Title: Lecture - Mathematical Physics I (Core), PHYS 777

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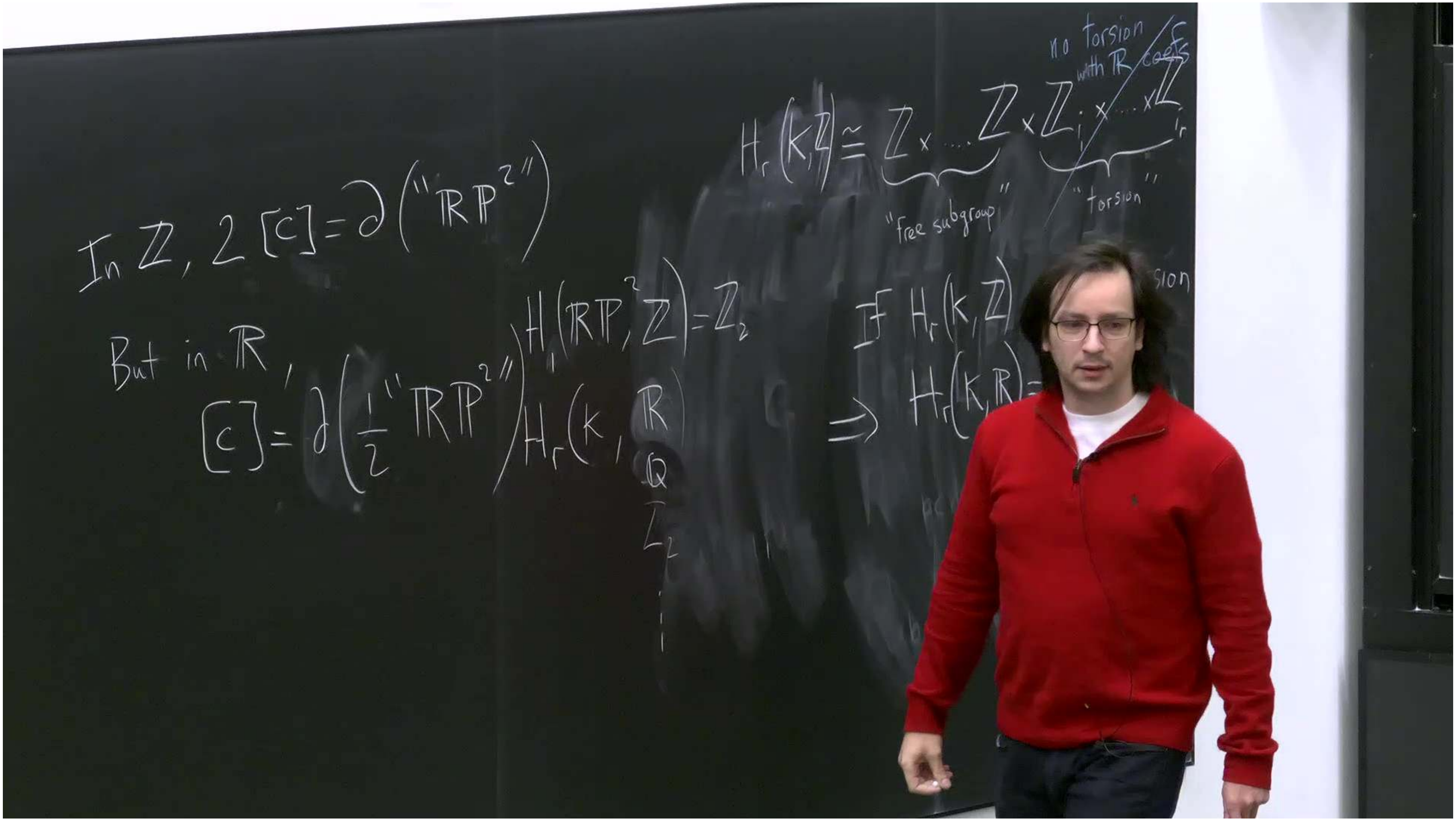

