Title: Lecture - Quantum Measurement and Continuous Markov Processes Mini-Course

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Collection/Series: Quantum Measurement and Continuous Markov Processes Mini-Course, Oct 27 - Dec 11, 2025

Subject: Mathematical physics, Other, Quantum Foundations, Quantum Information

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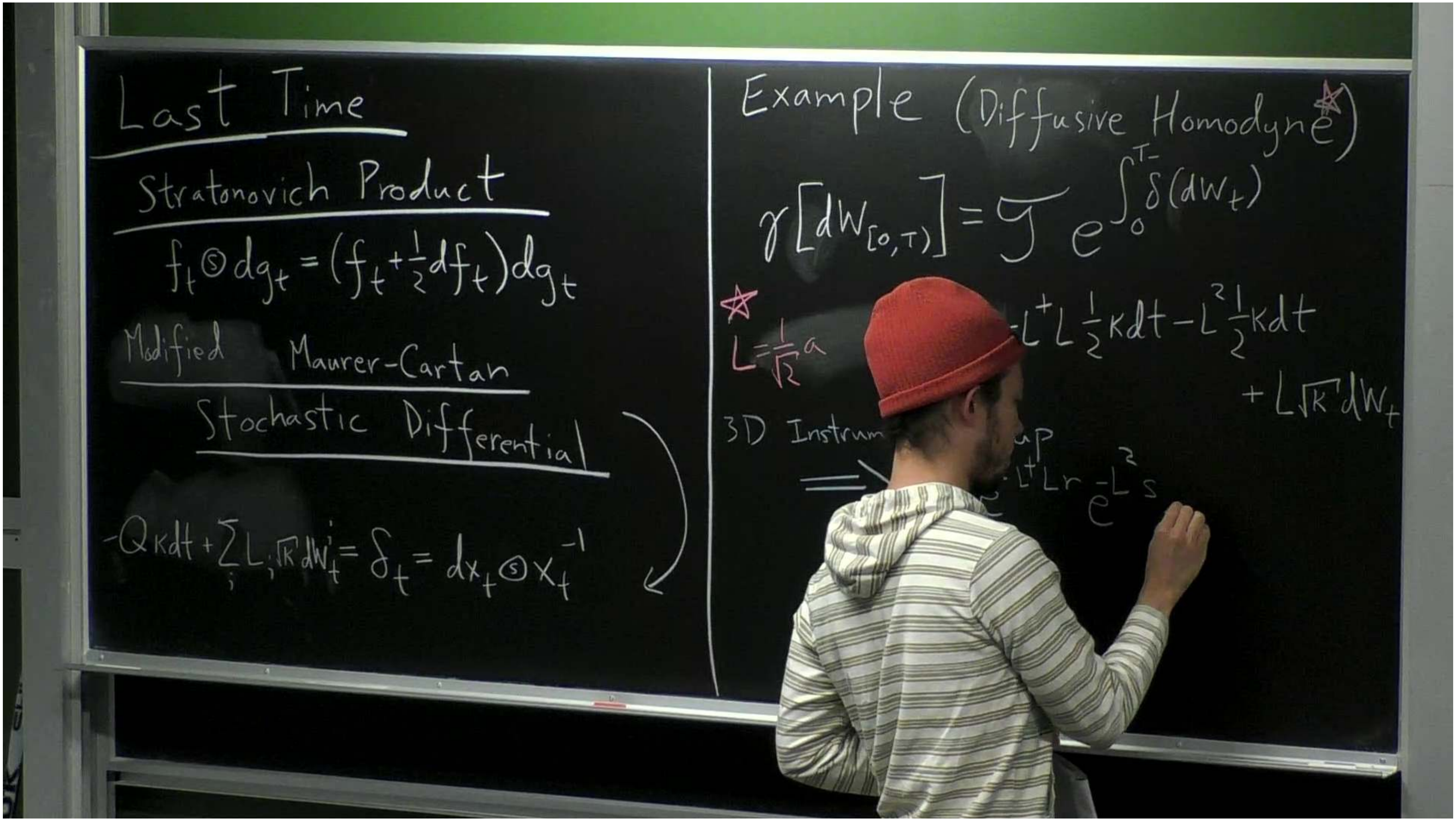
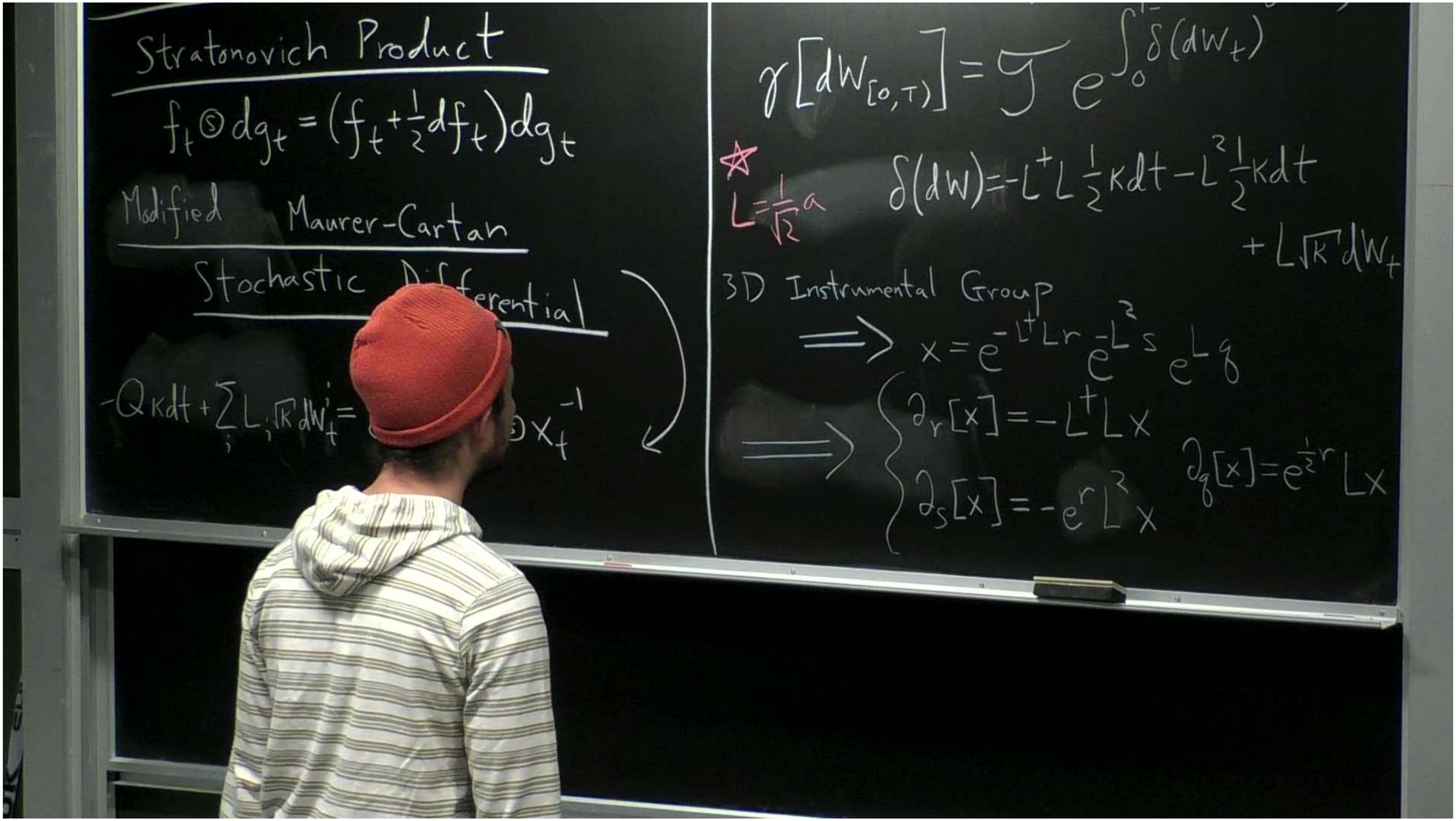

