$,

polars

$(\varphi, \cos\theta)$

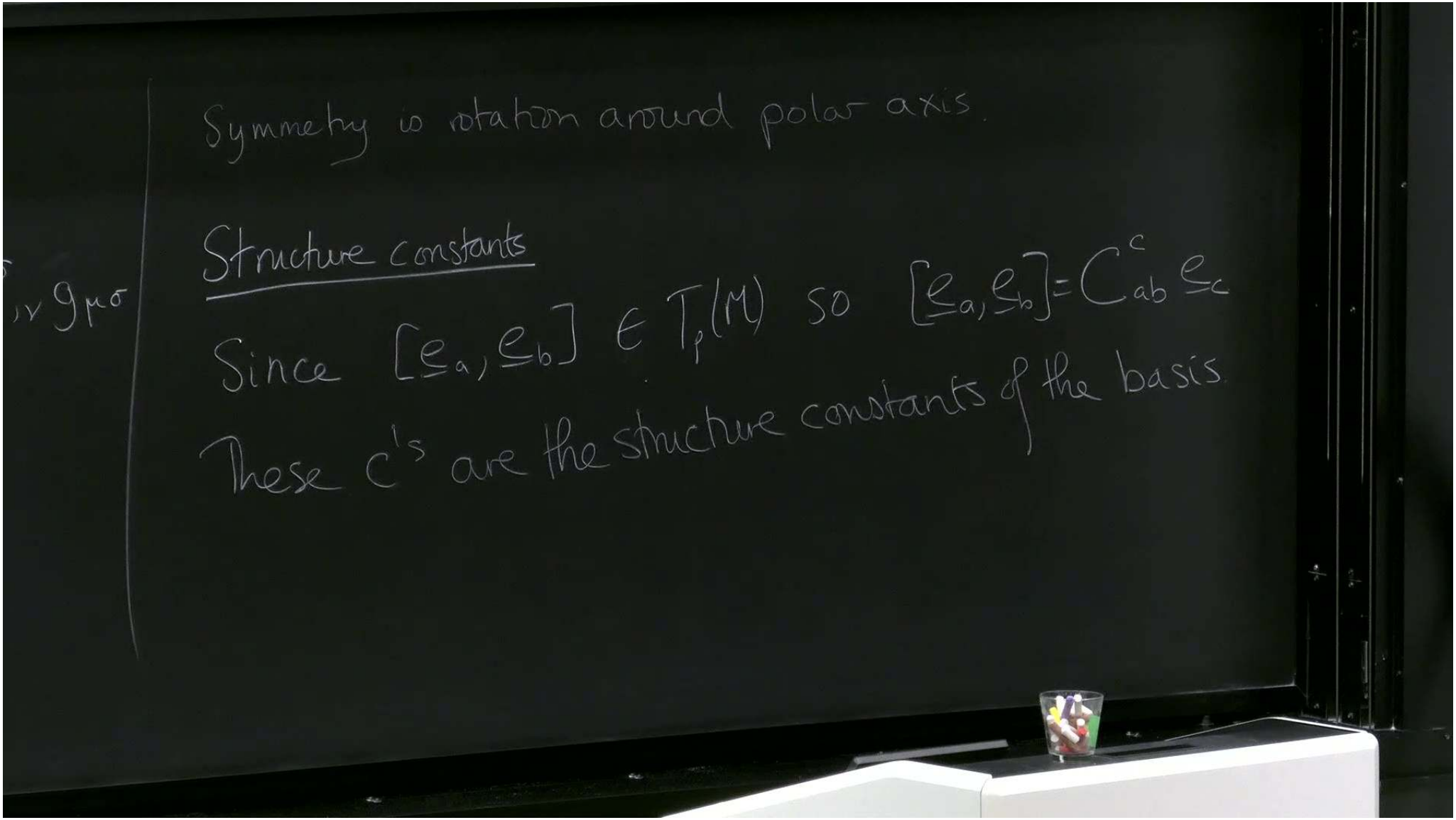
$\varphi^2 = d\Omega_{II}^2$

