Title: Lecture - Mathematical Physics I (Core), PHYS 777

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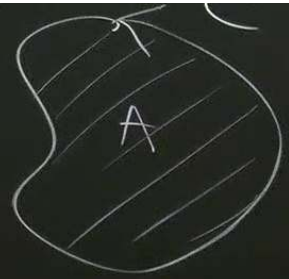
1. Homology and Cohomology

1.1 Simplicial complexes and Chains

On $\mathbb{R}^3$, $\oint_C \vec{F} \cdot d\vec{s} = \iint_A (\nabla \times \vec{F})\, dA$

Simplicial Complexes and Chains

On $\mathbb{R}^3$, $\oint_C \vec{F} \cdot d\vec{s} = \iint_A (\nabla \times \vec{F})\, dA$



On 2-torus T^2,

$\rightarrow C' = \partial$(some area)

$C \neq \partial$(some area)

Def: An r-simplex σ_r is a subset of $\mathbb{R}^{n+1}$ made out of $r+1$ points $P_0, P_1, P_2, \dots, P_r$ (NOT colinear) s.t.: $(r \leq n)$

$$\sigma_r := \langle P_0 P_1 \cdots P_r \rangle = \left\{ \sum_{i=0}^{r} a_i P_i \;\middle|\; a_i \geq 0, \sum_{i=0}^{r} a_i = 1 \right\}$$

Examples

- 0-simplex $\langle P_0 \rangle =$ P_0
- 1-simplex: $\langle P_0 P_1 \rangle =$

- 2-simplex: $\langle P_0 P_1 P_2 \rangle =$

If $\sigma_r = \langle p_0 \dots p_r \rangle$, then

$\sigma_s = \langle p_{i_0} \dots p_{i_s} \rangle$ for $s \leq r$

is called a s-face of σ_r.

Example:

$\sigma_3 =$

1 3-Face (tetrahedron itself)

4 2-faces (triangles)

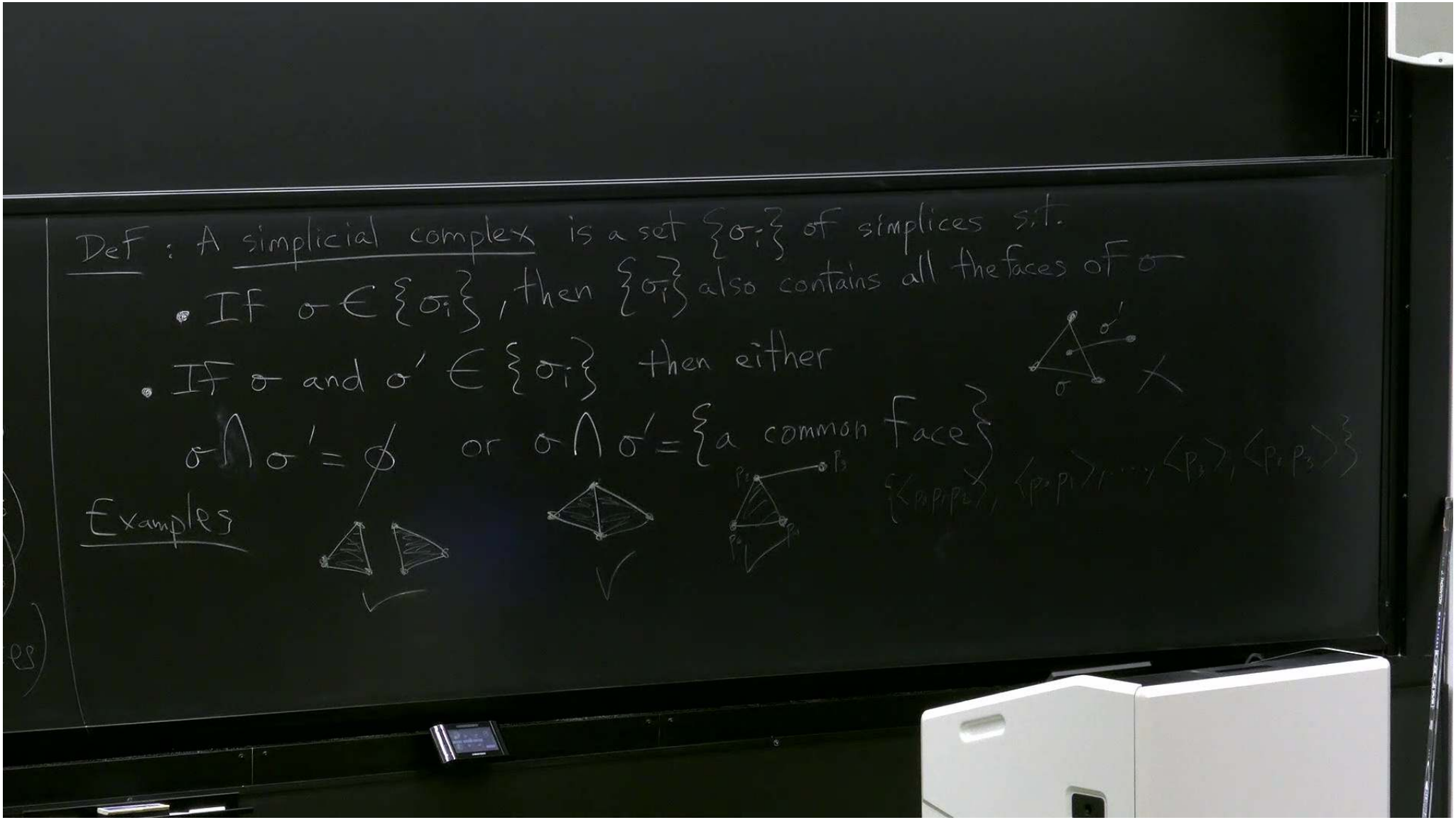
6 1-faces (edges)

