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Simultaneous Measurements of Noncommuting Observables. Positive Transformations and Instrumental Lie Groups

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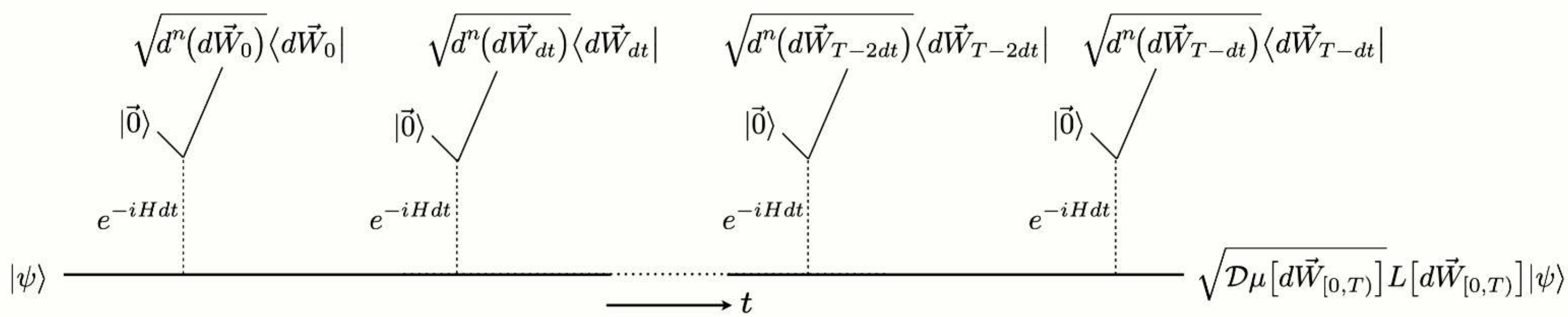
We formulate a general program for describing and analyzing continuous, differential weak, simultaneous measurements of noncommuting observables, which focuses on describing the measuring instrument *autonomously*, without states. The Kraus operators of such measuring processes are time-ordered products of fundamental *differential positive transformations*, which generate nonunitary transformation groups that we call *instrumental Lie groups*. The temporal evolution of the instrument is equivalent to the diffusion of a *Kraus-operator distribution function* defined relative to the invariant measure of the instrumental Lie group; the diffusion can be analyzed by Wiener path integration, stochastic differential equations, or a Fokker-Planck-Kolmogorov equation. This way of considering instrument evolution we call the *Instrument Manifold Program*. We relate the Instrument Manifold Program to state-based stochastic master equations. We then explain how the Instrument Manifold Program can be used to describe instrument evolution in terms of a universal cover we call the universal instrumental Lie group, which is independent not just of states, but also of Hilbert space. The universal instrument is generically infinite dimensional, in which situation the instrument's evolution is *chaotic*. Special simultaneous measurements have a finite-dimensional universal instrument, in which situation the instrument is considered to be *principal* and can be analyzed within the differential geometry of the universal instrumental Lie group. Principal instruments belong at the foundation of quantum mechanics. We consider the three most fundamental examples: measurement of a single observable, of position and momentum, and of the three components of angular momentum. As these measurements are performed continuously, they limit to strong simultaneous measurements. For a single observable, this gives the standard decay of co-



'IG. 1. Schematic of a sequence of indirect, differential weak measurements; full understanding comes after reading Secs. II A nd II B. A system in a state $|\psi\rangle$ is indirectly measured by a sequence of weak interactions $e^{-iH\,dt}$, where each set of meters is obɜrved after its interaction; that is, the system is continuously monitored. The incremental Kraus operator for the measurement t time t, given outcomes $d\vec{W}_t$, is $\sqrt{d^n(d\vec{W}_t)}\langle d\vec{W}_t|e^{-iH\,dt}|\vec{0}\,\rangle$; under the conditions outlined in Sec. II A, this Kraus operator ; the differential positive transformation of Eq. (1.1), that is, $\sqrt{d\mu(d\vec{W}_t)}\,L_{\vec{X}}(d\vec{W}_t)$, with $L_{\vec{X}}(d\vec{W}_t) = e^{-\vec{X}^2\kappa\,dt+\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t}$. The ıcremental Kraus operators "pile up" to become, at time T, the overall Kraus operator $\sqrt{\mathcal{D}\mu[d\vec{W}_{[0,T)}]}\,L[d\vec{W}_{[0,T)}]$, which is ritten as a time-ordered exponential in Eq. (1.2). The overall Kraus operator gives the unnormalized final state at time T, as hown in the figure. The collection of Kraus operators at time T, for all Wiener outcome paths $d\vec{W}_{[0,T)}$, defines an *instrument*, hich can be analyzed on its own, independent of system states—simply omit $|\psi\rangle$ from the figure—a style of analysis we call *ıstrument autonomy*. The Kraus operators move across the manifold of an *instrumental Lie group*, which is generated by the ıeasured observables; placing the instrument within its instrumental Lie group and analyzing its evolution there we call the *nstrument Manifold Program*.

with rate κ, $\vec{X}^2 = \vec{X}\cdot\vec{X}$, and $d\vec{W}_t = (dW_t^1, \ldots, dW_t^n)$ is the conjugate n-tuple of Wiener outcome increments that are registered by weak measurements. These differential positive transformations "pile up" as successive measurements are performed; at time T the instrument is the collection of Kraus operators,

$$\left\{ L\big[d\vec{W}_{[0,T)}\big] = \mathcal{T}\exp\bigg(\int_0^{T-dt} -\vec{X}^2\kappa\, dt + \vec{X}\cdot\sqrt{\kappa}\, d\vec{W}_t \bigg) \;:\; d\vec{W}_{[0,T)} \text{ is a Wiener path} \right\}, \tag{1.2}$$

where $\mathcal{T}$ denotes a time-ordered exponential. This scenario of piling up incremental Kraus operators is sketched in Fig. 1. These instruments are contained in the Lie group G infinitesimally generated by the measured observables, $\{X_1, \ldots, X_n\}$, and the quadratic term $\vec{X}^2$. We call G the *instrumental Lie group.* At every time T, the instrument (1.2) is equivalent to a *Kraus-operator distribution function*,

$$D_T(L) \equiv \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\; \delta\big(L, L\big[d\vec{W}_{[0,T)}\big]\big)\,, \tag{1.3}$$

where $\mathcal{D}\mu[d\vec{W}_{[0,T)}]$ is the Wiener path measure and $\delta\big(L, L\big[d\vec{W}_{[0,T)}\big]\big)$ is a Dirac δ-function with respect to the left-invariant measure of G. The Kraus-operator distribution function describes how the instrument is distributed in the instrumental Lie group. The Markovianity or group property of the instrument,

$$L[d\vec{W}_{[0,t+dt)}] = L(d\vec{W}_t)L[d\vec{W}_{[0,t)}]\,, \tag{1.4}$$

means that the Kraus-operator distribution function evolves according to a Fokker-Planck-Kolmogorov equation,