no torsion with $\mathbb{R}$ coefs
$H_r(K,\mathbb{Z}) \cong \mathbb{Z}\times \dots \mathbb{Z}\times\mathbb{Z}_{i_1}\times\dots\times\mathbb{Z}_{i_r}$
"free subgroup"
"torsion"
In $\mathbb{Z}$, $2[c]=\partial(``\mathbb{RP}^2")$
But in $\mathbb{R}$,
$[c]=\partial(\frac{1}{2}``\mathbb{RP}^2")$
$H_1(\mathbb{RP}^2,\mathbb{Z})=\mathbb{Z}_2$
$H_r(K,\mathbb{R}$
$\mathbb{Q}$
$\mathbb{Z}_2$
If $H_r(K,\mathbb{Z})$
$\Rightarrow H_r(K,\mathbb{R})$

1.3 : Differential forms (Math Notes IV)

Def: Let M be a smooth manifold.

An r-form is a totally antisymmetric $(0,r)$-tensor field ω:

$$\omega(\underbrace{X_1, \ldots, X_r}_{\text{vector fields}}) = (-1)^{\pi}\, \omega(X_{\pi(1)}, \ldots, X_{\pi(r)})$$

Ex: $r = 0$

Ex: • $r=0$: a scalar field $f: M \longrightarrow \mathbb{R}$

$x \longmapsto f(x)$

smooth functions on M

• $r=1$: a covector field $\alpha : \underbrace{\Gamma(TM)}_{\text{vector fields on } M} \longrightarrow C^{\infty}(M)$

$X \longmapsto \alpha(X)$ (Math Notes I)

Reminder:

A tangent vector at $p \in M$ is

$$X_p : C^\infty(M) \to \mathbb{R}$$
$$f \mapsto X_p f$$

In coord., $\{x^i\}$, $X_p f = \sum_{i=1}^{r} X^i \frac{\partial f}{\partial x^i}\Big|_p$

$$X_p = \sum_{i=1}^{r} X^i \frac{\partial}{\partial x^i}\Big|_p$$

A vector field is

$$X : C^\infty(M) \to C^\infty(M)$$
$$f \mapsto Xf$$

$$X = \sum_{i=1}^{r} X^i(x) \frac{\partial}{\partial x^i}$$

Co-vectors

Defined via maps from $T_p(M)$ to reals

$$\alpha_p : T_p(M) \longrightarrow \mathbb{R}$$

$$X_p \longmapsto \alpha_p(X_p) = \sum_{i=1}^{n} \alpha_i X^i$$

In coord. $\{x^i\}$,

$$\alpha_p = \sum_{i=1}^{n} \alpha_i \, dx^i \Big|_p$$

Ex: • $r=0$: a scalar field $f: M \longrightarrow \mathbb{R}$

$x \longmapsto f(x)$

smooth functions on M

• $r=1$: a covector field $\alpha : \underbrace{\Gamma(TM)}_{\text{vector fields on } M} \longrightarrow C^{\infty}(M)$

(1-Form on M)

$X \longmapsto \alpha(X) = \sum_{i=1}^{n} \alpha_i(x) X^i(x)$ in coord.

(Math Notes I)

We say $\alpha \in \Gamma(T^*M) := \Omega^1(M)$

• $r=2$: $\omega : \Gamma(TM$

<u>Ex:</u> • $r=0$: a scalar field $f: M \longrightarrow \mathbb{R}$

$x \longmapsto f(x)$

smooth functions on M

• $r=1$: a covector field $\alpha : \underbrace{\Gamma(TM)}_{\text{vector fields on } M} \longrightarrow C^{\infty}(M)$

(1-Form on M)

$X \longmapsto \alpha(X) = \sum_{i=1}^{n} \alpha_i(x) X^i(x)$ in coord.