Last Time
Stratonovich Product
$f_t \circledS dg_t = (f_t + \frac{1}{2} df_t) dg_t$
Modified Maurer-Cartan
Stochastic Differential
$-Q\kappa dt + \sum_i L_i \sqrt{\kappa} dW^i_t = \delta_t = dx_t \circledS x_t^{-1}$
Example (Diffusive Homodyne)
$\gamma[dW_{[0,T)}] = \mathcal{T} e^{\int_0^{T-} \delta(dW_t)}$
$L = \frac{1}{\sqrt{2}} a$
$-L^\dagger L \frac{1}{2} \kappa dt - L^2 \frac{1}{2} \kappa dt$
$+ L\sqrt{\kappa} dW_t$
3D Instrum
$\Longrightarrow$
$e^{-L^\dagger L r} e^{-L^2 s}$


Stratonovich Product
$f_t \circledS dg_t = (f_t + \frac{1}{2} df_t) dg_t$
Modified Maurer-Cartan
Stochastic Differential
$-Q\kappa dt + \sum_i L_i \sqrt{\kappa} dW_t^i =$
$\gamma[dW_{[0,T)}] = \mathcal{T} e^{\int_0^T \delta(dW_t)}$
$L = \frac{1}{\sqrt{2}} a$
$\delta(dW) = -L^\dagger L \frac{1}{2} \kappa dt - L^2 \frac{1}{2} \kappa dt + L\sqrt{\kappa} dW_t$
3D Instrumental Group
$\Longrightarrow x = e^{-L^\dagger L r} e^{-L^2 s} e^{L q}$
$\partial_r[x] = -L^\dagger L x$
$\partial_s[x] = -e^r L^2 x$
$\partial_q[x] = e^{\frac{1}{2} r} L x$

$$-Q\kappa\,dt + \sum_i L_i \sqrt{\kappa}\, dW_t^i = \delta_t = dx_t \circledS x_t^{-1}$$

$$\xRightarrow[\xi]{*} \begin{cases} \partial_r[x] = -L^\dagger L x \\ \partial_s[x] = -e^r L^2 x \end{cases} \qquad \partial_q[x] = e^{\frac{1}{2}r} L x$$

