Title: Lecture - Mathematical Physics I (Core), PHYS 777

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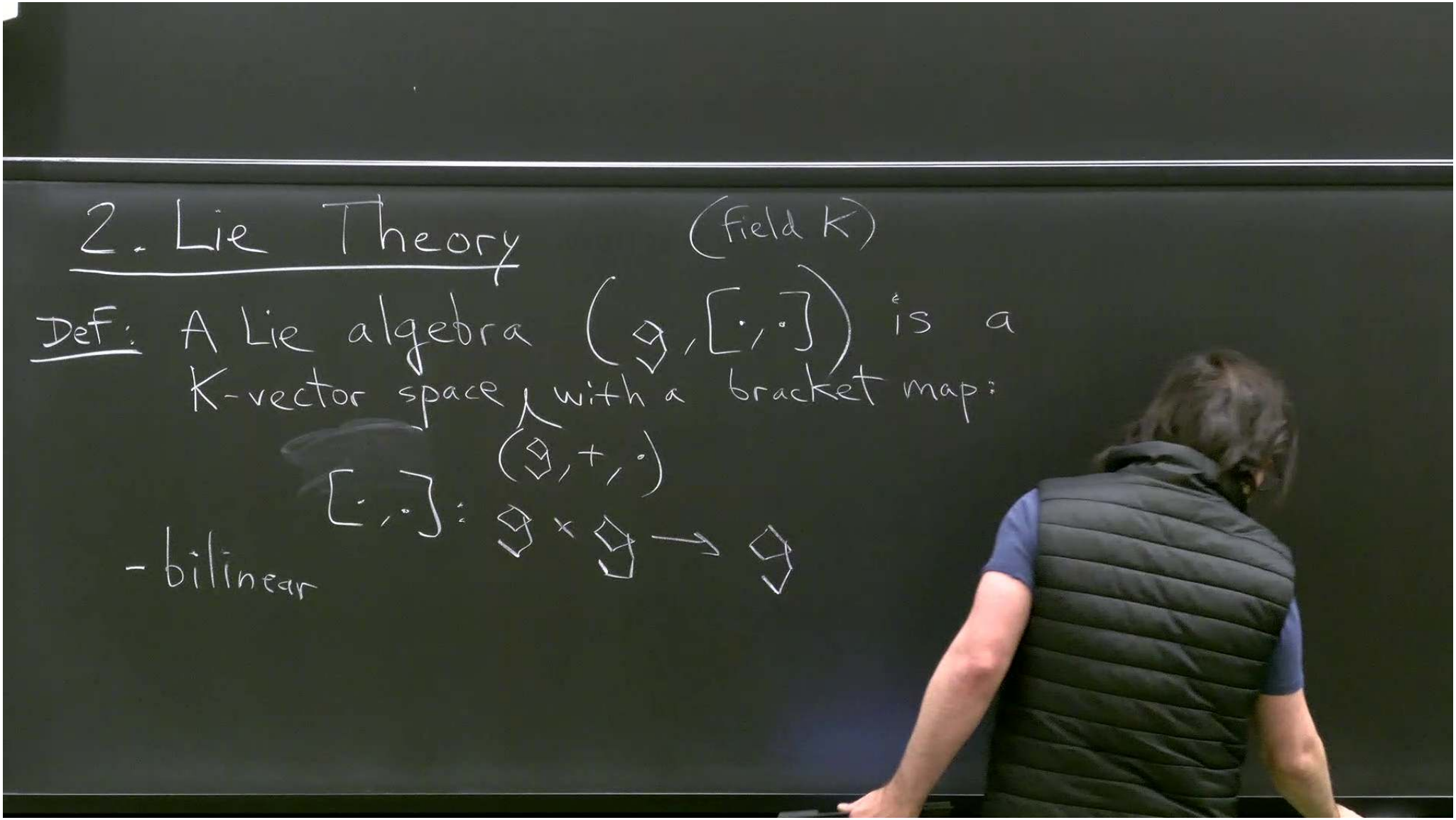
2. Lie Theory
(field K)
Def: A Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a
K-vector space $(\mathfrak{g}, +, \cdot)$ with a bracket map:
$[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
- bilinear

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- bilinear
- antisymmetric: $[x,y] = -[y,x]$
- Jacobi identity: $[[x,y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

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$[a,b]$ is NOT $ab - ba$

] is NOT
-ba

=0

Examples

$K = \mathbb{R}$

$[X,Y]f := X(Yf) - Y(Xf)$

1) $(\Gamma(TM), [\cdot,\cdot])$ is an ∞-dim Lie algebra

space of real vector fields

2) $(\mathbb{R}^3, [\![\cdot,\cdot]\!])$ is a 3-dim Lie algebra called Heisenberg algebra

$\{x,y,z\}$

$$\begin{cases} [\![x,y]\!] = z \\ [\![y,z]\!] = 0 \\ [\![z,x]\!] = 0 \end{cases}$$

3) $\mathfrak{gl}(n,\mathbb{C}) := (M_{n\times n}\text{ matrices}, [\cdot,\cdot])$ has dimension n^2