4 0-faces (vertices)

Def: A simplicial complex

- If $\sigma \in \{\sigma_i\}$,
- If σ and $\sigma' \in$

$\sigma \cap \sigma' = \emptyset$ or

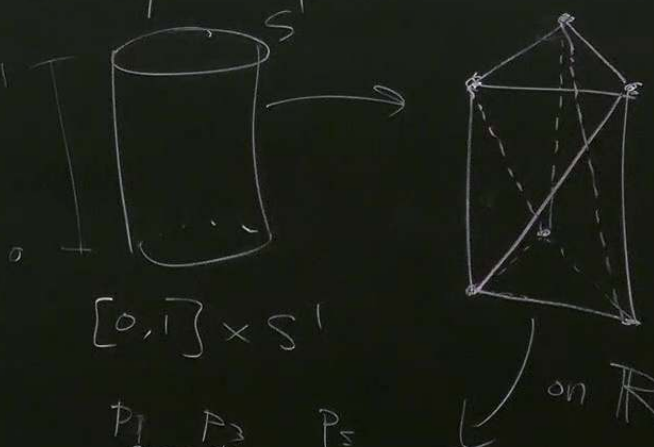

Def: A simplicial complex is a set $\{\sigma_i\}$ of simplices s.t.
• If $\sigma \in \{\sigma_i\}$, then $\{\sigma_i\}$ also contains all the faces of σ
• If σ and $\sigma' \in \{\sigma_i\}$ then either
$\sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma' = \{$a common face$\}$
Examples

A (topological) manifold M is called triangulable if $\exists$ a homeomorphism

$$f : \bigcup_{\sigma_i \in \{\sigma_i\}} \sigma_i \longrightarrow M$$

Ex: S^2 homeo

• cylinder



So far, $\langle p_0 p_1 \rangle =$ is unoriented

Now introduce an orientation:

$(p_0 p_1) =$ $(p_1 p_0) =$ $= -(p_0 p_1)$

In general, $(p_0 \cdots p_r) = (-1)^{|\pi|} (p_{\pi_1} \cdots p_{\pi_r})$

Let K be an oriented simplicial complex $\{\sigma_i\}$.