Today

$$\Omega_{d\vec{W}_{[0,t)}} = \sqrt{\mathcal{D}_\mu\left[d\vec{W}_{[0,t)}\right]}\; K_{\gamma[d\vec{W}_{[0,t)}]}$$

$\Downarrow$ Collect Equivalent Kraus Operators

$$\Omega_x = \sqrt{d_L x\, D_t(x)}\; K_x$$

"Kraus-Operator Density"

$*$ Last Time (cont.)

$$\xi:\; dx_t = dr_t \circledS \partial_r[x_t] + \dots$$

$$dx_t = \delta_t \circledS x_t^{-1} \Longrightarrow \begin{cases} dr_t = \frac{1}{2}\kappa\,dt \\ e^{r_t} \circledS ds_t = \frac{1}{2}\kappa\,dt \\ e^{\frac{1}{2}r_t} \circledS dq_t = \sqrt{\kappa}\,dW_t \end{cases}$$

$$\equiv$$

$$x_t = \gamma\left[dW_{[0,t)}\right]$$

Last Time

Stratonovich Product

$$f_t \circledS dg_t = \left(f_t + \tfrac{1}{2} df_t\right) dg_t$$

Modified Maurer-Cartan

Stochastic Differential

$$-Q\kappa dt + \sum_i L_i \sqrt{\kappa}\, dW_t^i = \delta_t = dx_t \circledS x_t^{-1}$$

Example (Diffusive Homodyne ☆)

$$\gamma\left[dW_{[0,T)}\right] = \mathcal{T}\, e^{\int_0^{T-} \delta(dW_t)}$$

☆ $L = \frac{1}{\sqrt{2}} a$

$$\delta(dW) = -L^\dagger L \tfrac{1}{2}\kappa dt - L^2 \tfrac{1}{2}\kappa dt + L\sqrt{\kappa}\, dW_t$$

3D Instrumental Group

$$\Longrightarrow \quad x = e^{-L^\dagger L r}\, e^{-L^2 s}\, e^{L q}$$

$$\xRightarrow[\xi]{*} \begin{cases} \partial_r[x] = -L^\dagger L x \\ \partial_s[x] = -e^r L^2 x \end{cases} \qquad \partial_q[x] = e^{\frac{1}{2}r} L x$$

Today ☆

$*$ Last Time (cont.)

ξ : $dx_t = dr_t \circledS \partial [x_t] + \ldots$

$$\partial_s[x] = -e^r L^2 x$$

Today ☆

$$\Omega_{d\vec{W}_{[0,t)}} = \sqrt{\mathcal{D}_\mu\left[d\vec{W}_{[0,t)}\right]}\, K_{\gamma[d\vec{W}_{[0,t)}]}$$

$\Downarrow$ Collect Equivalent Kraus Operators

$$\Omega_x = \sqrt{d_L x\, D_t(x)}\, K_x$$

"Kraus-Operator Density" ☆

Last Time (cont.)

$$dx_t = dr_t \circledS \partial_r[x_t] + \dots$$

$$\boxed{dx_t = \delta_t \circledS x_t^{-1} \;\equiv\; x_t = \gamma\left[dW_{[0,t)}\right]} \Longrightarrow \begin{cases} dr_t = \frac{1}{2}\kappa\, dt \\ e^{r_t} \circledS ds_t = \frac{1}{2}\kappa\, dt \\ e^{\frac{1}{2}r_t} \circledS dq_t = \sqrt{\kappa}\, dW_t \end{cases}$$

Returning to the process.

$\partial_s[x] = -e^r L^3 x$

New Tool

Def$^{\underline{n}}$ Let $\mathbb{X} \in \mathfrak{g}$ be an element of a (finite-dimensional) Lie algebra $\mathfrak{g}$ w/ Universal covering group $G = e^{\mathfrak{g}}$, and let $x \in G$ be a point. Then the right-invariant derivative of a function f in the direction $\underline{\mathbb{X}}$ at a point x is

$$\underset{\leftarrow}{\mathbb{X}}[f](x) = \lim_{h \to 0} \frac{f(e^{h\mathbb{X}} x) - f(x)}{h}$$

Today ☆

$\Omega_{d\vec{W}} = \boxed{\mathcal{D}_\mu [d\vec{W} \ldots} \; K$

* Last Time (cont.)