"general linear algebra" $[A,B] = A\cdot B - B\cdot A$

4) $\mathfrak{sl}(n,\mathbb{C})$:= traceless matrices of $\mathfrak{gl}(n,\mathbb{C})$ has dim n^2-1

"special linear algebra"

$$\mathrm{Tr}([A,B]) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA)$$
$$= 0 \quad (\mathrm{Tr}\text{ is cyclic})$$

$\Rightarrow [A,B]$ is in $\mathfrak{sl}(n,\mathbb{C})$

5) Lain

Poincaré algebra is a 10-dim real Lie algebra ↳ 6 Lorentz ($M_{\mu\nu}$ antisym)
4 Translations (P_μ)

$[P_\mu, P_\nu] = 0$

$[M_{\mu\nu}, P_\sigma]$ ... $[M_{\mu\nu}, M_{\rho\sigma}] = \ldots$

Poisson bracket

$(C^\infty(M), \{\cdot,\cdot\})$ where $M = T^*Q$ is a phase space manifold is a Lie algebra

$\Rightarrow [A,B]$ is in $S(n,\mathbb{C})$ (Γ is cyclic)

is a Li

2.1 Can we hope to classify (complex) Lie algebras?

Def A Lie algebra $(R,[\cdot,\cdot])$ is SOLVABLE if

Terminates

$$R \supseteq [R,R] \supseteq [[R,R],[R,R]] \supseteq \cdots \supseteq \{0\}$$

$$[R,R] := \{[r_1,r_2] \mid r_1, r_2 \in R\}$$

is a Lie algebra

Examples

- An abelian Lie algebra $\mathfrak{g}$ is solvable

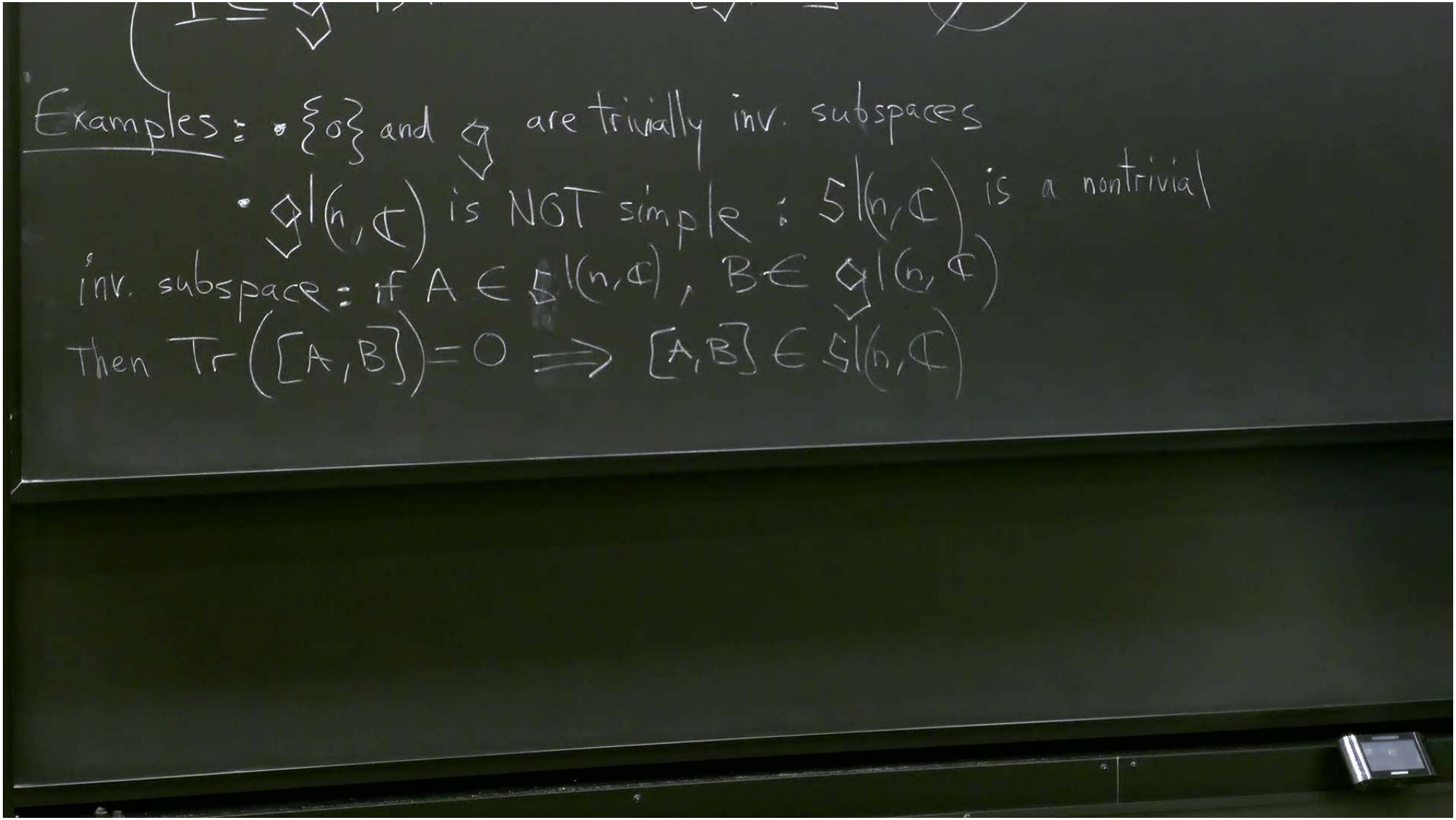
$$[\mathfrak{g}, \mathfrak{g}] = \{0\}$$