(Math Notes I)

We say $\alpha \in \Gamma(T^*M) := \Omega^1(M)$

• $r=2$: $\omega : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^{\infty}(M)$

$\omega \in \Omega^2(M) \; \left(= \Gamma\left(\Lambda^2 T^*M\right)\right)$

Ex:

- In Electromagnetism,
  "gauge potential" A is a 1-form
  "Field strength" F is a 2-form
- In classical physics,
  T^*M is phase space

Notation: $\Omega^r(M)$ is the set of r-forms on M.

Def: $\wedge : \Omega^r(M) \times \Omega^s(M) \longrightarrow \Omega^{r+s}(M)$

$$\omega\ ,\ \sigma \longmapsto \omega \wedge \sigma$$

defined by

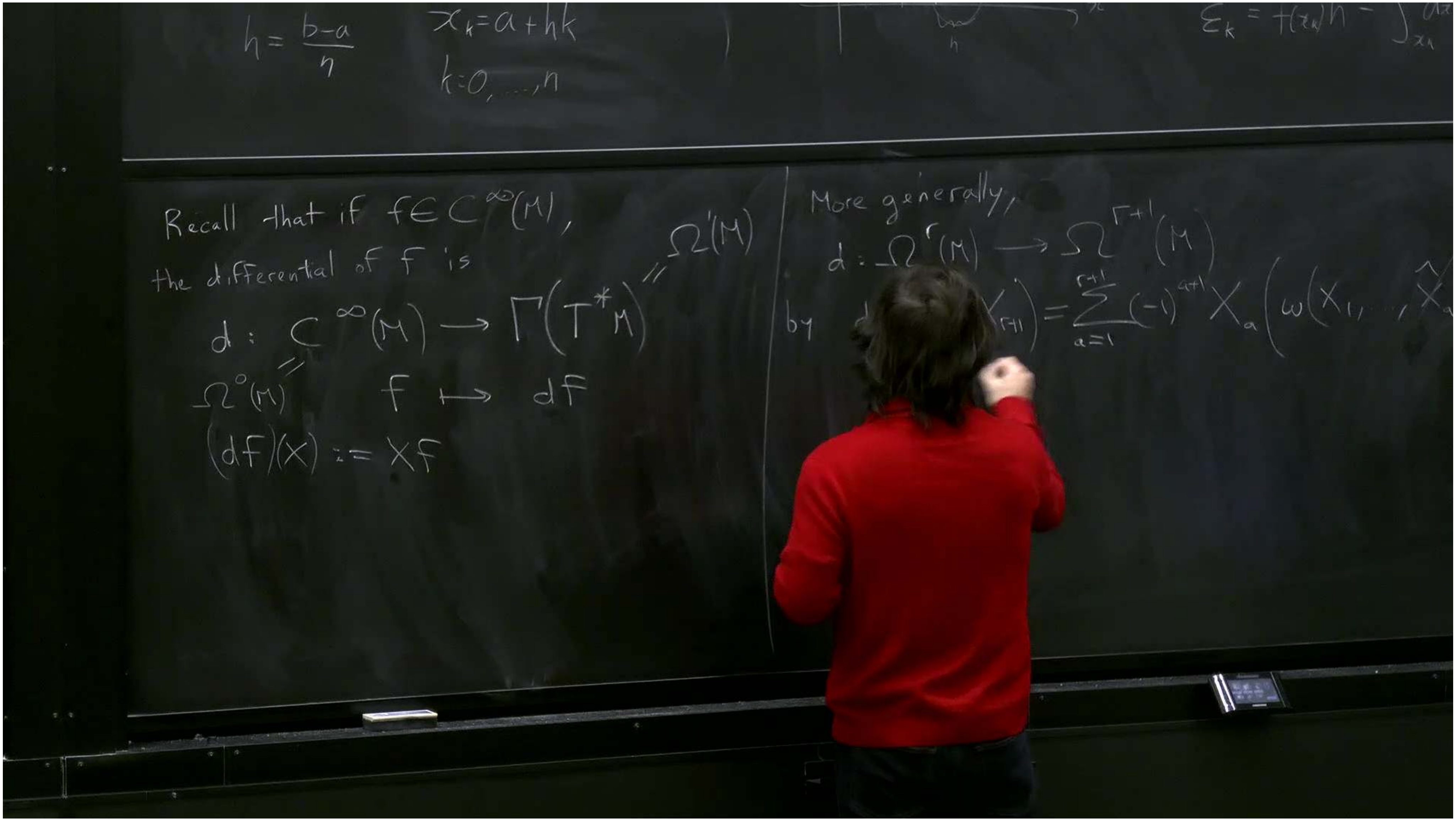
$$(\omega \wedge \sigma)(X_1, \ldots, X_{r+s}) = \frac{1}{r!}\frac{1}{s!} \sum_{\pi \in \mathrm{perm}(r+s)} (-1)^{\pi} (\omega \otimes \sigma)(X_{\pi(1)}, \ldots, X_{\pi(r+s)})$$