$dx_t = dr_t \circledS \partial_r[x_t] + \ldots$

Translation (Ultra-)operators

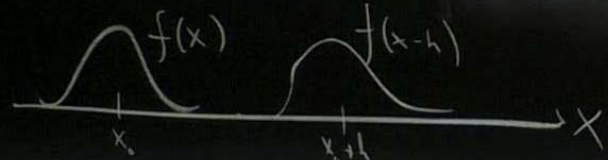
Left-translation

$$L_g[f](x) = f(g^{-1}x)$$

Right Translation

$$R_g[f](x) = f(xg^{-1})$$

Think of how $f(x-h)$ is translated like



Two Properties

① $R_g \circ \underleftarrow{X} \circ R_g^{-1} = \underleftarrow{X}$

"$\underleftarrow{X}$ is right-translation invariant"

② If f is analytic

$$f(e^{h\overline{X}}x) = e^{-h\underleftarrow{X}}[f](x)$$

Think of

$$f(x+a) = e^{a\frac{d}{dx}}[f](x).$$

"Noncommutative Taylor's Theorem"

arXiv:2510.13980

$$f(x+a) = e^{a\frac{d}{dx}}[f](x).$$

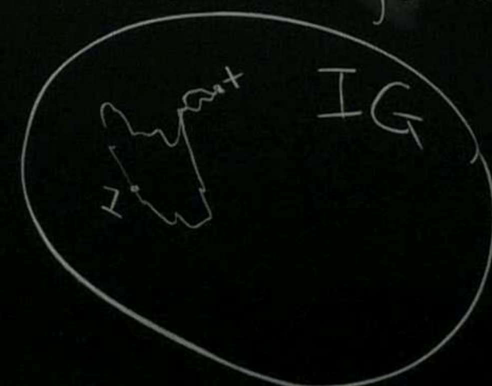
"Noncommutative Taylor's Theorem"

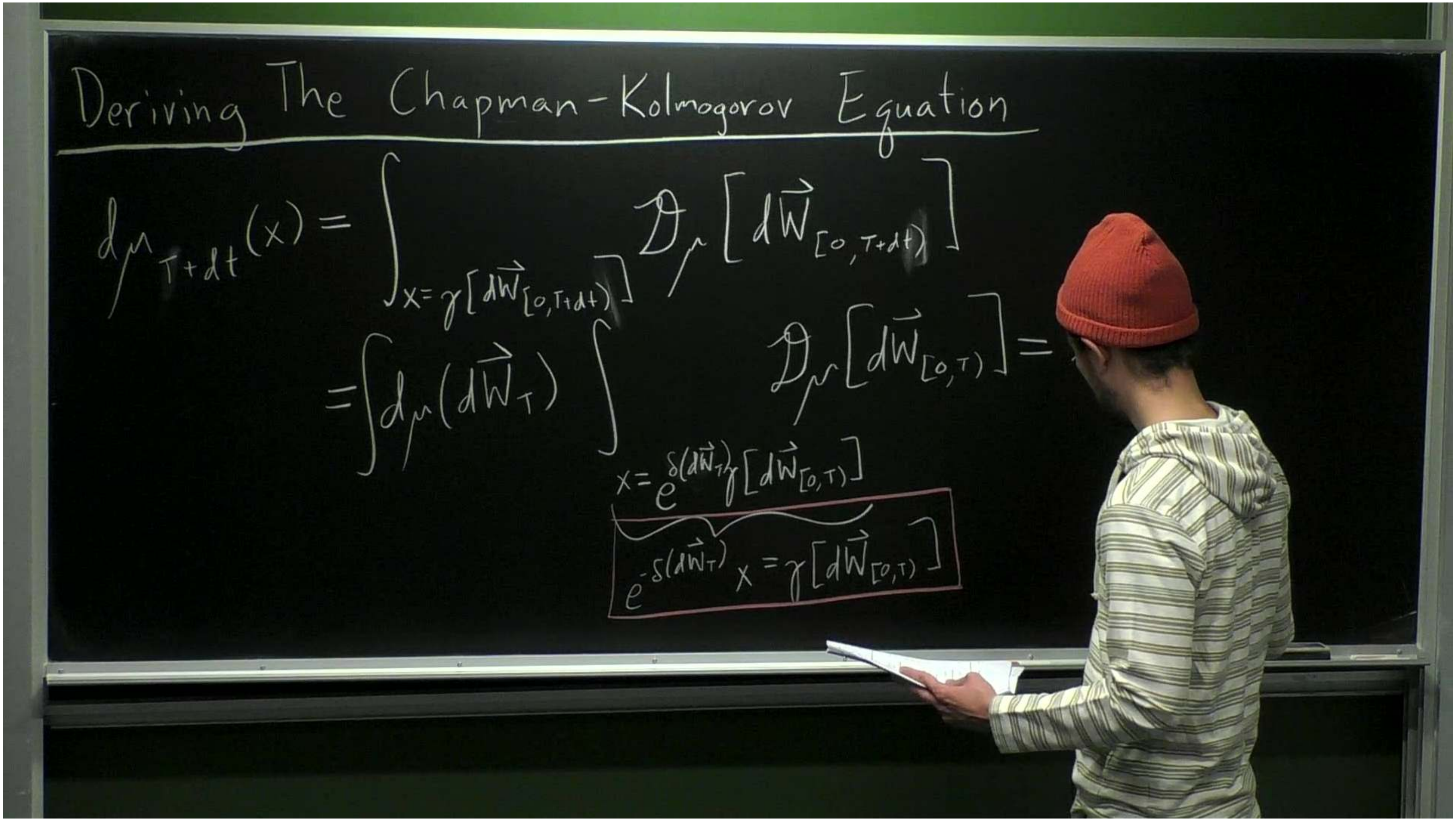
Similarly left-inv derivatives   But we won't need these