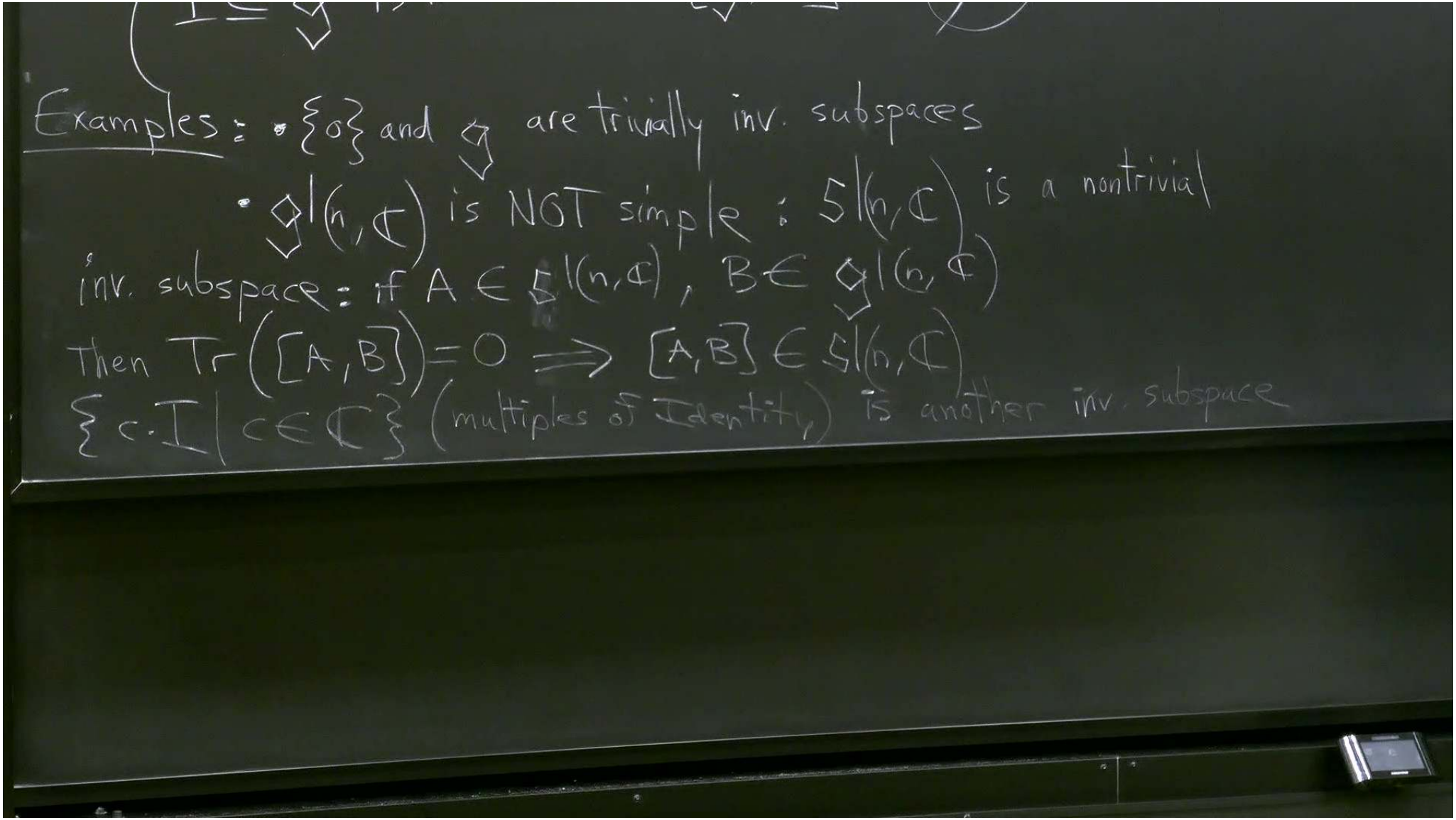
- $\mathfrak{sl}(n, \mathbb{C})$ is NOT solvable: $[\mathfrak{sl}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C})] = \mathfrak{sl}(n, \mathbb{C})$