$$\frac{1}{\kappa}\frac{\partial}{\partial t}D_t(L) = \bigg(\underset{\longleftarrow}{\vec{X}}^2 + \frac{1}{2}\sum_i \underset{\longleftarrow}{X_\mu}\underset{\longleftarrow}{X_\mu} \bigg)[D_T](L)\,, \tag{1.5}$$

where $\underset{\longleftarrow}{X}$ denotes a right-invariant derivative, operation of the SP QM process is a Wiener-like path integral,

$$\mathcal{Z}_T \equiv \int \mathcal{D}\mu\left[dw_{[0,T)}\right] \mathcal{O}\Big(L\left[dw_{[0,T)}\right]\Big)\,,$$

olutely trivial to solve,

$$\mathcal{Z}_T = \left(\int d\mu(dw)\, \mathcal{O}\left(e^{-H_\mathrm{o} 2\kappa dt + Q\sqrt{\kappa}dW_t^q + P\sqrt{\kappa}dW_t^p}\right)\right)^{\circ T/dt} = e^{-\frac{1}{2}\kappa T(\mathrm{a}}$$

of this article, however is entirely in the manifold diffusion process defined b
d to define sample-paths in a finite-dimensional manifold. The infinitesima
P, and H_o. By simply considering their Lie brackets to first order,

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$$[Q, P] = i\Omega\,,$$
$$[H_\mathrm{o}, Q] = -iP\,,$$
$$[H_\mathrm{o}, P] = iQ\,,$$

$$\big[[H_\mathrm{o}, Q], Q\big] = -\Omega\,,$$
$$\big[[H_\mathrm{o}, P], P\big] = -\Omega\,,$$
$$\big[H_\mathrm{o}, [H_\mathrm{o}, Q]\big] = Q\,,$$
$$\big[H_\mathrm{o}, [H_\mathrm{o}, P]\big] = P\,,$$

A. Differential weak measurements and incremental Kraus operators

A differential weak measurement of multiple observables is made by doing a sequence of indirect weak measurements of the several observables; these indirect measurements are implemented by coupling independent Gaussian meters to the system, one for each observable. We call this a “differential weak measurement” because the Kraus operators are differentially close to the identity; these incremental Kraus operators can then be regarded as fundamental, infinitesimally generated *differential positive transformations* of a differentiable manifold. Although a differential measurement is definitely weak, there are measurements that are generally construed as weak but that have some

Kraus operators that are not close to the identity (e.g., jump processes). The “weak” in differential weak measurement is thus both insufficient by itself and unnecessary when preceded by “differential”; it is included to throw a lifeline to conventional usage.

The key accomplishment of this section is to show that at the level of differential weak measurements, the commutators of the observables can be ignored, so there is no temporal order to the measurements of the several observables and these measurement can be regarded as occurring *simultaneously*.

$$L_X(dW) \equiv e^{X\sqrt{\kappa}\,dW - X^2\kappa\,dt}\,, \tag{2.8}$$

brings the Kraus operator (2.4) into the form

$$\sqrt{dq}\,\langle q|e^{-iH\,dt/\hbar}|0\rangle = \sqrt{d\mu(dW)}\,L_X(dW)\,. \tag{2.9}$$

The (completely positive) superoperator for outcome dW,

$$d\mathcal{Z}_X(dW) = d\mu(dW)\,L_X(dW)\odot L_X(dW)^\dagger\,, \tag{2.10}$$

we call an *instrument element*. We stress that the outcome increment dW is literally the outcome of the measurement, scaled to have a variance dt. We also note that the exponential expressions here are exact in the sense that they hold even when dt is not infinitesimal. The set of instrument elements corresponding to all outcomes is the *instrument* [31, 32, 81]. Here we also introduce the "odot" ($\odot$) notation [82–84] for a superoperator, defined by

$$A\odot B^\dagger(C) = ACB^\dagger\,. \tag{2.11}$$

The $\odot$ is literally a tensor product, but if one doesn't want to think about that, one can think of the $\odot$ as just a placeholder for an operator on which the superoperator acts. We say a bit more about the odot notation below.

Integrating the instrument elements over outcomes gives the (unconditional) *quantum operation* associated with the instrument,

$$\begin{aligned}\mathcal{Z}_{X,dt} &\equiv \int d\mathcal{Z}_X(dW)\\ &= \int d\mu(dW)\,L_X(dW)\odot L_X(dW)^\dagger\end{aligned}$$

freedom was used by Gross *et al.* [86], who replaced Gaussian meters with qubit meters in a state-based formulation of continuous measurements. The conditions for the emergence of Gaussian behavior should rightly be the subject of further investigation.

The incremental Kraus operators (2.29) and the overall Kraus operators (2.41) were derived above from a meter model in which a measurement of position, a continuous variable, is made on each of the meters; von Neumann essentially introduced this meter model and called it an indirect measurement [9]. We ask the reader now to join us in a **shift in perspective**, the **first** of three: *regard the incremental Kraus operators for simultaneous measurements of noncommuting observables, $L_{dt} = e^{\delta}$ of (2.29), not as derived objects, but as the fundamental differential positive transformations, more fundamental in quantum measurement theory than von Neumann projectors. The forward generator δ plays the role for positive transformations that anti-Hermitian Hamiltonian generators, $-iH\,dt$, play in generating unitary transformations.* Continuously measuring commuting observables leads, over time, to von Neumann's original conception of eigenstates of Hermitian operators as measurement outcomes. The perspectival shift is that Hermitian operators now play the more important role of generating positive transformations, acting via exponentiation of the forward generator δ to produce the incremental Kraus operators. For noncommuting observables, these incremental Kraus operators, piled up over time, lead to ... — well, that is the subject of the rest of this paper.

Although several researchers have hinted at or touched on the significance of positive transformations [38, 45, 47], especially those who work or comment on linear quantum trajectories [23, 25, 46, 48], none has had a complete understanding of how differential weak, simultaneous measurements lead to the differential positive transformations, $L_{dt} = e^{\delta}$ of (2.29), nor of how these transformations pile up to construct instrument manifolds.

The overall Kraus operator (2.41) is the solution to the SDE

$$\begin{aligned} dL_t\, L_t^{-1} &= L_{t+dt}L_t^{-1} - 1 \\ &= L_{\vec{X}}(d\vec{W}_t) - 1 \\ &= \delta_t + \frac{1}{2}\delta_t^2 \end{aligned} \tag{2.43}$$

$$[\underleftarrow{X}, \underleftarrow{Y}][f] = \left(([Y,X]L)_{jk} \frac{\partial}{\partial L_{jk}} + ([Y,X]L)^*_{jk} \frac{\partial}{\partial L^*_{jk}} \right) f = \underleftarrow{[Y,X]}[f] \,, \tag{2.75}$$

thus giving a commutator antihomomorphism,

$$[\underleftarrow{X}, \underleftarrow{Y}] = -\underleftarrow{[X,Y]} \,. \tag{2.76}$$

The right-invariant derivatives inherit the commutators of the path generators X and Y, with a minus sign coming from the right invariance. Although Eqs. (2.74)–(2.76) are instructive in showing how the commutators emerge as vector fields from their action on an arbitrary function, the relation (2.76) follows immediately from letting the derivatives act on linear functions, as in Eq. (2.73).