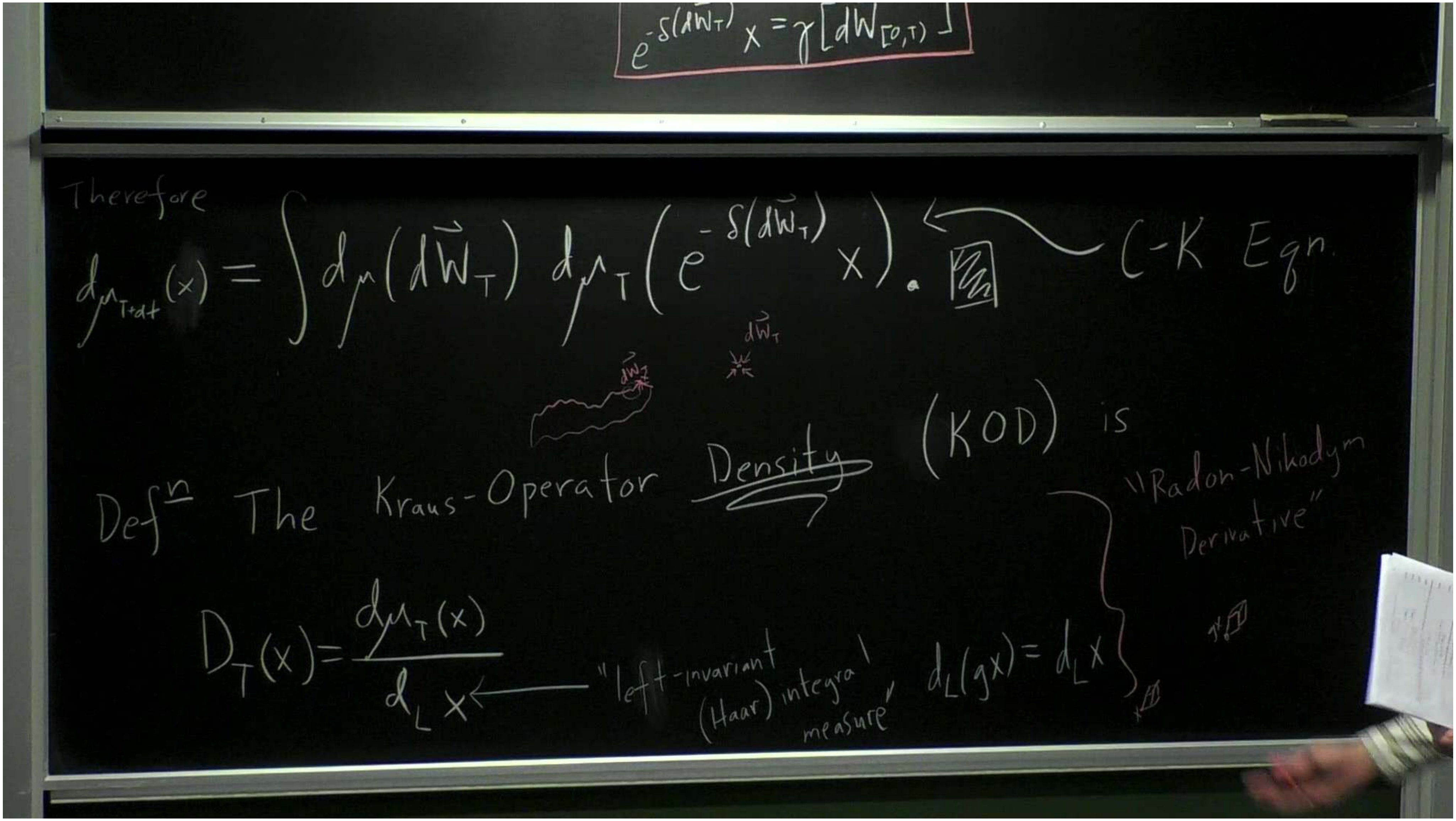
$$\underrightarrow{X}[f](x) = \lim_{h\to 0} \frac{f(xe^{hX}) - f(x)}{h} \implies f(xe^{hX}) = e^{h\underrightarrow{X}}[f](x)$$

Def$^{\underline{n}}$ The Kraus-Operator Distribution for a diffusive measurement for a time T is

$$d\mu_T(x) = \int_{x=\gamma[d\vec{W}_{[0,T)}]} \mathcal{D}\mu[d\vec{W}_{[0,T)}].$$