Ex: If $\alpha, \beta \in \Omega^1(M)$,

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$$

Note $$(\alpha \wedge \beta)(X_1, X_2) = \begin{vmatrix} \alpha(X_1) & \alpha(X_2) \\ \beta(X_1) & \beta(X_2) \end{vmatrix}$$

Prop: $(\omega \wedge \sigma) = (-1)^{rs}(\sigma \wedge \omega)$



Recall that if f ∈ C^∞(M),
the differential of f is
d : C^∞(M) → Γ(T*M)
Ω^0(M)
Ω^1(M)
f ↦ df
(df)(X) := Xf
More generally,
d : Ω^r(M) → Ω^{r+1}(M)
by
h = (b−a)/n
x_k = a + hk
k = 0, …, n

$\Omega^1(M)$

More generally,

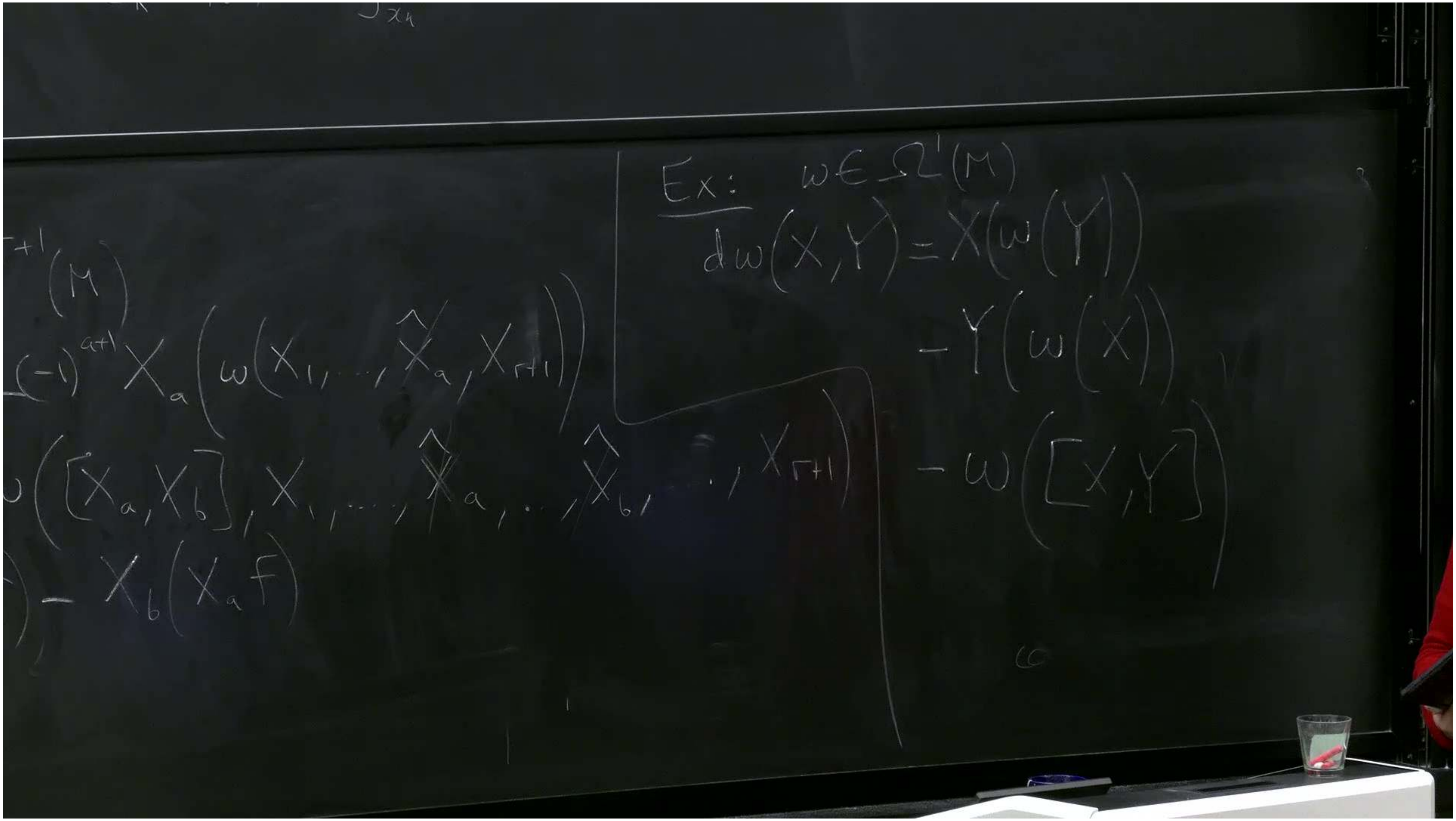
$$d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$$