It is useful to appreciate that for left-invariant derivatives, defined by

$$\underrightarrow{X}[f](L) \equiv \frac{df(Le^{hX})}{dh}\bigg|_{h=0} = \lim_{h\to 0} \frac{f\big(Le^{hX}\big) - f(L)}{h} \,, \tag{2.77}$$

we have

$$\mathcal{D}p\big[d\vec{W}_{[0,T)}\big|\rho_0\big]\,\rho\big[d\vec{W}_{[0,T)}\big|\rho_0\big] = \mathcal{D}\mu[d\vec{W}_{[0,T)}]\,\tilde{\rho}\big[d\vec{W}_{[0,T)}\big|\rho_0\big]\,. \tag{2.96}$$

The last of these associates the two density operators with their conjugate measures, which is key to the unravelings we turn to now.

The Wiener differential unraveling (2.47) and the KOD unraveling (2.61) are state-independent unravelings of the unconditional quantum operation $\mathcal{Z}_T$. State-based unravelings start by applying $\mathcal{Z}_T$, in the form of these two unravelings, to the initial state to get an unconditional, normalized final state $\mathcal{Z}_T(\rho_0)$. For each unraveling, there are two ways to proceed, by using unnormalized or normalized states and their conjugate distributions. The result is four state-based unravelings:

$$\mathcal{Z}_T(\rho_0) = \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\; L[d\vec{W}_{[0,T)}]\rho_0 L[d\vec{W}_{[0,T)}]^\dagger = \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\;\tilde{\rho}\big[d\vec{W}_{[0,T)}\big|\rho_0\big]\,, \tag{2.97}$$

$$\mathcal{Z}_T(\rho_0) = \int \mathcal{D}p\big[d\vec{W}_{[0,T)}\big|\rho_0\big]\;\rho\big[d\vec{W}_{[0,T)}\big|\rho_0\big]\,, \tag{2.98}$$

$$\mathcal{Z}_T(\rho_0) = \int d\mu(L)\,D_T(L)\,L\rho_0 L^\dagger = \int d\mu(L)\,D_T(L)\,\tilde{\rho}(L|\rho_0)\,, \tag{2.99}$$

$$\mathcal{Z}_T(\rho_0) = \int d\mu(L)\,D_T(L)\;\mathrm{tr}(L^\dagger L\rho_0)\,\rho(L|\rho_0) = \int dp_T(L|\rho_0)\,\rho(L|\rho_0)\,. \tag{2.100}$$

The first two of these unravelings are differential and thus serve as the basis for developing SDEs for an evolving quantum state, a development we take up below. The first is a state-based version of the Wiener differential unraveling (2.47)—just put ρ_0 in place of the $\odot$; it gives rise to linear quantum trajectories and a linear SDE. The second unravels $\mathcal{Z}_T(\rho_0)$ into normalized states and thus leads to stochastic master equations; notable is that to get to the stochastic master equation, one must decompose into incremental time steps both the Born-rule measure $\mathcal{D}p\big[d\vec{W}_{[0,T)}\big|\rho_0\big]$ and the normalized state $\rho\big[d\vec{W}_{[0,T)}\big|\rho_0\big]$. We call this second unraveling, that of Eq. (2.98), the *Born-rule differential unraveling.*

The third and fourth unravelings are based on the KOD unraveling (2.61). The third is a direct expression of the KOD unraveling—just put ρ_0 in place of the $\odot$. It introduces an overall unnormalized linear state, preserving term $\vec{X}^2$. Piling up incremental Kraus operators leads to the overall Kraus operator $L_T = L[d\vec{W}_{[0,T)}]$, which is written as a time-ordered product in Eq. (2.41). This product can be reduced to a product of finitely many exponential factors, each of which, by the Magnus expansion [102, 103], has an argument given by a series of integrals of the operators $\{X_1, \ldots, X_n, \vec{X}^2\} = \{\vec{X}, \vec{X}^2\}$ and their successive commutators. This is to say that the overall Kraus operator is an element of the instrumental Lie group $G = e^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra generated by the set $\{\vec{X}, \vec{X}^2\}$. Below we get to the instrumental groups, both universal and $\mathcal{H}$-specific (or quantum), in two steps, which highlight the difference between the measured observables and the quadratic generator.

First, however, we bring forward some general properties, which are based on the fact that the real vector space $\mathfrak{g}$ is the direct sum of a subspace $\mathfrak{g}_{\mathsf{l}}$ of Hermitian generators and a subspace $\mathfrak{g}_{\mathsf{o}}$ of anti-Hermitian generators:

$$\mathfrak{g} = \mathfrak{g}_{\mathsf{o}} \oplus \mathfrak{g}_{\mathsf{l}} \,. \tag{2.126}$$

The two subspaces satisfy

$$[\mathfrak{g}_{\mathsf{o}}, \mathfrak{g}_{\mathsf{o}}] \subseteq \mathfrak{g}_{\mathsf{o}} \,, \tag{2.127}$$
$$[\mathfrak{g}_{\mathsf{o}}, \mathfrak{g}_{\mathsf{l}}] \subseteq \mathfrak{g}_{\mathsf{l}} \,, \tag{2.128}$$
$$[\mathfrak{g}_{\mathsf{l}}, \mathfrak{g}_{\mathsf{l}}] \subseteq \mathfrak{g}_{\mathsf{o}} \,, \tag{2.129}$$

thus identifying $\mathfrak{g}_{\mathsf{o}} \subset \mathfrak{g}$ as a Cartan pair. Equation (2.127) implies that $\mathfrak{g}_{\mathsf{o}}$ is a Lie subalgebra, which generates the subgroup $G_{\mathrm{o}} = e^{\mathfrak{g}_{\mathsf{o}}}$ of unitary transformations within G. In contrast, Eq. (2.129) indicates that the Hermitian subspace $\mathfrak{g}_{\mathsf{l}}$ is not a subalgebra; $\mathfrak{g}_{\mathsf{l}}$ generates the positive transformations, which are not a subgroup of G, but should be thought of as a base manifold $\mathcal{E}$ within G. The incremental Kraus operator $L_{dt} = e^{\delta}$ is a differential positive transformation, and the forward generator δ is an element of $\mathfrak{g}_{\mathsf{l}}$. Equation (2.128) says that unitary conjugation of a positive transformation gives another positive transformation; conjugation of the base manifold $\mathcal{E}$ by an element of the unitary subgroup G_{o} is a rotation of the base manifold.