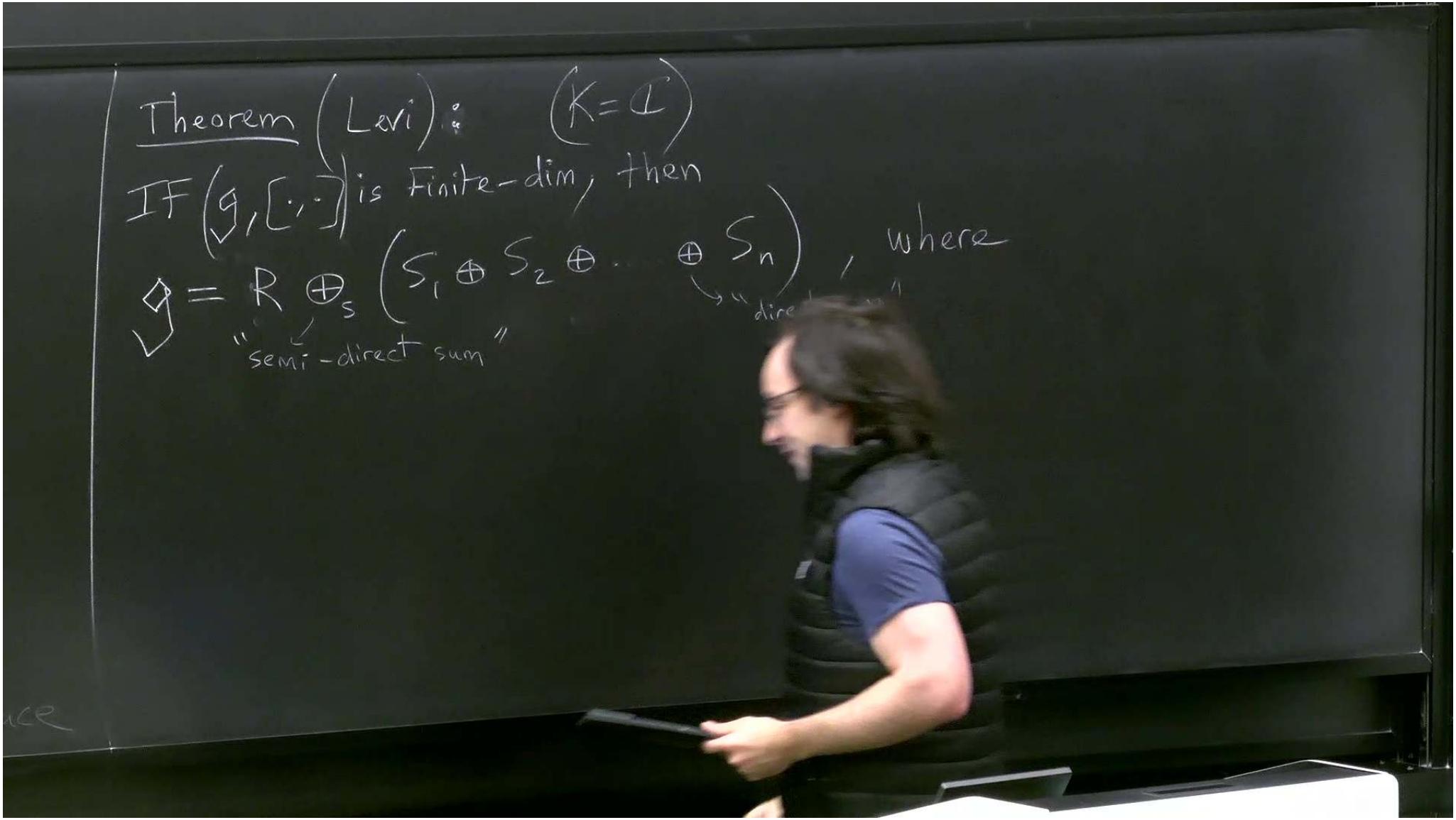
Def: A nonabelian Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is SIMPLE if it has no nontrivial invariant subspaces ("ideals")