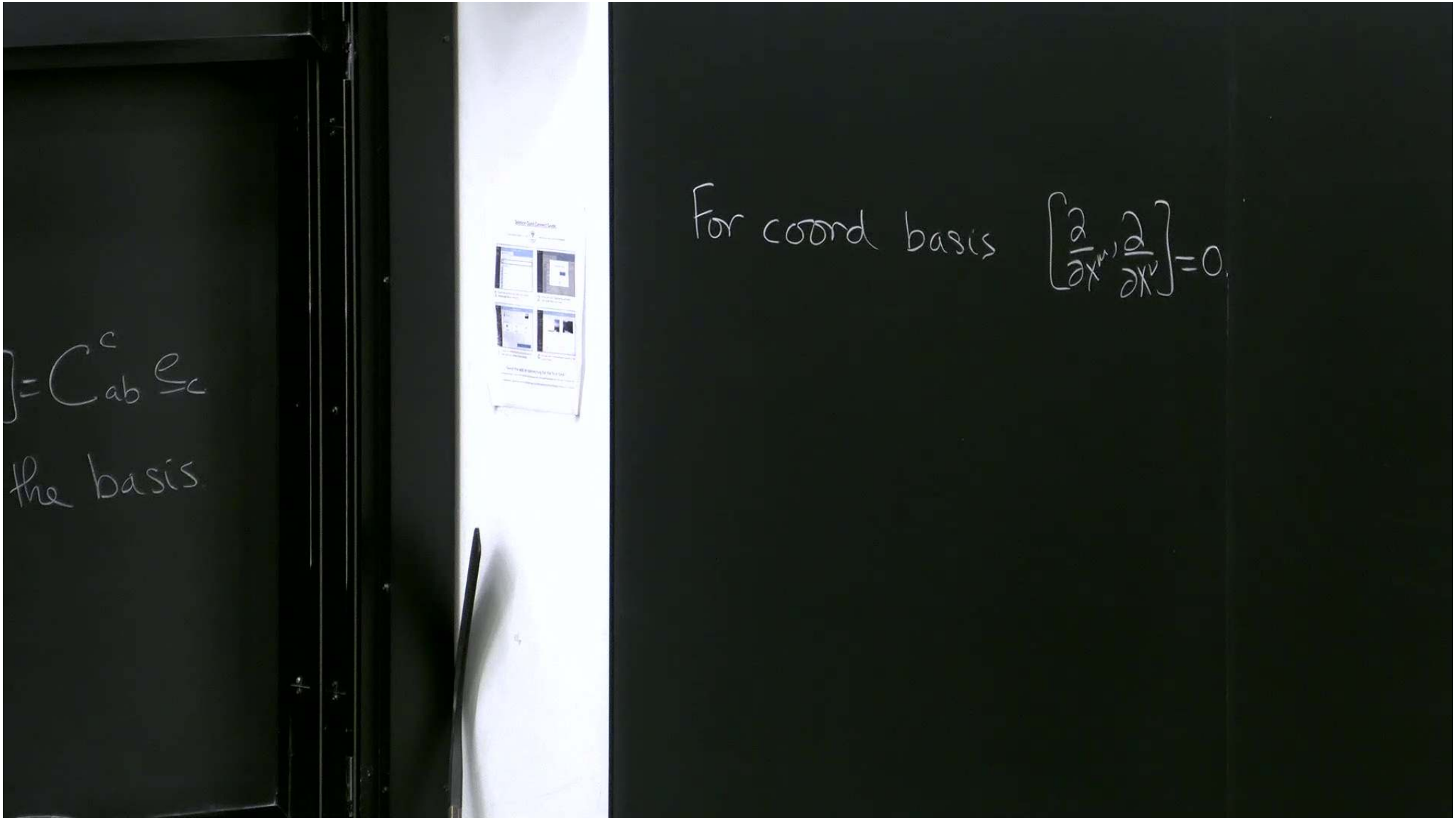
$$k_\varphi = \frac{\partial}{\partial\varphi}$$