by
$$d\omega(X_1,\dots,X_{r+1}) = \sum_{a=1}^{r+1} (-1)^{a+1} X_a\left(\omega(X_1,\dots,\hat{X}_a,\dots,X_{r+1})\right)$$
$$+ \sum_{a<b} (-1)^{a+b}\, \omega\left([X_a,X_b], X_1,\dots,\hat{X}_a,\dots,\hat{X}_b,\dots,X_{r+1}\right)$$

$$[X_a,X_b]f := X_a(X_b f) - X_b(X_a f)$$

Ex: $\omega \in \Omega^1(M)$

$d\omega(X$



$\mathrm{Ex:}\ \omega \in \Omega^1(M)$
$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$
$(-1)^{a+1} X_a(\omega(X_1, \dots, \hat{X}_a, X_{r+1}))$
$\omega([X_a, X_b], X_1, \dots, \hat{X}_a, \dots, \hat{X}_b, \dots, X_{r+1})$
$- X_b(X_a f)$

Notation: $\Omega^r(M)$ is the set of r-forms on M.

$C^\infty(M)$-bilinear

Def: $\wedge : \Omega^r(M) \times \Omega^s(M) \longrightarrow \Omega^{r+s}(M)$

$\omega, \sigma \longmapsto \omega \wedge \sigma$

$(\omega_1 + \omega_2) \wedge \sigma = \omega_1 \wedge \sigma + \omega_2 \wedge \sigma$

$\omega \wedge (\sigma_1 + \sigma_2) = \omega \wedge \sigma_1 + \omega \wedge \sigma_2$

defined by

$$(\omega \wedge \sigma)(X_1, \dots, X_{r+s}) = \frac{1}{r!}\frac{1}{s!} \sum_{\pi \in \mathrm{perm}(r+s)} (-1)^\pi (\omega \otimes \sigma)\left(X_{\pi(1)}, \dots, X_{\pi(r+s)}\right)$$

Ex: I

$\alpha \wedge \beta =$

Note $(\alpha \wedge$

Prop: (

Ex: If $\alpha, \beta \in \Omega^1(M)$,

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$$

Note $$(\alpha \wedge \beta)(X_1, X_2) = \begin{vmatrix} \alpha(X_1) & \alpha(X_2) \\ \beta(X_1) & \beta(X_2) \end{vmatrix}$$