The Kraus operators are points in the group manifold G and, at the same time, in the way of groups, they are also transformations of G. Any Kraus operator has a unique group-theoretic polar decomposition $L = W\sqrt{E}$, as in commutators in the iterative process (2.133) is within the *universal enveloping algebra* $U_{\mathfrak{f}}$ of the observable Lie algebra $\mathfrak{f}$ [54, 105]; this is the associative algebra that is free of constraints, *except* for the commutators coming from $\mathfrak{f}$. In general, when one works in the universal enveloping algebra, the iterations (2.133) do not close, so $\mathfrak{g} = \Delta^{(\infty)}$ is an infinite-dimensional Lie algebra, and the corresponding Lie group $G = e^{\mathfrak{g}}$ is also infinite dimensional. We call G the *universal instrumental Lie group.*

Working within $A_{\mathcal{H}}$ yields a $\mathcal{H}$-specific instrumental Lie algebra $\mathfrak{h}$ and an $\mathcal{H}$-specific quantum instrumental group $e^{\mathfrak{h}}$, whereas working within the universal enveloping algebra $U_{\mathfrak{f}}$ gives the Hilbert-space-independent Lie algebra $\mathfrak{g}$ and the universal instrumental group $G = e^{\mathfrak{g}}$. It is instructive to consider the difference between $\mathfrak{h}$ and $\mathfrak{g}$. The quadratic term is quadratic in the "linear" measured observables, and its matrix commutators generally generate higher and higher powers of the elements of $\mathfrak{f}$. When working with matrices on a finite-dimensional $\mathcal{H}$, sufficiently high powers are constrained to be related to lower powers by the dimensionality of $\mathcal{H}$, so the iterative process (2.133) closes after a finite number of steps. This is particularly obvious in the extreme case that $\mathfrak{f} = \mathfrak{gl}(\mathcal{H}, \mathbb{C})$; then $\vec{X}^2$ is already in $\mathfrak{f}$, so the iterative process goes nowhere and $\mathfrak{h} = \mathfrak{f} = \mathfrak{gl}(\mathcal{H}, \mathbb{C})$. In contrast, when working in the universal enveloping algebra $U_{\mathfrak{f}}$, where the associative algebra is constrained only by the commutators coming from $\mathfrak{f}$, high powers of elements of $\mathfrak{f}$ are not constrained to be related to lower powers, so the iterative process defining $\mathfrak{g}$ can and generally does go on forever. This universal iterative process yields the universal instrumental Lie algebra $\mathfrak{g}$ and the corresponding Lie group, the universal instrumental group $G = e^{\mathfrak{g}}$, which is a kind of universal covering group that unifies all the $\mathcal{H}$-specific quantum instrumental groups. We summarize this as the **third perspectival shift**: *detach the instrument from Hilbert space and place it in its proper home, the universal instrumental Lie group, where the three faces of the stochastic trinity can be applied universally.*

Only very special instruments have a finite-dimensional universal instrumental group; we call these *principal* (universal) instruments. These are pre-quantum [106], Hilbert-space-independent objects that structure any Hilbert space in which they reside. The cases 1-2-3 of Sec. III are all principal instruments. Universal instruments that are not principal instruments we call *chaotic* (universal) instruments.

We need to examine more carefully the relation between the Lie algebras and the Lie groups and between the ($\mathcal{H}$-specific) quantum and universal realizations. There is an associative-algebra homomorphism $\hat{\pi} : U_{\mathfrak{f}} \to A_{\mathcal{H}}$, meaning that the map respects the algebraic properties:

$$\hat{\pi}(z_1 \mathsf{x}_1 + z_2 \mathsf{x}_2) = z_1 \hat{\pi}(\mathsf{x}_1) + z_2 \hat{\pi}(\mathsf{x}_2) , \tag{2.134}$$

in which they reside. The cases 1-2-3 of Sec. III are all principal instruments. Universal instruments that are not principal instruments we call *chaotic* (universal) instruments.

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$$\hat{\pi}(z_1 \mathsf{x}_1 + z_2 \mathsf{x}_2) = z_1 \hat{\pi}(\mathsf{x}_1) + z_2 \hat{\pi}(\mathsf{x}_2)\,, \tag{2.134}$$

$$\hat{\pi}(\mathsf{x}_1 \mathsf{x}_2) = \hat{\pi}(\mathsf{x}_1)\hat{\pi}(\mathsf{x}_2)\,, \tag{2.135}$$

$$\hat{\pi}(\mathsf{x}^\dagger) = \hat{\pi}(\mathsf{x})^\dagger\,, \tag{2.136}$$

for any $\mathsf{x}_1, \mathsf{x}_2 \in U_{\mathfrak{f}}$ and $z_1, z_2 \in \mathbb{C}$. Restricting the domain of this map to the universal instrumental Lie algebra $\mathfrak{g}$ gives a Lie-algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{h}$ that projects the universal instrumental Lie algebra $\mathfrak{g}$ onto the $\mathcal{H}$-specific instrumental Lie algebra $\mathfrak{h}$. The kernel of this projection map,

$$\ker \pi = \pi^{-1}(0) = \{\mathsf{x} \in \mathfrak{g} \mid \pi(\mathsf{x}) = 0\} \equiv \mathfrak{k}\,, \tag{2.137}$$

is an ideal of $\mathfrak{g}$, since $[\mathsf{k}, \mathsf{g}] \in \mathfrak{k}$ for any $\mathsf{k} \in \mathfrak{k}$ and $\mathsf{g} \in \mathfrak{g}$. The Lie group $e^{\mathfrak{k}}$ is a normal subgroup of $G = e^{\mathfrak{g}}$. The quotient group $G/e^{\mathfrak{k}}$ is not, however, $e^{\mathfrak{h}}$ because $e^{\mathfrak{h}}$ knows that elements of $\mathfrak{h}$ other than 0 exponentiate to the identity.

To go further, we extend π to a group projection map $\Pi : G \to e^{\mathfrak{h}}$, defined by $\Pi(e^{\mathsf{g}}) = e^{\pi(\mathsf{g})}$ for any $\mathsf{g} \in \mathfrak{g}$. It is important to realize that Π is the associative-algebra projection map $\hat{\pi}$ restricted to G:

$$\Pi(e^{\mathsf{g}}) = e^{\pi(\mathsf{g})} = e^{\hat{\pi}(\mathsf{g})} = \hat{\pi}(e^{\mathsf{g}})\,. \tag{2.138}$$

The kernel of this map,

$$\ker \Pi = \Pi^{-1}(1) = \{\mathsf{g} \in G \mid \Pi(\mathsf{g}) = 1\} \equiv K\,, \tag{2.139}$$

is a normal subgroup of G, as one sees easily by applying the projection map (2.138). Moreover, it is also easy to see that the quotient group, $H = G/K$, is isomorphic to $\Pi(G) = e^{\mathfrak{h}}$,

$$\frac{1}{\kappa}\frac{\partial D_t(x)}{\partial t} = \Delta[D_t](x)\,, \qquad \text{with} \qquad \Delta \equiv \underleftarrow{\vec{X}}^2 + \frac{1}{2}\sum_\mu \underleftarrow{X_\mu}\underleftarrow{X_\mu}\,, \tag{3.19}$$

where the derivatives, with underarrows pointing to the left, are right-invariant derivatives.