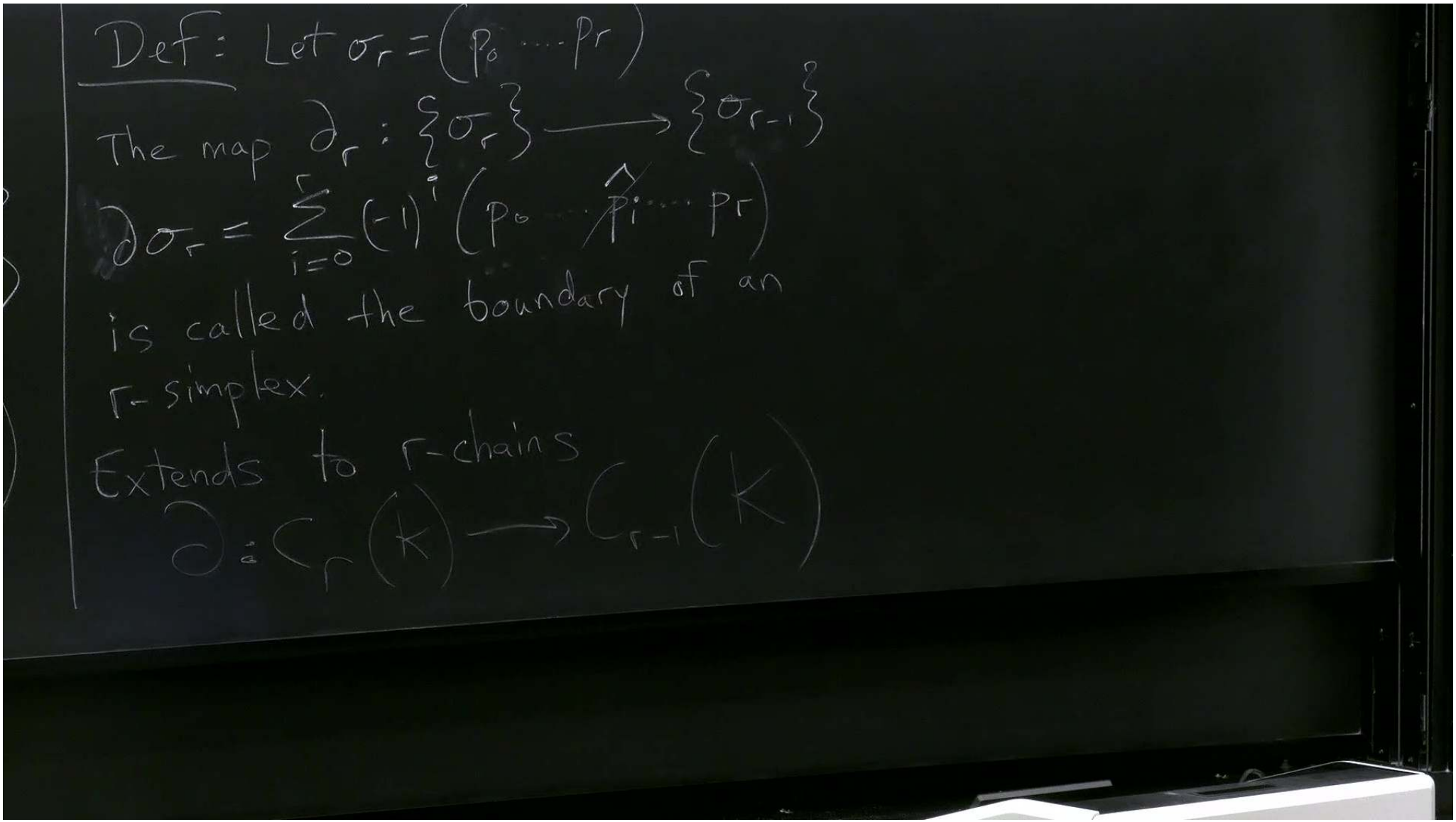
Def: The r-th chain group $C_r(K, \mathbb{Z})$ is the set