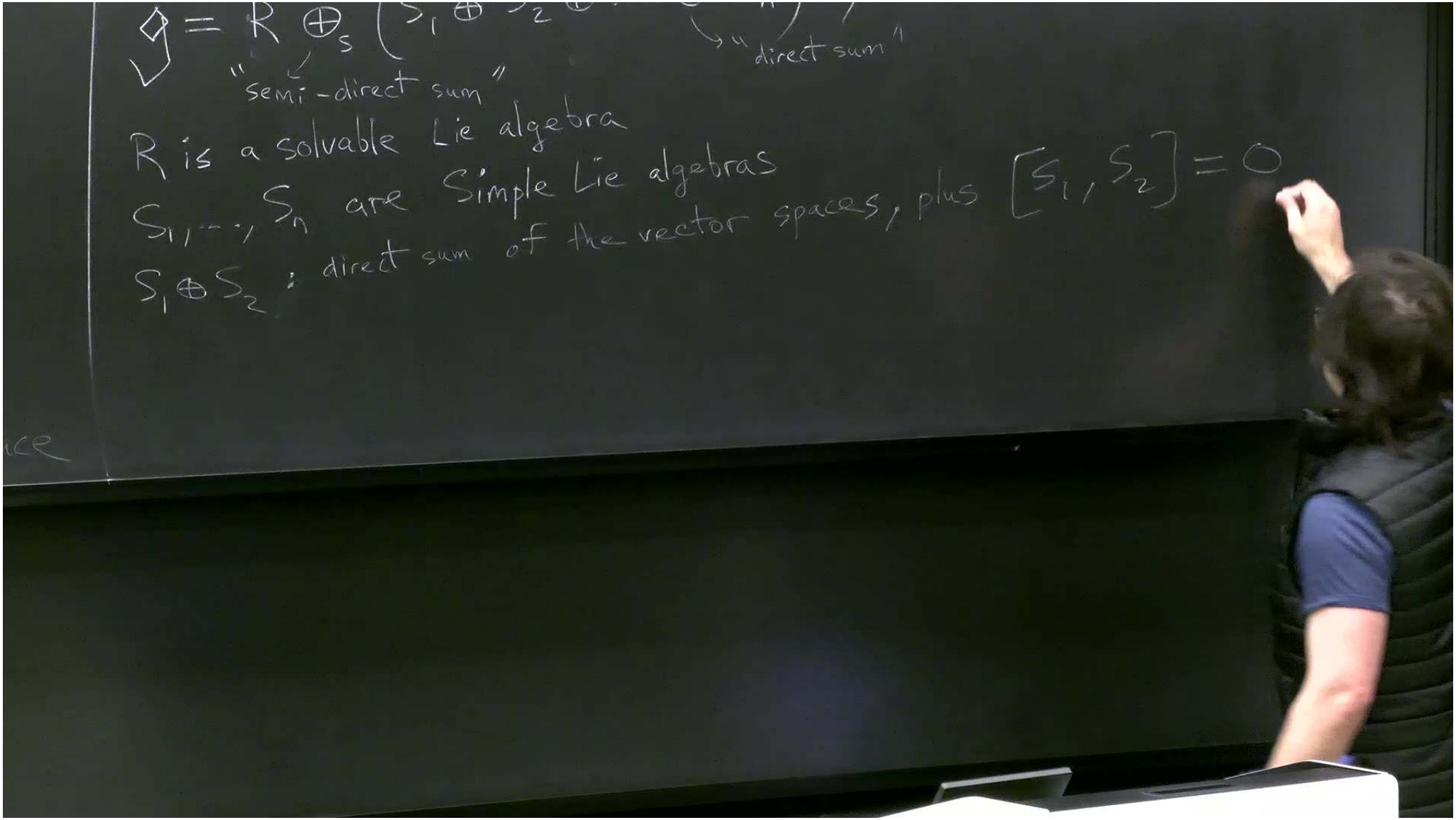
($I \subseteq \mathfrak{g}$ is invariant if $[\mathfrak{g}, I] \subseteq I$)

Examples:
- $\{0\}$ and $\mathfrak{g}$ are trivially inv. subspaces
- $\mathfrak{gl}(n,\mathbb{C})$ is NOT simple: $\mathfrak{sl}(n,\mathbb{C})$ is a nontrivial inv. subspace


Examples: • $\{0\}$ and $\mathfrak{g}$ are trivially inv. subspaces
• $\mathfrak{gl}(n,\mathbb{C})$ is NOT simple; $\mathfrak{sl}(n,\mathbb{C})$ is a nontrivial
inv. subspace: if $A \in \mathfrak{sl}(n,\mathbb{C})$, $B \in \mathfrak{gl}(n,\mathbb{C})$
Then $\mathrm{Tr}([A,B]) = 0 \Rightarrow [A,B] \in \mathfrak{sl}(n,\mathbb{C})$


Examples: • $\{0\}$ and $\mathbb{C}^n$ are trivially inv. subspaces
• $\mathfrak{gl}(n,\mathbb{C})$ is NOT simple: $\mathfrak{sl}(n,\mathbb{C})$ is a nontrivial
inv. subspace: if $A \in \mathfrak{sl}(n,\mathbb{C})$, $B \in \mathfrak{gl}(n,\mathbb{C})$
Then $\mathrm{Tr}([A,B]) = 0 \Rightarrow [A,B] \in \mathfrak{sl}(n,\mathbb{C})$
$\{c \cdot I \mid c \in \mathbb{C}\}$ (multiples of Identity) is another inv. subspace


Theorem (Levi): (K = C)
IF (g, [·,·]) is Finite-dim, then
g = R ⊕_s (S_1 ⊕ S_2 ⊕ .. ⊕ S_n), where
"semi-direct sum"


$\mathfrak{g} = R \oplus_s (S_1 \oplus S_2 \oplus \cdots)$
"semi-direct sum"
"direct sum"
R is a solvable Lie algebra
$S_1, \cdots, S_n$ are simple Lie algebras
$S_1 \oplus S_2$: direct sum of the vector spaces, plus $[S_1, S_2] = 0$