2. Cartan coördinate systems for principal instruments

The Cartan or "KAK" decomposition is the universal analog of a singular-value decomposition. More specifically, every continuous matrix group is a representation of a universal Lie group in which case the analogy is literally that the singular-value decomposition of a representation is a representation of the Cartan decomposition. What this means is that the terms of a Cartan decomposition are more about how the dimensions of the Lie group are connected than they are about the Hilbert space that may carry it. Applied to our three cases, the Cartan decompositions are:

1. For measurement of a single observable, the instrumental Lie group is $G \cong \mathbb{R}^2$ of Eq. (3.7). The K in the Cartan decomposition is $K = \{1\}$ and

$$x = e^{-X^2 r + Xa}\,. \tag{3.20}$$

The invariant measure is the familiar Cartesian measure,

$$d\mu(x) = dr\,da\,. \tag{3.21}$$

strumental Lie group is the 7D IWH of Eq. (3.8). The K in the Cartan d

$$x = \left(D_\beta e^{i1\phi}\right) e^{-H_\circ r - 1\ell} D_\alpha^{-1} \,,$$

$$D_\alpha = e^{-iP\alpha_1 + iQ\alpha_2} \,, \quad \text{with} \quad \alpha \equiv \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) \,,$$

splacement operator (and similarly for β). The Haar measure in Cartan coör

$$d^7\mu(x) = \frac{d^2\beta}{\_} d\phi \, dr \, \sinh^2 r \, d\ell \, \frac{d^2\alpha}{\_} \,,$$

umental Lie group is the 7D ISpin(3) of Eq. (3.9). The K in the Cartan d
and

$$x = \ (D_{\hat{m}} e^{-iJ_z \psi}) e^{-\vec{J}^2 \ell + J_z a} D_{\hat{n}}^{-1} \,,$$

$$^{\phi} e^{-iJ_y \theta} = e^{-i\theta(J_y \cos\phi - J_x \sin\phi)} e^{-iJ_z \phi} \,, \quad \text{with} \quad \hat{n} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$$

placement operator (and similarly for $\hat{m}$). The Haar measure in Cartan coör

$$d^7\mu(x) = d^2\mu(\hat{m}) \, \frac{d\psi}{4\pi} \, d\ell \, da \, \sinh^2 a \, d^2\mu(\hat{n}) \,,$$

$$\delta_t = -\vec{J}^2 \kappa\, dt + J_x \sqrt{\kappa}\, dW^x + J_y \sqrt{\kappa}\, dW^y + J_z \sqrt{\kappa}\, dW^z \,. \tag{3.72}$$

l cover of ISM is the 7-dimensional Lie group we call the *Instrumental Spin Group* $\mathrm{ISpin}(3) = \mathrm{Spin}(3, \mathbb{C}) \times$

oints $x \in \mathrm{ISpin}(3)$ can be coördinated by the Cartan decomposition (3.25), and the Haar measure in inates is given by Eq. (3.27).

-form SDEs for the time-ordered exponential 3.15), with ISM forward generator 3.72, it is convenient to ıly partially, writing $x \in \mathrm{ISpin}(3)$ as

$$x = V e^{J_z a - \vec{J}^2 \ell} U \,. \tag{3.73}$$

.te SDEs for the center coördinate ℓ and the ruler/purity a are

$$d\ell_t = \kappa\, dt \,, \tag{3.74}$$

$$da_t = \kappa\, dt \,\coth a_t + \sqrt{\kappa}\, dY_t^z \,, \tag{3.75}$$

s for the past and future unitaries U and V, written as MMCSDs, are

$$dU_t U_t^{-1} - \frac{1}{2}(dU_t U_t^{-1})^2 = \left(-iJ_x \sqrt{\kappa}\, dY_t^y + iJ_y \sqrt{\kappa}\, dY_t^x\right) \operatorname{csch} a_t \,, \tag{3.76}$$

$$dV_t^{-1} V_t - \frac{1}{2}(dV_t^{-1} V_t)^2 = \left(-iJ_x \sqrt{\kappa}\, dY_t^y + iJ_y \sqrt{\kappa}\, dY_t^x\right) \coth a_t \,. \tag{3.77}$$

ıe derivatives ∇_μ, the Kolmogorov forward generator is

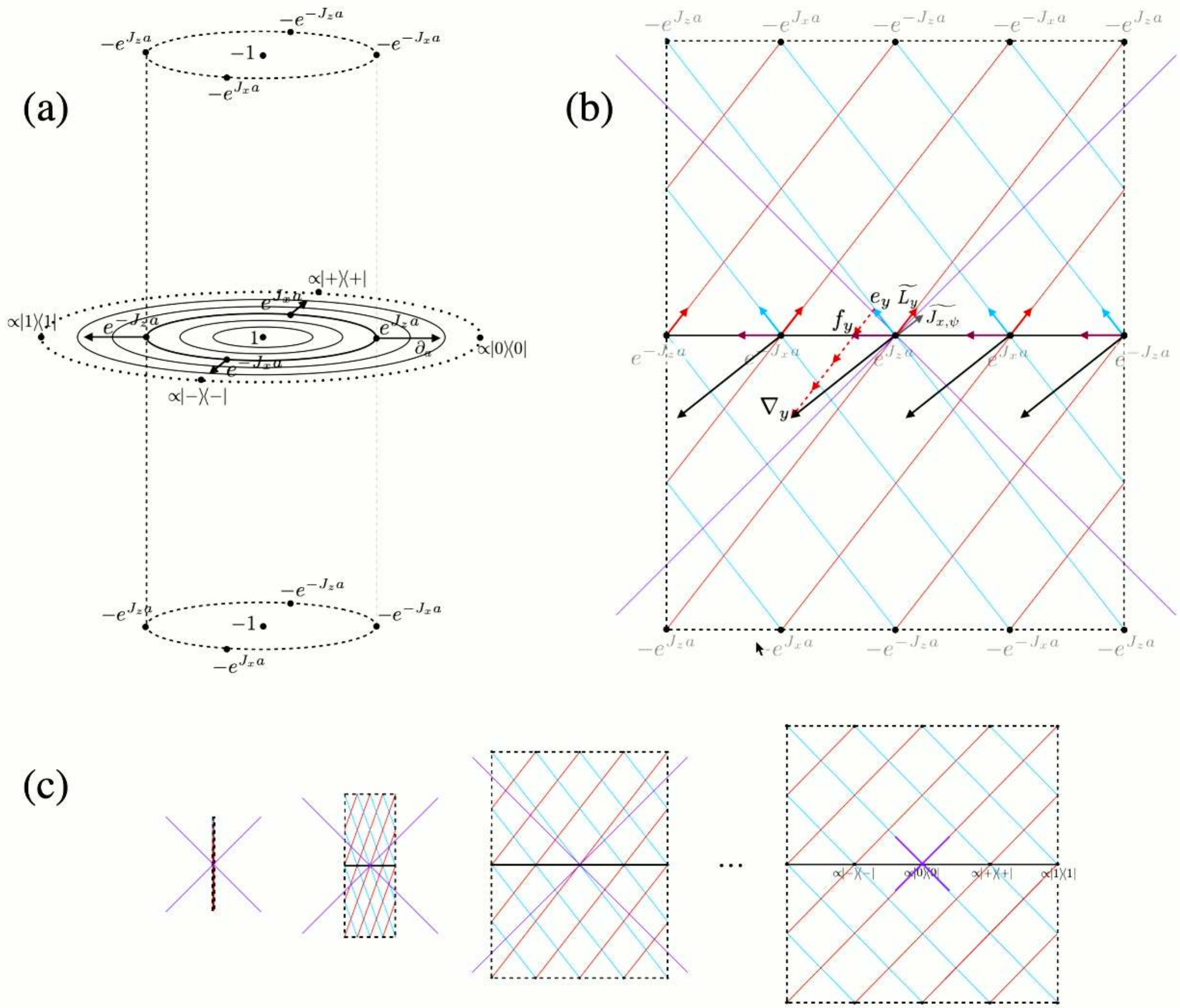
$$\Delta = \underleftarrow{\vec{J}^2} + \frac{1}{2}\left(\partial_a^2 + 2\coth a\,\partial_a + \nabla_x\nabla_x + \nabla_y\nabla_y\right). \tag{3.87}$$