Deriving The Chapman-Kolmogorov Equation
$d\mu_{T+dt}(x) = \int_{x=\gamma[d\vec{W}_{[0,T+dt)}]} \mathcal{D}\mu\left[d\vec{W}_{[0,T+dt)}\right]$
$= \int d\mu(d\vec{W}_T) \int_{x=e^{\delta(d\vec{W}_T)}\gamma[d\vec{W}_{[0,T)}]} \mathcal{D}\mu\left[d\vec{W}_{[0,T)}\right] =$
$\boxed{e^{-\delta(d\vec{W}_T)} x = \gamma[d\vec{W}_{[0,T)}]}$


$e^{-S(d\vec{W}_T)}x = \gamma[dW_{[0,T)}]$
Therefore
$d\mu_{T+dt}(x) = \int d\mu(d\vec{W}_T)\, d\mu_T(e^{-S(d\vec{W}_T)}x)$
C-K Eqn.
$d\vec{W}_T$
Def$^{\underline{n}}$ The Kraus-Operator Density (KOD) is
"Radon-Nikodym Derivative"
$D_T(x) = \frac{d\mu_T(x)}{d_L x}$
"left-invariant (Haar) integral measure"
$d_L(gx) = d_L x$

$$x = e^{\delta(d\vec{W}_T)}\gamma[dW_{[0,T)}]$$

$$\boxed{e^{-\delta(d\vec{W}_T)}\,x = \gamma[d\vec{W}_{[0,T)}]}$$

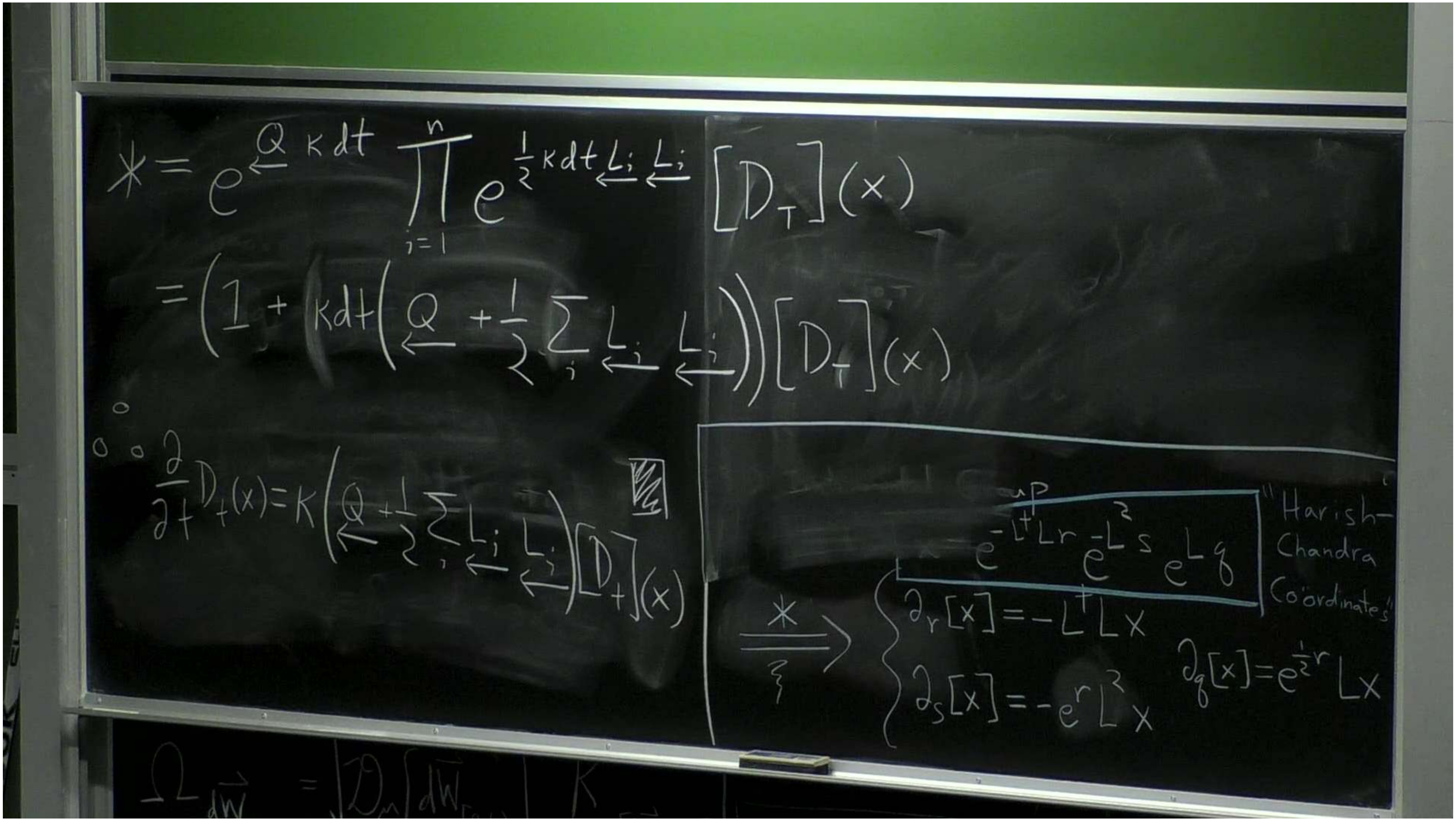
Deriving the Fokker-Planck-Kolmogorov forward equation for the KOD