Prop: $(\omega \wedge \sigma) = (-1)^{rs} (\sigma \wedge \omega)$

$f \in C^\infty(M)$

$$(f\omega \wedge \sigma) = f(\omega \wedge \sigma)$$
$$= \omega \wedge (f\sigma)$$

defined by

$$(\omega \wedge \sigma)(X_1, \ldots, X_{r+s}) = \frac{1}{r!}\frac{1}{s!} \sum_{\pi \in \mathrm{perm}(r+s)} (-1)^{\pi} (\omega \otimes \sigma)(X_{\pi(1)}, \ldots, X_{\pi(r+s)})$$

$+\,\omega \wedge \sigma_2$

Ex: 2 of Maxwell's equations are $dF = 0$

Recall that if $f \in C^\infty(M)$, the differential of f is

$$d : C^\infty(M) \longrightarrow \Gamma(T^*M) = \Omega^1(M)$$

$$\Omega^0(M) = C^\infty(M), \qquad f \longmapsto df$$

$$(df)(X) := Xf$$

- $d\omega$ is antisym, $C^\infty(M)$-multilinear
- $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^r \omega \wedge d\sigma$ (ω: r-form, σ: s-form)

More generally,

$$d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$$

by

$$d\omega(X_1, \dots, X_{r+1}) = \sum_{a=1}^{r+1} (-1)^{a+1} X_a \big(\omega \dots$$

$$+ \sum_{a<b} (-1)^{a+b} \omega\big([X_a, X_b], \dots$$

$$[X_a, X_b] f := X_a(X_b f) - X_b(X_a \dots$$

$$= \omega_\wedge \sigma_1 + \omega_\wedge \sigma_2$$

defined by

$$(\omega_\wedge \sigma)(X_1, \dots, X_{r+s}) = \frac{1}{r!}\frac{1}{s!} \sum_{\pi \in \text{perm}(r+s)} (-1)^\pi \, (\omega \otimes \sigma)(X_{\pi(1)}, \dots, X_{\pi(r+s)})$$

Prop

Ex: 2 of Maxwell's equations are $dF = 0$

$$F = F_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$= \frac{1}{2} F_{\mu\nu} \left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \right)$$

$$= \frac{1}{2} F_{\mu\nu}\, dx^\mu {}_\wedge dx^\nu$$

A general r-form can be written in basis

$$\omega = \frac{1}{r!}\, \omega_{\mu_1 \cdots \mu_r}(x) \underbrace{dx^{\mu_1} {}_\wedge \cdots {}_\wedge dx^{\mu_r}}_{\text{basis of 1-forms}}$$

(summation implied)

$\to$ F can be written as $F = dA$ on $\mathbb{R}^4$.

What if not $\mathbb{R}^4$? Not always true!

r+s)

in basis
of 1-forms

$\ldots \wedge dx^{\mu_r}$

tion implied

on $\mathbb{R}^4$.

true!

1.4. de Rham Cohomology

Def: A form $\omega \in \Omega^r(M)$ is

- closed if $d\omega = 0$
- exact if $\omega = d\sigma$, $\exists \sigma \in \Omega^{r-1}(M)$

Prop: $d^2 = 0$

Proof: Let $\omega \in \Omega^r(M)$.