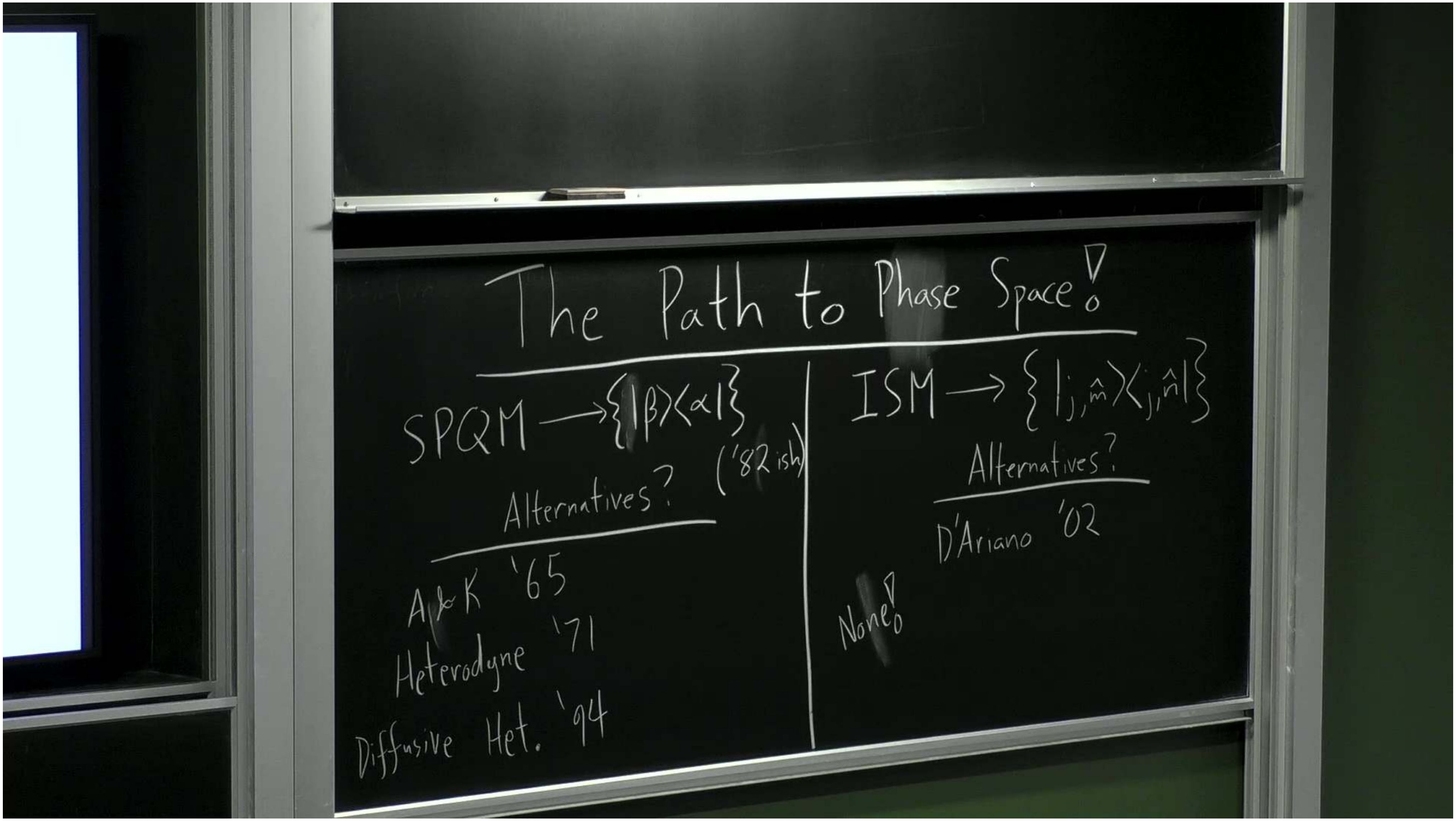
have the full solution for the KOD $D_T(x)$, but we can consider the distribution function obtained by ver the future unitary,

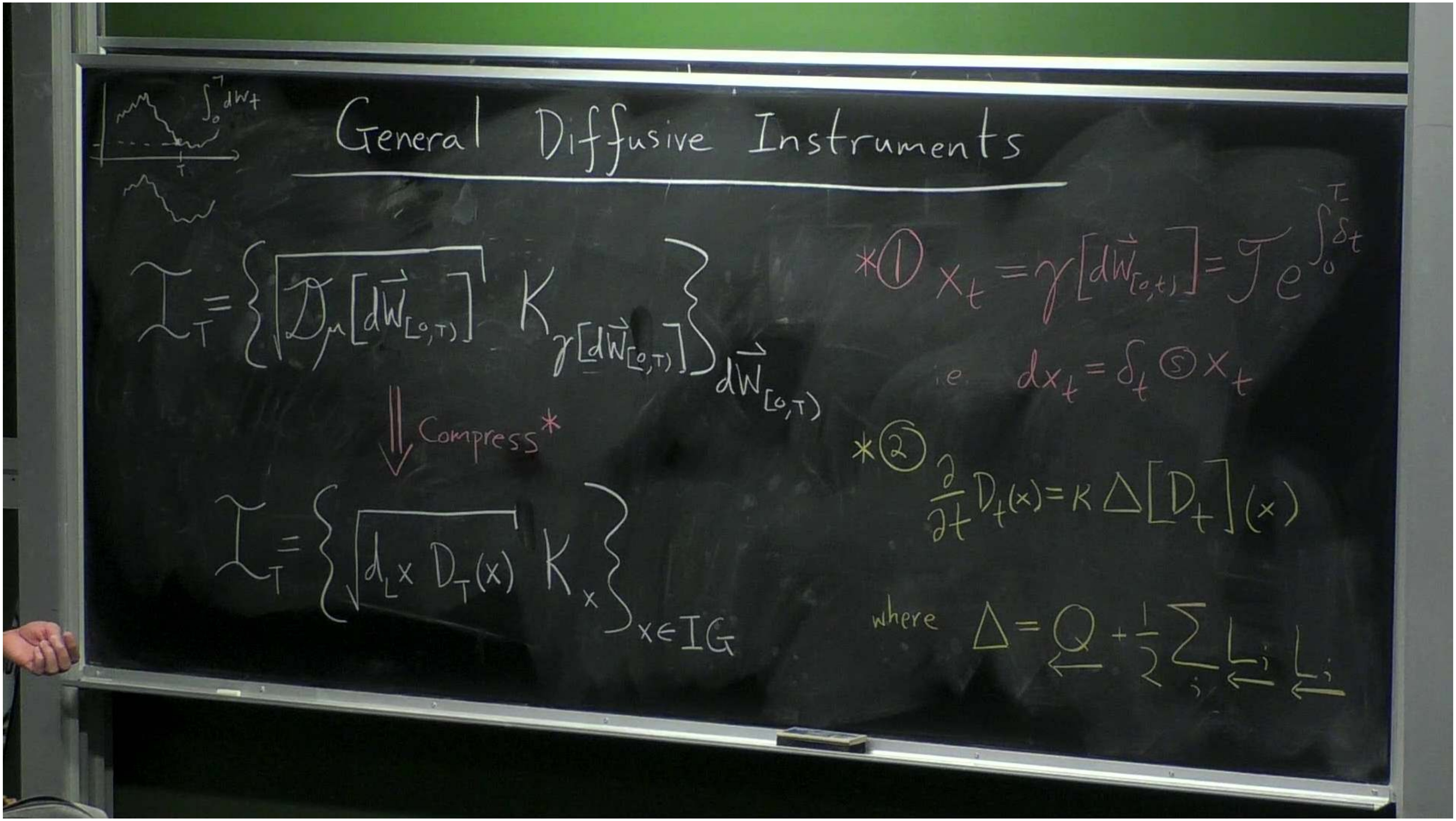
$$D_T(Kx) \equiv \int_K d^3\mu(V)\, D_T(x)\,, \tag{3.88}$$