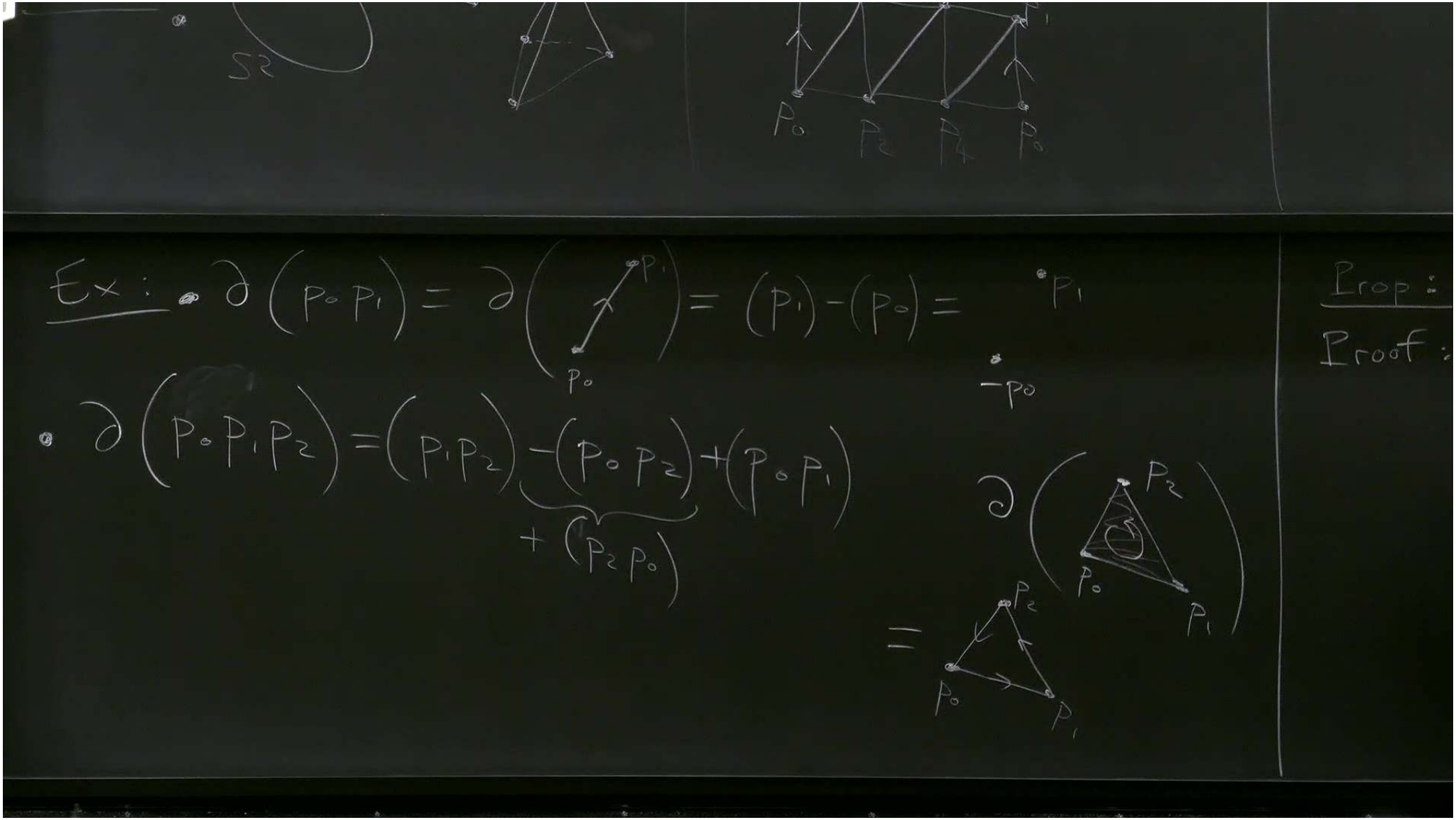
$$\left\{ \sum_{i=1}^{|\{\sigma_{r_i}\}|} n_i \sigma_{r_i} \;\middle|\; n_i \in \mathbb{Z},\ \sigma_{r_i} \in \{\sigma_{r_i}\} \right\}$$

$c \in \{\dots\}$ is called an r-chain.

Ex:

$c = (P_0 P_1) + (P_2 P_3) + (P_3 P_4)$ is a 1-chain


Def: Let $\sigma_r = (p_0 \cdots p_r)$
The map $\partial_r : \{\sigma_r\} \longrightarrow \{\sigma_{r-1}\}$
$\partial\sigma_r = \sum_{i=0}^{r} (-1)^i (p_0 \cdots \hat{p_i} \cdots p_r)$
is called the boundary of an
r-simplex.
Extends to r-chains
$\partial : C_r(K) \longrightarrow C_{r-1}(K)$


S2
P0
P2
P4
P6
Ex:
∂(P0 P1) = ∂(P1 / P0) = (P1) − (P0) =
P1
−P0
∂(P0 P1 P2) = (P1 P2) − (P0 P2) + (P0 P1)
+ (P2 P0)
∂
P2
P0
P1
=
P2
P0
P1
Prop:
Proof:

$(p_{\pi_1} \cdots p_{\pi_{r+1}})$

π is a permutation

Prop: $\partial^2 = 0$

Proof: $\partial(p_0 \cdots p_{r+1}) = \sum_{i=0}^{r+1} (-1)^i (p_0 \cdots \hat{p_i} \cdots p_{r+1})$

$$\partial\partial(p_0 \cdots p_{r+1}) = \sum_{i=0}^{r+1} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j (p_0 \cdots \hat{p_j} \cdots \hat{p_i} \cdots p_{r+1}) + \sum_{j=i+1}^{r+1} (-1)^{j-1} (p_0 \cdots \hat{p_i} \cdots \hat{p_j} \cdots p_{r+1}) \right]$$