In local coord.,

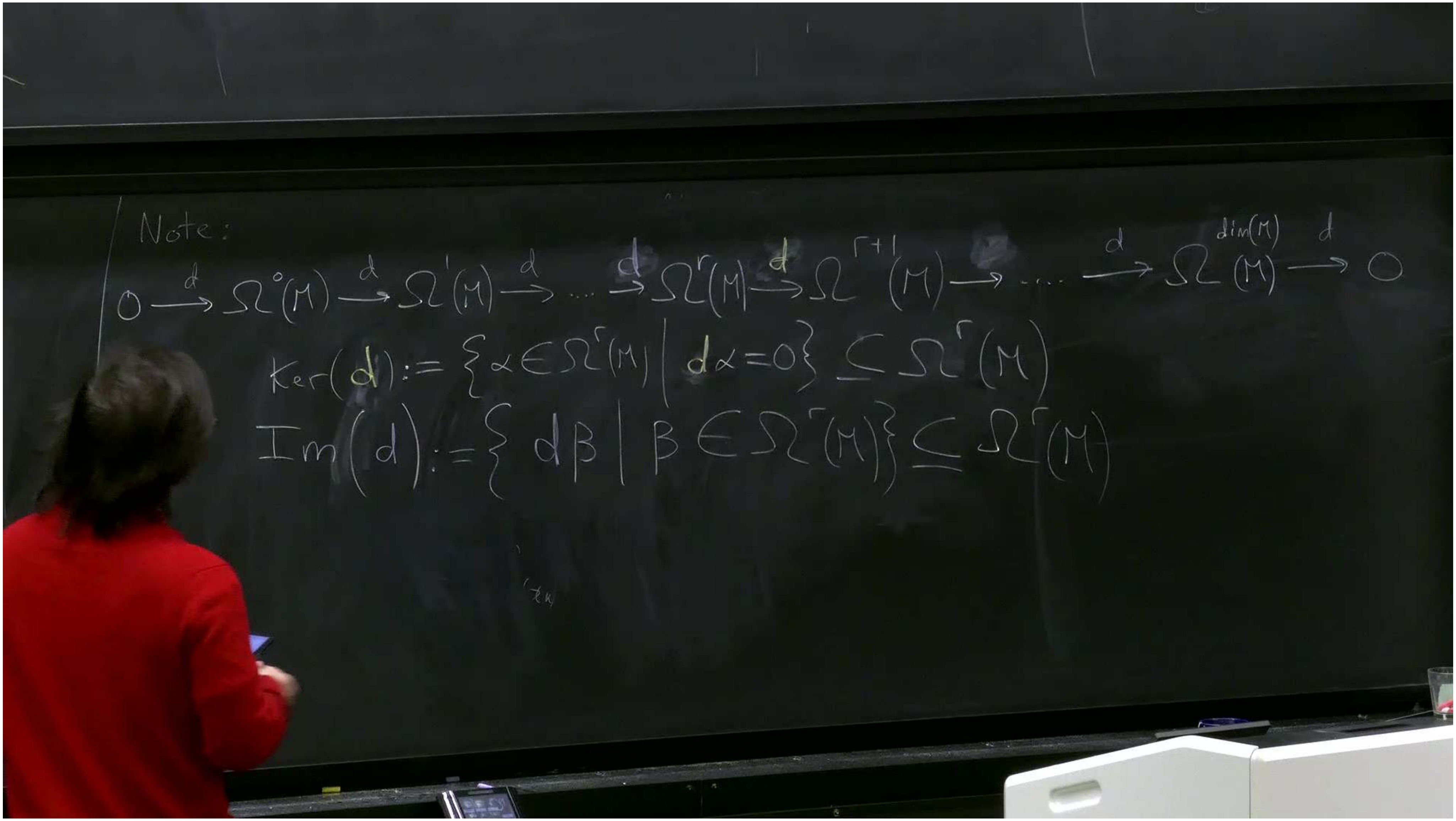
$$d\omega = \frac{1}{r!} \frac{\partial}{\partial x^\nu} \omega_{\mu_1 \cdots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

$$d(d\omega) = \frac{1}{r!} \underbrace{\frac{\partial}{\partial x^\gamma} \frac{\partial}{\partial x^\nu}}_{\text{symmetric}} \omega_{\mu_1, \cdots, \mu_r} \overbrace{dx^\gamma \wedge dx^\nu}^{\text{antisym}} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

$$= 0$$

Note:

$$0 \longrightarrow \Omega^0(M) \longrightarrow \Omega^1(M) \longrightarrow \cdots$$