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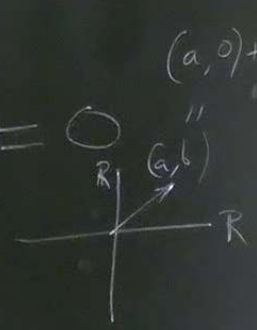
$(a,0)+(0,b)$



ace

$\mathfrak{g} = R \oplus_s (S_1 \oplus S_2 \oplus \cdots \oplus S_n)$

"direct sum"

"semi-direct sum"

R is a solvable Lie algebra

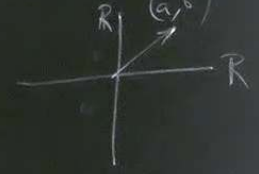
$S_1, \ldots, S_n$ are simple Lie algebras

$S_1 \oplus S_2$: direct sum of the vector spaces, plus $[S_1, S_2] = 0$

$R \oplus_s S_1$: direct sum of the vect. spaces, plus

$$[S_1, R] \subseteq R$$

$(a,0)+(0,b)$



↳ Classifying solvable Lie algebras is "wild" (hopeless)

Def: If $\mathfrak{g}$ has no solvable part, then

$\mathfrak{g} = S_1 \oplus \dots \oplus S_n$ is called SEMI-SIMPLE

Can we hope to classify S_i's? YES!

$\Rightarrow [A,B]$ is in $S(\Gamma, \mathcal{A})$ (Γ is cyclic)

is a Li

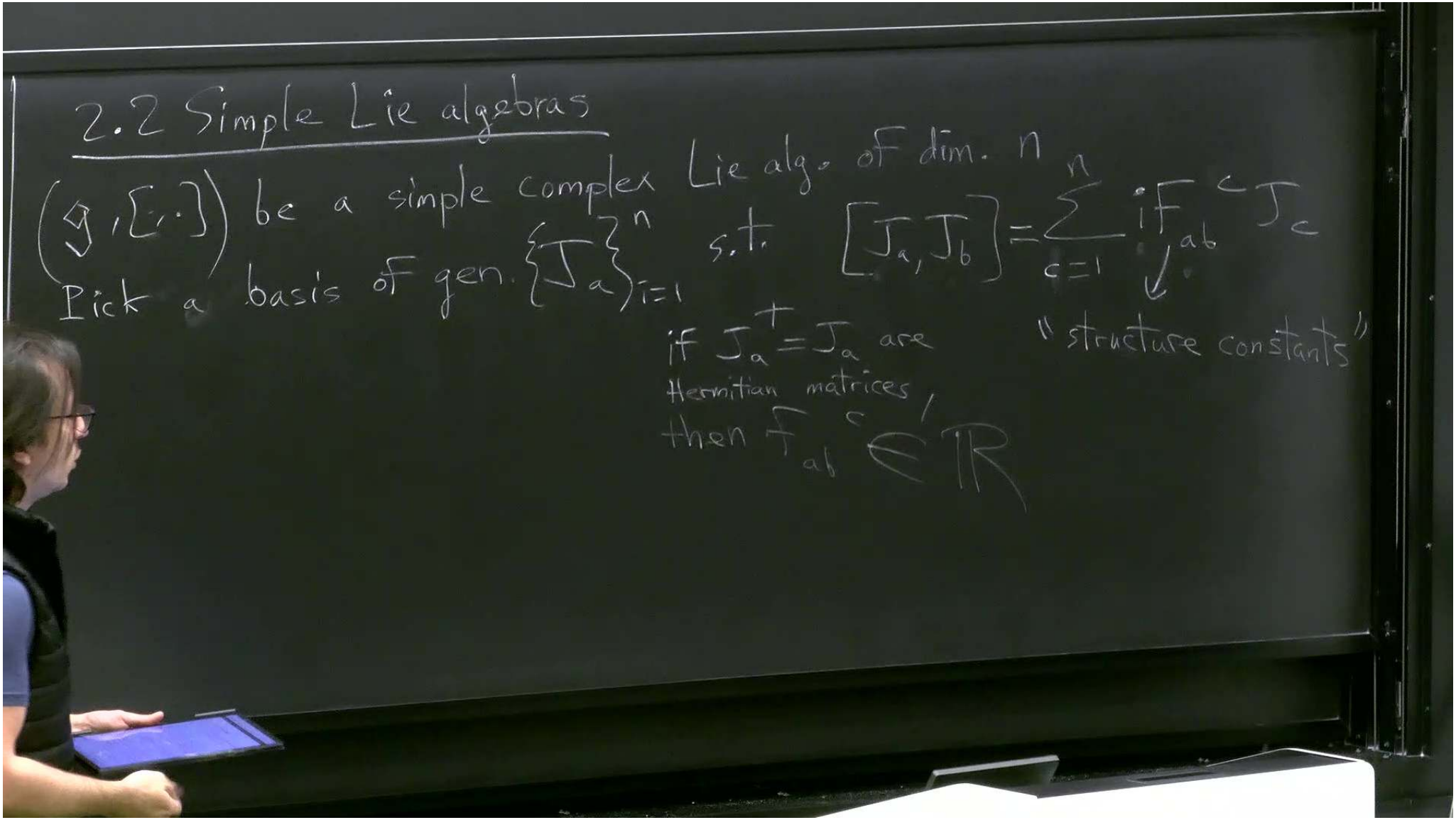
2.1 Can we hope to classify (complex) Lie algebras?