$= \cdots = 0$

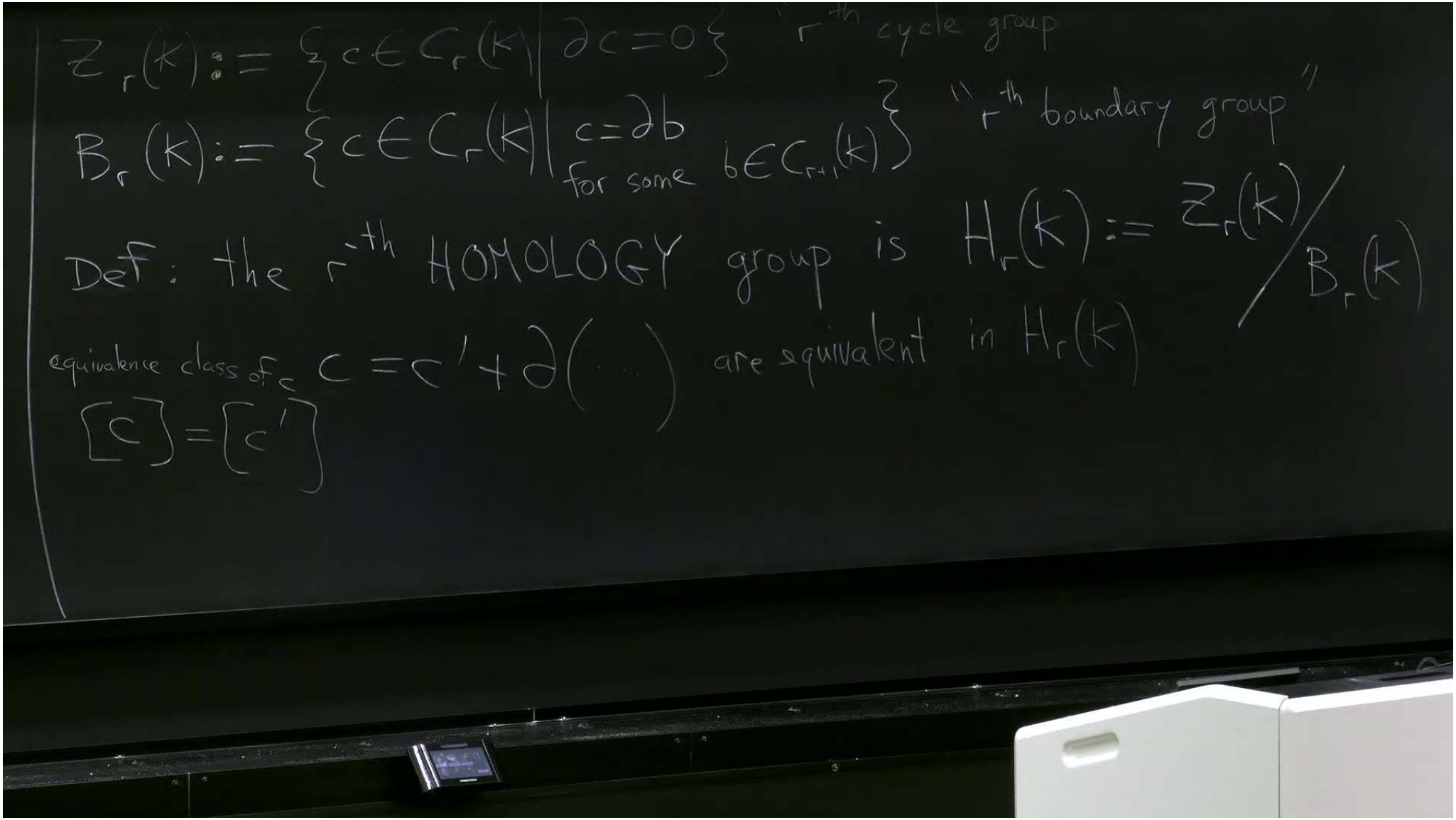
p_0 p_2 is a 1-chain

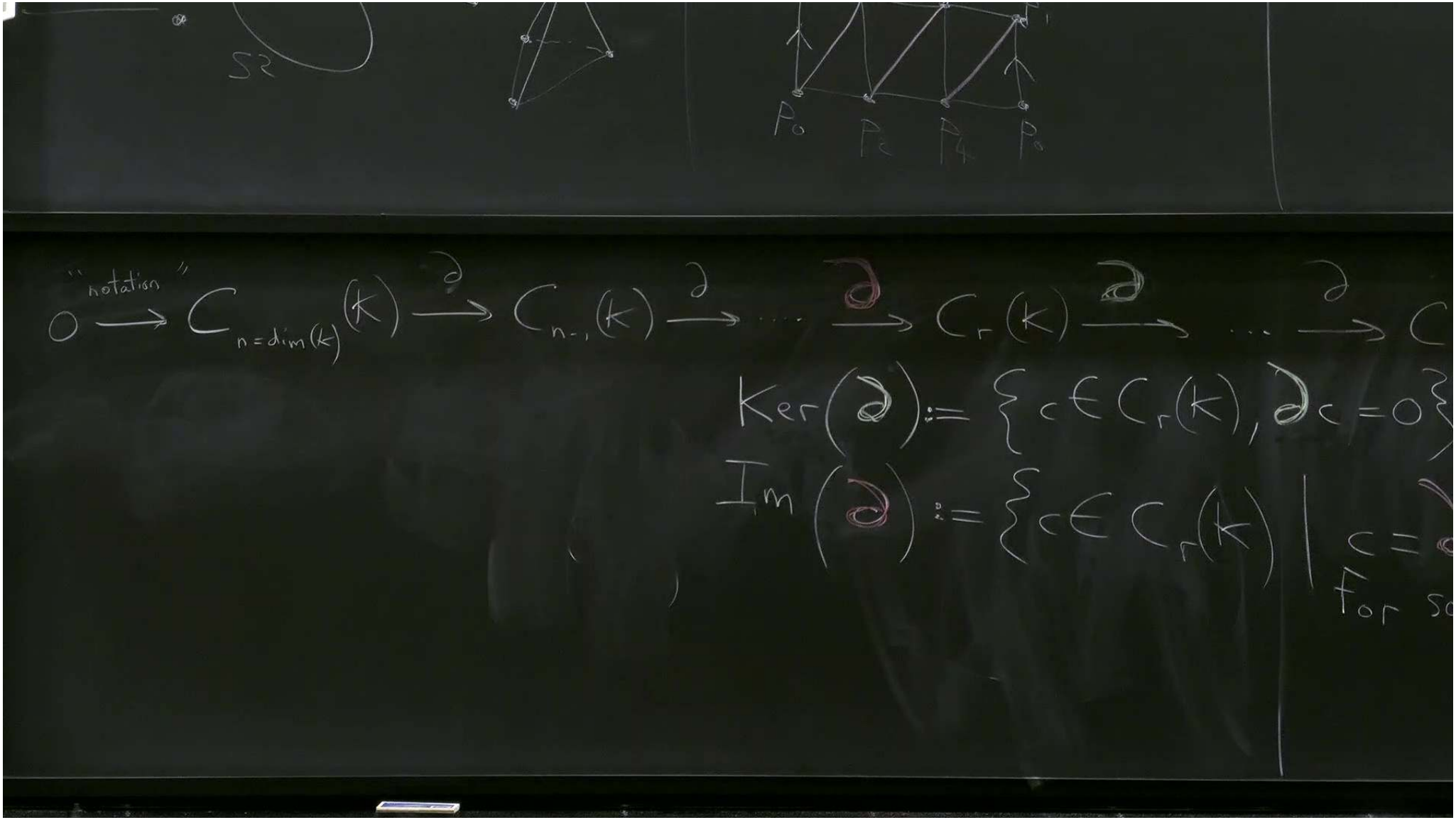
So $\partial^2 = 0$ on simplices,

So $\partial^2 = 0$ on r-chains

Def: An r-chain $c \in C_r(K)$ is called

- An r-cycle if $\partial c = 0$
- An r-boundary if $\exists b \in C_{r+1}(K)$ s.t. $c = \partial b$


$Z_r(k) := \{c \in C_r(k) \mid \partial c = 0\}$ "r^{th} cycle group"
$B_r(k) := \{c \in C_r(k) \mid c = \partial b \text{ for some } b \in C_{r+1}(k)\}$ "r^{th} boundary group"
Def: the r^{th} HOMOLOGY group is $H_r(k) := Z_r(k)/B_r(k)$
equivalence class of c $c = c' + \partial(\ldots)$ are equivalent in $H_r(k)$
$[c] = [c']$


S2
P0
P2
P4
P6
"notation"
$0 \longrightarrow C_{n=\dim(K)}(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_r(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C$
$\ker(\partial) := \{c \in C_r(K), \partial c = 0\}$
$\mathrm{Im}(\partial) := \{c \in C_r(K) \mid c =$
for so