$$D_{T+dt}(x) = \int \underbrace{d\mu(d\vec{W})}_{d\mu(dW^1)\cdots d\mu(dW^i)} \underbrace{D_T(e^{-\delta(d\vec{W})}x)}_{e^{\overleftarrow{\delta(d\vec{W})}}[D_T](x)}$$

$$\underbrace{e^{\overleftarrow{\delta(d\vec{W})}}}_{e^{-\sum_i \overleftarrow{L_i}\sqrt{\kappa}\,dW^i + \overleftarrow{Q}\kappa dt}}$$

$$= e^{\overleftarrow{Q}\kappa dt} \prod_{i=1}^{n} d\mu(dW^i)\, e^{-\overleftarrow{L_i}\sqrt{\kappa}\,dW^i}\,[D_T](x)$$

$$d\mu(dW) = \frac{d(dW)}{\sqrt{2\pi dt}}\,e^{-\frac{dW^2}{2dt}}$$


$* = e^{\overleftarrow{Q}\kappa dt} \prod_{i=1}^{n} e^{\frac{1}{2}\kappa dt \overleftarrow{L_i}\overleftarrow{L_i}} [D_T](x)$
$= \left(1 + \kappa dt\left(\overleftarrow{Q} + \frac{1}{2}\sum_i \overleftarrow{L_i}\,\overleftarrow{L_i}\right)\right)[D_T](x)$
$\frac{\partial}{\partial t} D_t(x) = \kappa\left(\overleftarrow{Q} + \frac{1}{2}\sum_i \overleftarrow{L_i}\,\overleftarrow{L_i}\right)[D_t](x)$
$\boxed{e^{-L^\dagger L r}\, e^{-L^2 s}\, e^{L q}}$
"Harish-Chandra Coördinates"
$\overset{*}{\underset{\xi}{\Longrightarrow}} \begin{cases} \partial_r[x] = -L^\dagger L x \\ \partial_s[x] = -e^{r} L^2 x \end{cases}$
$\partial_q[x] = e^{\frac{1}{2}r} L x$

Initial condition $D_0(x) = \delta(x)$

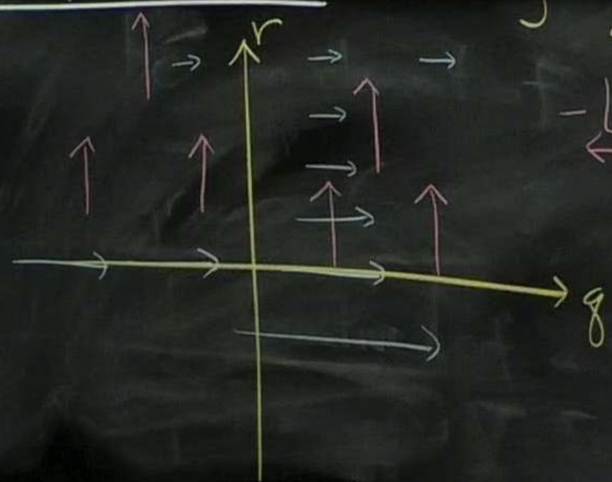
$\uparrow$ Group's δ-function

$\partial_s[x] = -e^r L^2 x$

Back to Example 3

using $\overleftarrow{Y}[x] = Yx$

$$\begin{cases} \partial_r = -\overleftarrow{L^\dagger L} \\ e^{-r}\partial_s = -\overleftarrow{L}^2 \\ e^{-\frac{1}{2}r}\partial_q = \overleftarrow{L} \end{cases}$$



$-\overleftarrow{L^\dagger L}$

$\overleftarrow{L}$

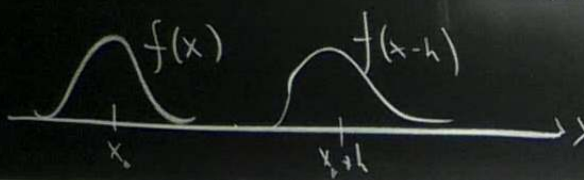
Today ☆

$\Omega_{d\vec{W}} = \int \mathcal{D}_m [d\vec{W}_{[\ldots]}] \; K$

* Last Time (cont.)

$dx_t = dr_t \circledS \partial_r[x_t] + \ldots$

is translated like



for more, see
arXiv:2510.13980

$f(e^{h\Delta}x) = e^{h\overleftarrow{\Delta}}[f](x)$

Think of

$f(x+a) = e^{a\frac{d}{dx}}[f](x)$.