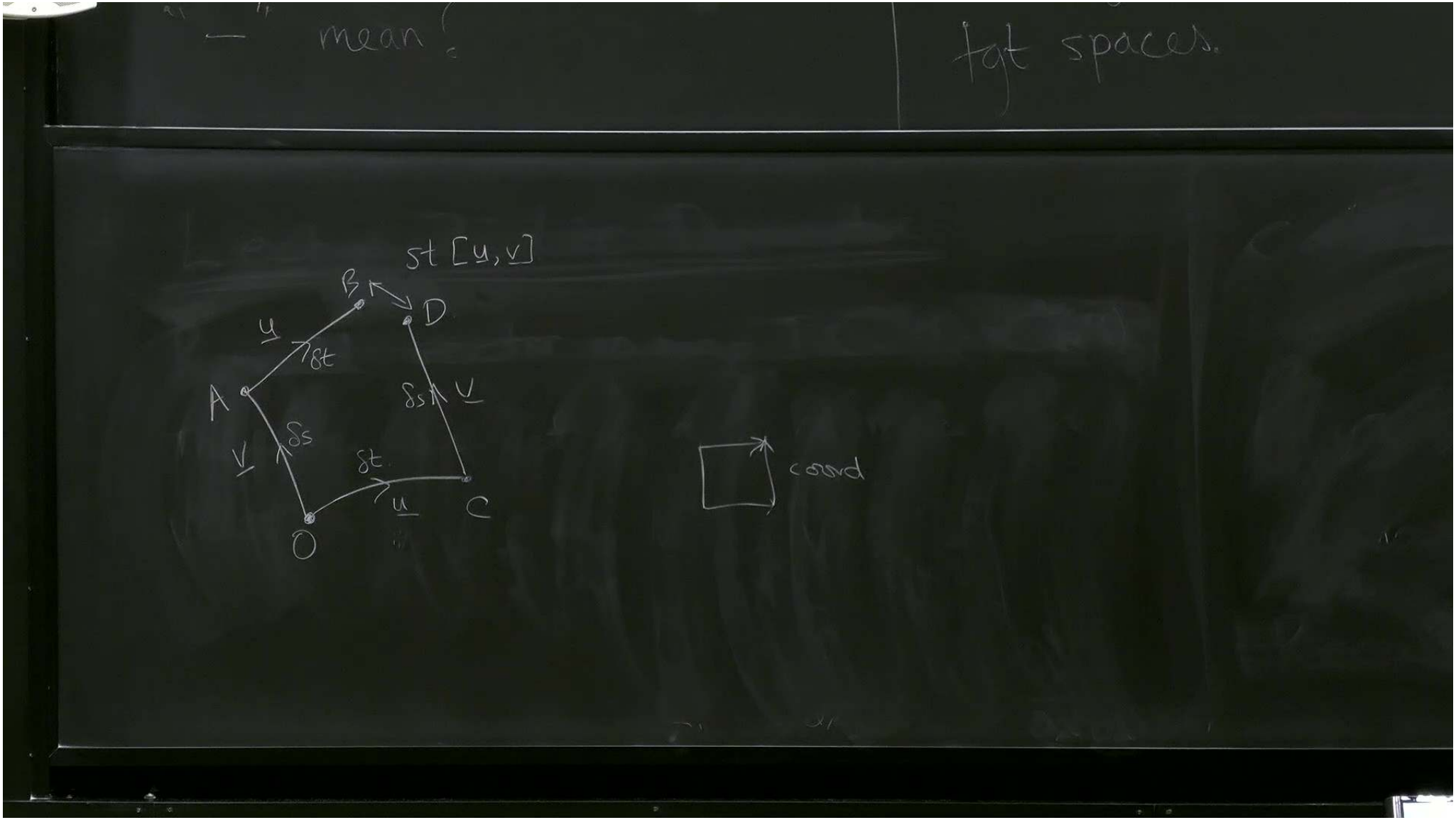
$$(\mathcal{L}_k g)_{\mu\nu} = k^\sigma g_{\mu\nu,\sigma} + k^\sigma_{,\mu} g_{\sigma\nu} + k^\sigma_{,\nu} g_{\mu\sigma}$$

but $\frac{\partial}{\partial\varphi} g_{\mu\nu} = 0$ & $k^\sigma_{,\mu} = 0$

$$\Rightarrow \mathcal{L}_k g = 0$$


Symmetry is rotation around polar axis.
Structure constants
Since $[e_a, e_b] \in T_p(M)$ so $[e_a, e_b] = C^c_{ab} e_c$
These c's are the structure constants of the basis


$]=C^{c}_{ab}e_c$
the basis
For coord basis $\left[\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right]=0$.


mean!
tgt spaces.
st [u, v]
B
D
u
δt
A
δs
v
δs
v
δt
u
C
O
coord