Note:
$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r(M) \xrightarrow{d} \Omega^{r+1}(M) \longrightarrow \cdots \xrightarrow{d} \Omega^{\dim(M)}(M) \xrightarrow{d} 0$
$\mathrm{Ker}(d) := \{\alpha \in \Omega^r(M) \mid d\alpha = 0\} \subseteq \Omega^r(M)$
$\mathrm{Im}(d) := \{d\beta \mid \beta \in \Omega^r(M)\} \subseteq \Omega^r(M)$

$$(\omega \wedge \sigma)(X_1, \ldots, X_{r+s}) = \frac{1}{r!}\frac{1}{s!}\sum_{\pi \in \mathrm{perm}(r+s)} (-1)^{\pi} (\omega \otimes \sigma)\left(X_{\pi(1)}, \ldots, \pi_{(r+s)}\right)$$

Prop:

Then $d^2 = 0 \iff \mathrm{Im}(d) \subseteq \mathrm{Ker}(d)$ $\quad (d(d\beta) = 0)$

$\iff$ "exact forms are closed" but are closed forms exact?

Def

Poincaré Lemma: If $M = \mathbb{R}^n$, then closed forms are exact $(\mathrm{Im}(d) = \mathrm{Ker}(d$

Notations: $Z^r(M) = \{\omega \in \Omega^r(M) \mid d\omega = 0\}$ is the r^{th} cocycle group

$B^r(M) = \{\omega \in \Omega^r(M) \mid \omega = d\beta,\ \beta \in \Omega^{r-1}(M)\}$ is the r^{th} coboundar

$Z \quad r$

defined by

$$(\omega\wedge\sigma)(X_1,\dots,X_{r+s}) = \frac{1}{r!}\frac{1}{s!}\sum_{\pi\in \mathrm{perm}(r+s)} (-1)^{\pi}\,(\omega\otimes\sigma)(X_{\pi(1)},\dots,X_{\pi(r+s)})$$

$-\omega\wedge\sigma_1+\omega\wedge\sigma_2$

Prop: $(\omega\wedge\sigma)=(-1)^{rs}($

Then $d^2=0 \iff \mathrm{Im}(d)\subseteq \mathrm{Ker}(d)$ $\quad (d(d\beta)=0)$

$\iff$ "exact forms are closed" but are closed forms exact?

Poincaré Lemma: If $M=\mathbb{R}^n$, then closed forms are exact $(\mathrm{Im}(d)=\mathrm{Ker}(d))$

Notations: $Z^r(M)=\{\omega\in\Omega^r(M)\mid d\omega=0\}$ is the r^{th} cocycle group

$B^r(M)=\{\omega\in\Omega^r(M)\mid \omega=d\beta,\ \beta\in\Omega^{r-1}(M)\}$ is the r^{th} coboundary group

1.4. de Rham Co

Def: A form $\omega\in$

- closed if
- exact if

$$d^2=0 \Longleftrightarrow B^r(M) \subseteq Z^r(M)$$

Def: The r^{th} de Rham COHOMOLOGY group is the vector space

$$H^r(M) := Z^r(M) / B^r(M)$$

2 closed forms ω, σ are equivalent if

$$\omega = \sigma + d(\cdot)$$

bilinear

$$(\omega_1+\omega_2)\wedge\sigma = \omega_1\wedge\sigma + \omega_2\wedge\sigma$$

$$\omega\wedge(\sigma_1+\sigma_2) \quad \omega\wedge\sigma_1 + \omega\wedge\sigma_2$$

$$\left(X_{\pi(1),\dots,\pi(r+s)} \right)$$

$$d\omega = \frac{1}{r!}\frac{\partial}{\partial x^\nu}\omega_{\mu_1\cdots\mu_r}\, dx^\nu \wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}$$

$$d(d\omega) = \frac{1}{r!}\underbrace{\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial x^\nu}}_{\text{symmetric}}\omega_{\mu_1\cdots\mu_r}\,\overbrace{dx^\gamma\wedge dx^\nu}^{\text{antisym}}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}$$

$$= 0$$

$$d^2 x^\mu = 0$$

Then

$$d(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^r\,\omega\wedge d\sigma$$

$$dF = \frac{\partial F}{\partial x^\nu}dx^\nu$$