stands for unitary group SU(2) of future unitaries V. This distribution function governs completeness, om the KOD unraveling (3.17),

$$\begin{aligned} 1 = (1)\mathcal{Z}_T &= \int_{\mathrm{ISpin}(3)} d\mu(x)\, D_T(x)\, R(x)^\dagger R(x) \\ &= \int_{\mathrm{ISpin}(3)} d\mu(x)\, D_T(x)\, e^{-\vec{J}^2 2\ell} U^\dagger e^{J_z 2a} U \\ &= \int d\ell\, da \sinh^2 a\, d^2\mu(\hat{n}) \left(\int_K d^3\mu(V)\, D_T(x)\right) e^{-\vec{J}^2 2\ell} D_{\hat{n}} e^{J_z 2a} D_{\hat{n}}^\dagger \\ &= \int d\ell\, da \sinh^2 a\, d^2\mu(\hat{n})\, D_T(Kx)\, e^{-\vec{J}^2 2\ell} D_{\hat{n}} e^{J_z 2a} D_{\hat{n}}^\dagger\,. \end{aligned} \tag{3.89}$$


(a)
(b)
(c)
$-e^{-J_z a}$
$-e^{J_z a}$
$-e^{-J_x a}$
-1
$-e^{J_x a}$
$\propto|+\rangle\langle+|$
$\propto|1\rangle\langle 1|$
$\propto|0\rangle\langle 0|$
$\propto|-\rangle\langle-|$
1
∂_a
e_y
$\widetilde{L_y}$
f_y
$\widetilde{J_{x,\psi}}$
∇_y
$\propto|-\rangle\langle-|$
$\propto|0\rangle\langle 0|$
$\propto|+\rangle\langle+|$
$\propto|1\rangle\langle 1|$


The Path to Phase Space!
SPQM $\longrightarrow \{|\beta\rangle\langle\alpha|\}$
('82 ish)
Alternatives?
A & K '65
Heterodyne '71
Diffusive Het. '94
ISM $\longrightarrow \{|j,\hat{m}\rangle\langle j,\hat{n}|\}$
Alternatives?
D'Ariano '02
None!



General Diffusive Instruments
Compress*
where

$\Downarrow$ Compress*

$$\mathcal{I}_T = \left\{ \sqrt{d_L x \, D_T(x)} \, K_x \right\}_{x \in IG}$$

*② $\frac{\partial}{\partial t} D_t(x) = \kappa \Delta[D_t](x)$

where $\Delta = \underleftarrow{Q} + \frac{1}{2} \sum_i \underleftarrow{L_i} \, \underleftarrow{L_i}$

Simultaneous P&Q M't

("SPQM")

$$\delta_t^{(SPQM)} = -2H_0 \kappa \, dt + Q\sqrt{\kappa}\, dW_t^q + P\sqrt{\kappa}\, dW_t^p$$

$$\Delta^{(SPQM)} = 2\underleftarrow{H_0} + \frac{1}{2}\left(\underleftarrow{Q}\,\underleftarrow{Q} + \underleftarrow{P}\,\underleftarrow{P}\right)$$

$$IWH = \left\{ \left(D_\beta e^{i I \phi}\right) e^{-H_0 r - I\ell} D_\alpha^{-1} \right\}$$