π is a permutation

$\partial \to C_0(K) \xrightarrow{\partial} 0$

$\{c = 0\} \subseteq C_r(K)$

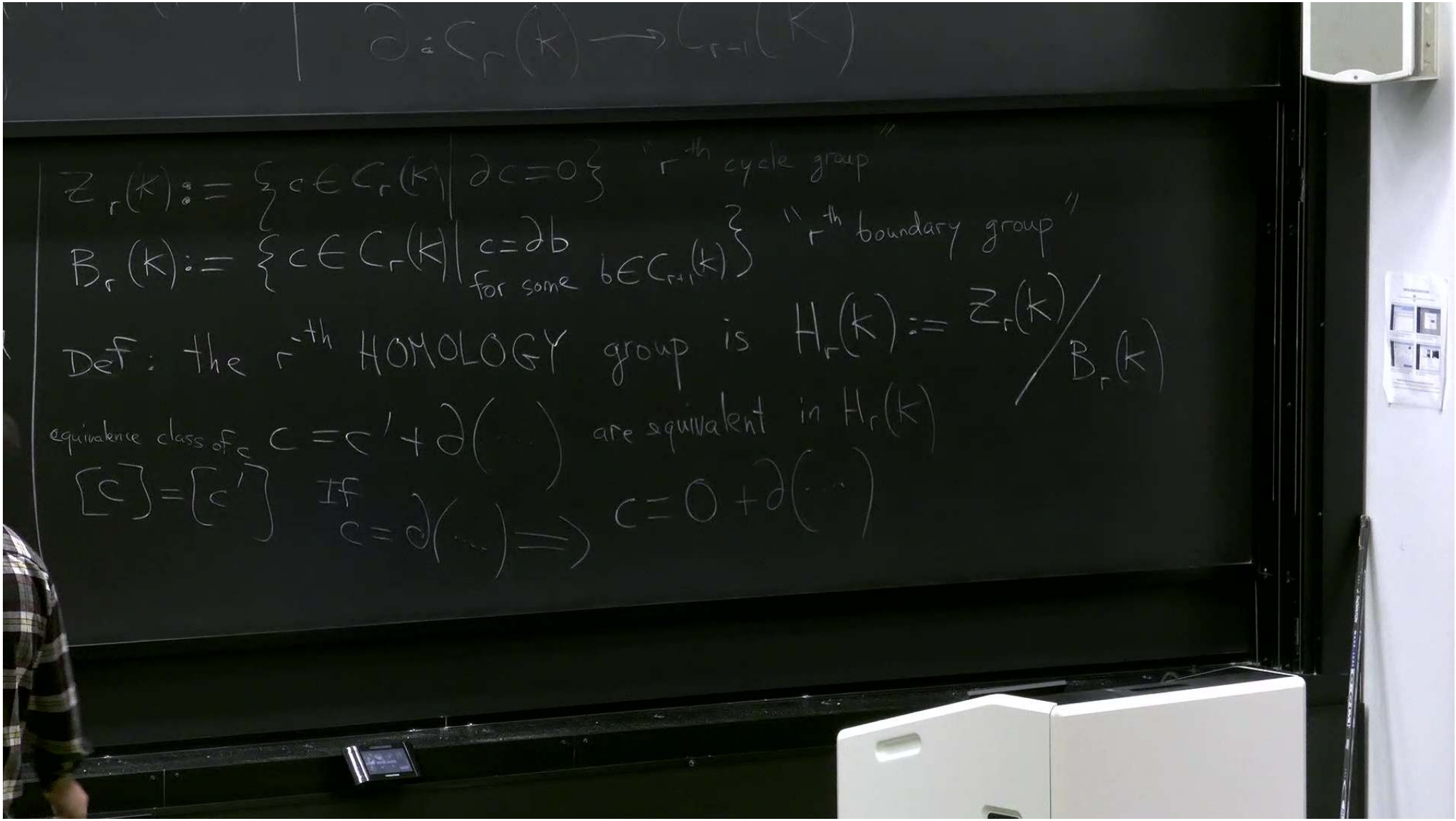
$\{c = \partial b \text{ for some } b \in C_{r+1}(K)\} \subseteq C_r(K)$