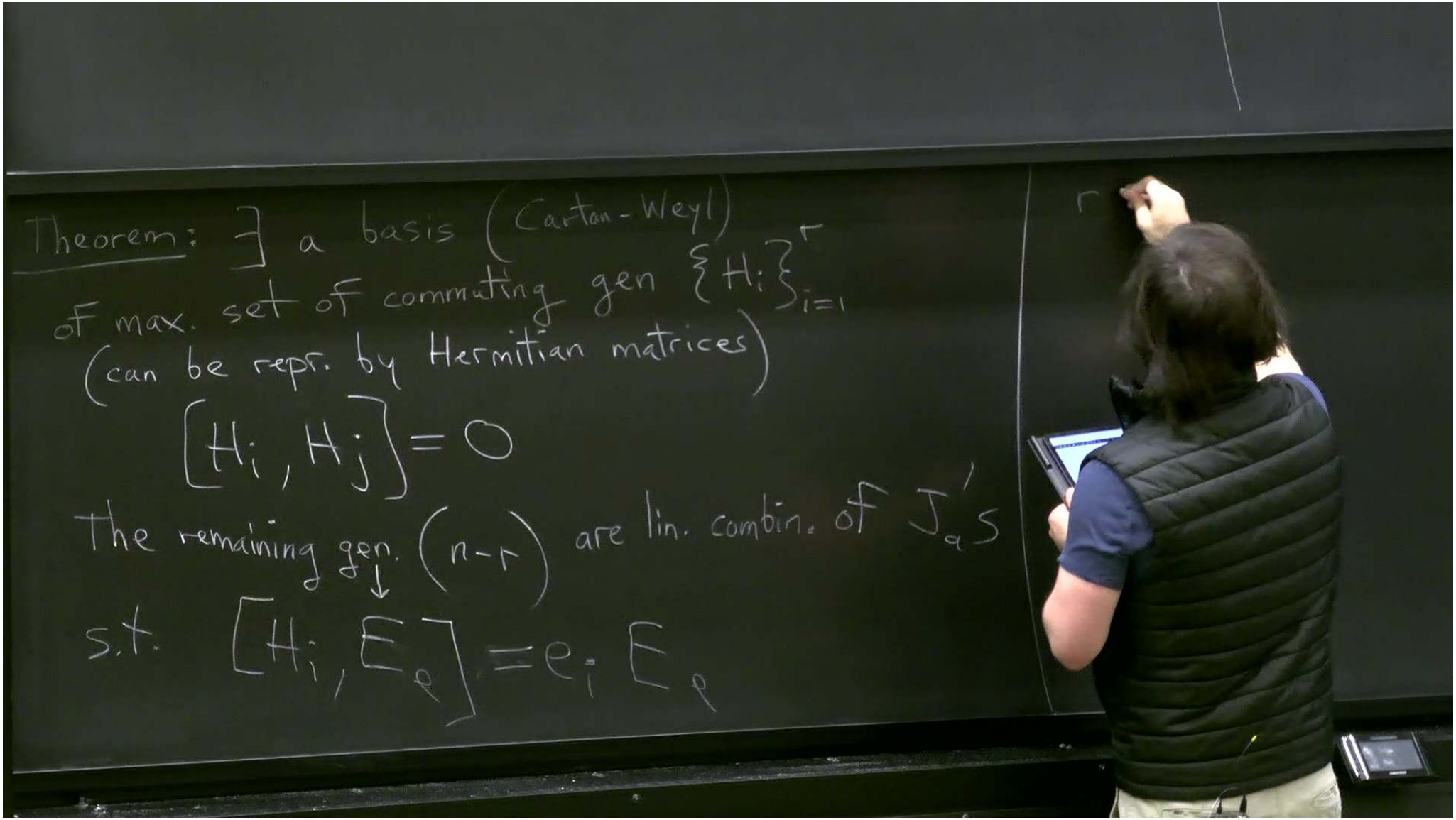
Def A Lie algebra $(R, [\cdot,\cdot])$ is SOLVABLE if