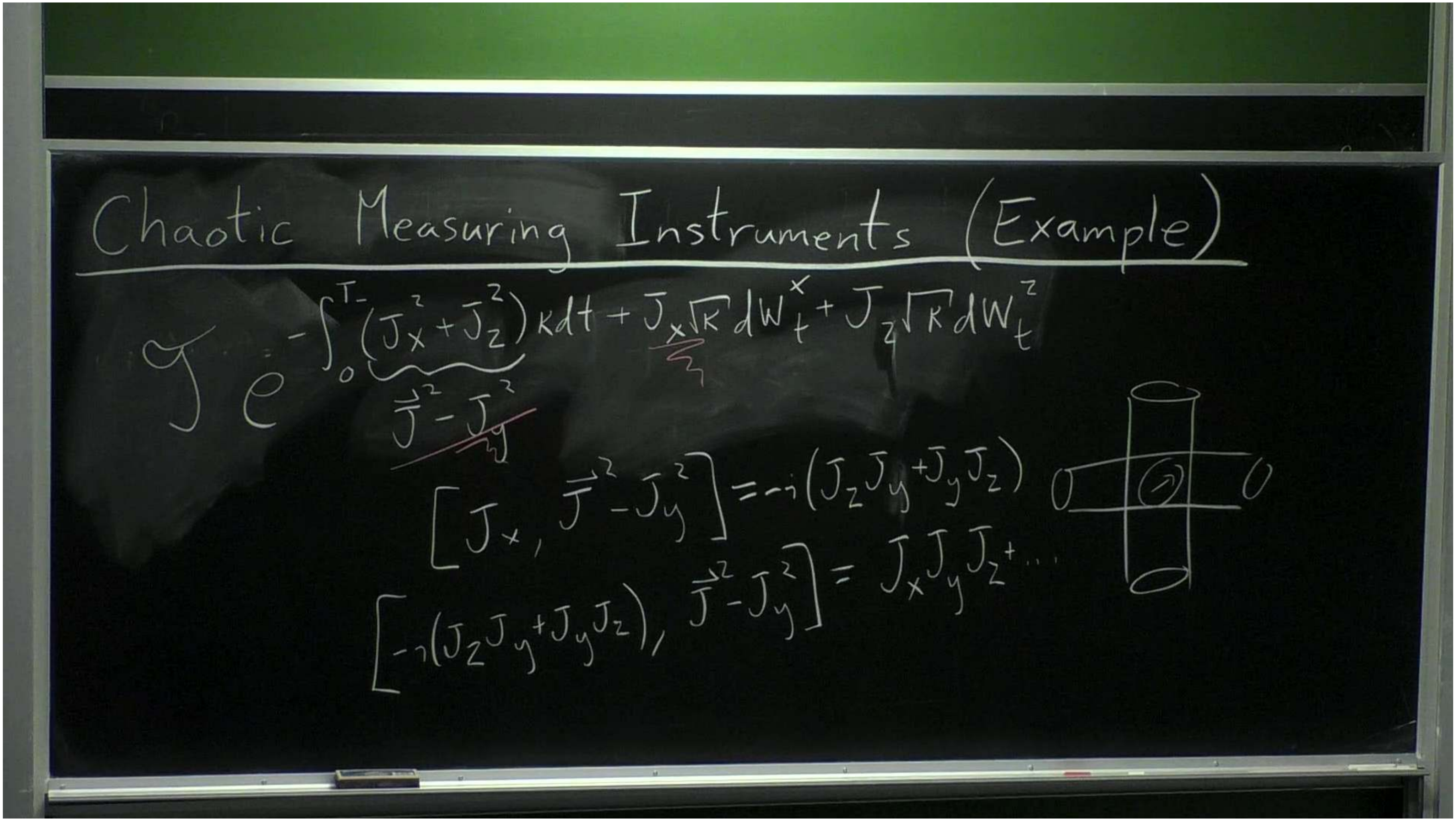
Isotropic Spin M't

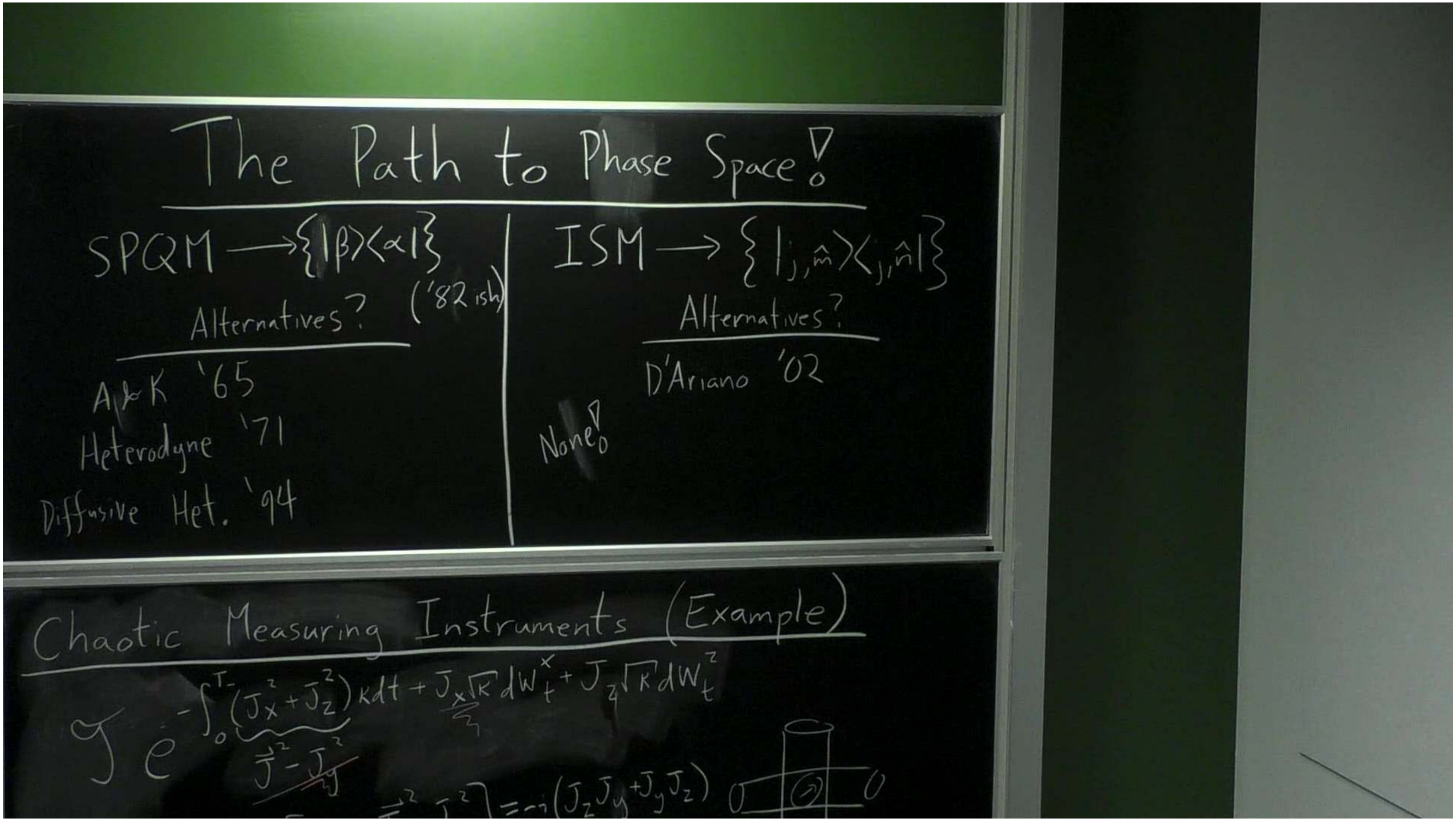
("ISM")

$$\delta_t^{(ISM)} = -\vec{J}^2 \kappa \, dt + \vec{J} \cdot \sqrt{\kappa}\, dW_t$$

$$\Delta^{(ISM)} = \underleftarrow{\vec{J}^2} + \frac{1}{2} \sum_{k \in \{x,y,z\}} \underleftarrow{J_k}\,\underleftarrow{J_k}$$

$$ISpin = \left\{ \left(D_{\hat{m}} e^{-iJ_z \psi}\right) e^{-\vec{J}^2 \ell + J_z a} D_{\hat{n}}^{-1} \right\}$$


Chaotic Measuring Instruments (Example)
$\mathcal{T} e^{-\int_0^{T} \underbrace{(J_x^2 + J_z^2)}_{\vec{J}^2 - J_y^2} \kappa dt + J_x \sqrt{\kappa} dW_t^x + J_z \sqrt{\kappa} dW_t^z}$
$[J_x, \vec{J}^2 - J_y^2] = -i(J_z J_y + J_y J_z)$
$[-i(J_z J_y + J_y J_z), \vec{J}^2 - J_y^2] = J_x J_y J_z + \dots$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