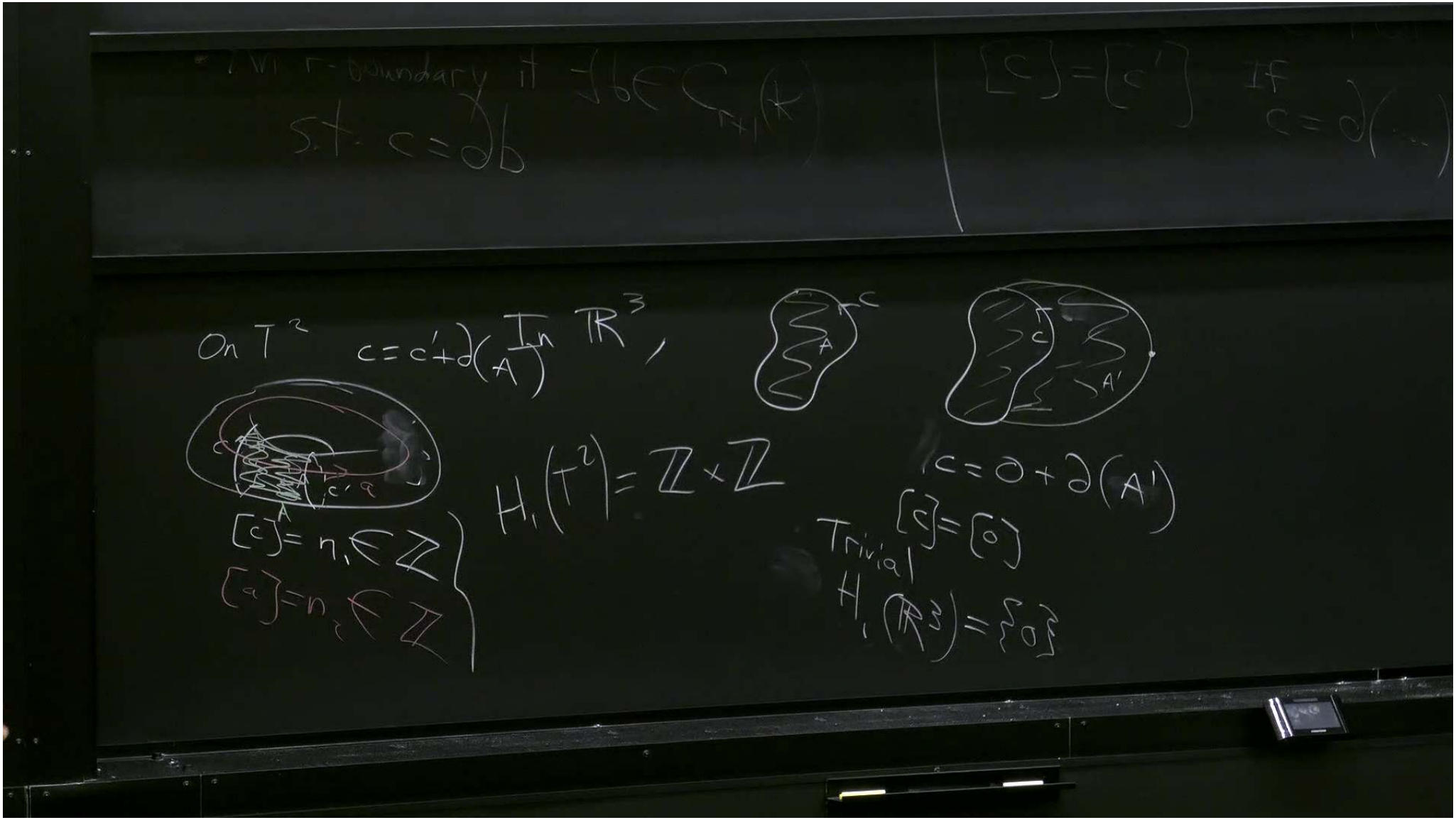
$\partial^2 = 0 \quad (\partial c = \partial(\partial b) = \partial^2 b = 0)$

$\iff \mathrm{Im}(\partial) \subseteq \ker(\partial)$

$\iff$ "boundaries are cycles"

Theorem: $H_r(K)$ are topological invariants


$\partial : C_r(K) \longrightarrow C_{r-1}(K)$
$Z_r(K) := \{c \in C_r(K) \mid \partial c = 0\}$ "r^{th} cycle group"
$B_r(K) := \{c \in C_r(K) \mid c = \partial b \text{ for some } b \in C_{r+1}(K)\}$ "r^{th} boundary group"
Def: the r^{th} HOMOLOGY group is $H_r(K) := Z_r(K) / B_r(K)$
equivalence class of c $c = c' + \partial(\dots)$ are equivalent in $H_r(K)$
$[c] = [c']$ If $c = \partial(\dots) \Rightarrow c = 0 + \partial(\dots)$


s.t. $c = \partial b$
$[c] = [c']$ If $c = \partial(\ldots)$
On T^2 $c = c' + \partial(A)$ In $\mathbb{R}^3$,
$H_1(T^2) = \mathbb{Z} \times \mathbb{Z}$
$[c] = n_1 \in \mathbb{Z}$
$[a] = n_2 \in \mathbb{Z}$
$c = 0 + \partial(A')$
Trivial
$[c] = [0]$
$H_1(\mathbb{R}^3) = \{0\}$