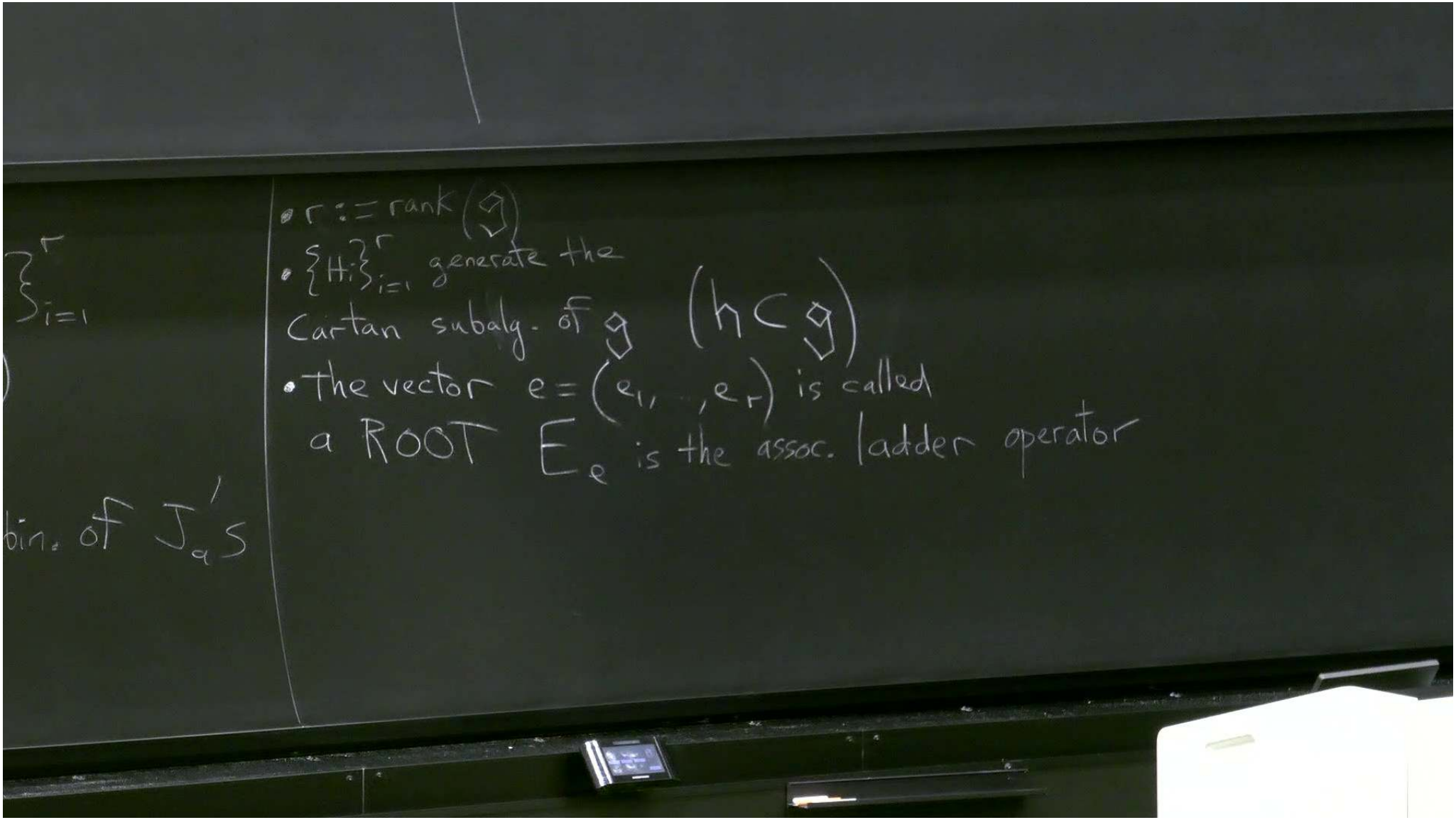
$$R \geq [R,R] \geq [[R,R],[R,R]] \geq \cdots \geq \{0\}$$

(Terminates)

$$[R,R] := \{[r_1, r_2] \mid r_1, r_2 \in R\}$$


2.2 Simple Lie algebras
$(\mathfrak{g}, [\cdot,\cdot])$ be a simple complex Lie alg. of dim. n
Pick a basis of gen. $\{J_a\}_{i=1}^{n}$ s.t. $[J_a, J_b] = \sum_{c=1}^{n} i f_{ab}^{\ c} J_c$
if $J_a^\dagger = J_a$ are Hermitian matrices, then $f_{ab}^{\ c} \in \mathbb{R}$
"structure constants"


Theorem: ∃ a basis (Cartan-Weyl)
of max. set of commuting gen $\{H_i\}_{i=1}^{r}$
(can be repr. by Hermitian matrices)
$[H_i, H_j] = 0$
The remaining gen. $(n-r)$ are lin. combin. of J_a's
s.t. $[H_i, E_\rho] = e_i E_\rho$
r


$\}_{i=1}^{r}$
bin. of J_a's
• $r := \text{rank}(\mathfrak{g})$
• $\{H_i\}_{i=1}^{r}$ generate the Cartan subalg. of $\mathfrak{g}$ $(\mathfrak{h} \subset \mathfrak{g})$
• the vector $e = (e_1, \ldots, e_r)$ is called a ROOT E_e is the assoc. ladder operator

$\Rightarrow [A,B]$ is in $\mathfrak{sl}(n,\mathbb{R})$ (T_F is cyclic)

is a Li

- The roots are nonzero
- $e(H_i) = e_i$, so $e \in h^*$

Prop: If e is a root, then $-e$ is also a root