"Noncommutative Taylor's Theorem"

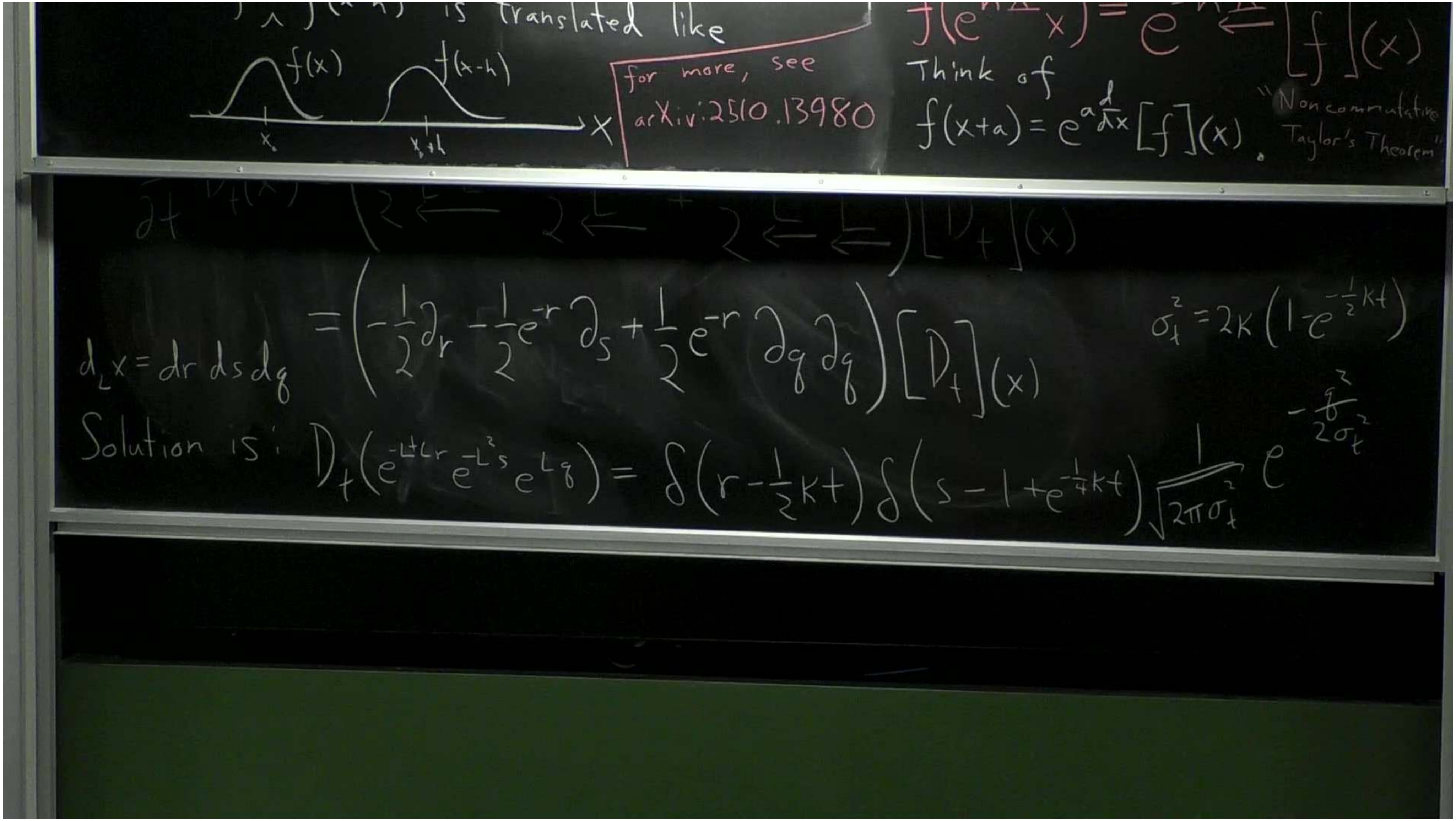
Example 3 (cont.)

Remember $Q = \frac{1}{2}L^{\dagger}L + \frac{1}{2}L^{2}$

FPKE is

$$\frac{\partial}{\partial t}D_t(x) = \left(\frac{1}{2}\underleftarrow{L^{\dagger}L} + \frac{1}{2}\underleftarrow{L^{2}} + \frac{1}{2}\underleftarrow{L}\,\underleftarrow{L}\right)[D_t](x)$$

$$= \left(-\frac{1}{2}\partial_r - \frac{1}{2}e^{-r}\partial_s + \frac{1}{2}e^{-r}\partial_q\partial_q\right)[D_t](x)$$


is translated like
f(x)
f(x-h)
x₀
x₀+h
x
for more, see
arXiv:2510.13980
Think of
f(x+a) = e^{a d/dx}[f](x).
"Noncommutative Taylor's Theorem"
d_L x = dr ds dq
= (-1/2 ∂_r - 1/2 e^{-r} ∂_s + 1/2 e^{-r} ∂_q ∂_q)[D_t](x)
σ_t² = 2κ(1 - e^{-1/2 κt})
Solution is: D_t(e^{-L+Lr} e^{-L² s} e^{Lq}) = δ(r - 1/2 κt) δ(s - 1 + e^{-1/4 κt}) 1/√(2πσ_t²) e^{-q²/2σ_t²}