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Undistillability implies PPT? A point of view from partial trace inequalities

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Overview

1. The Distillability Problem
2. Partial Trace Inequalities
3. Partial Traces Beyond Normal Matrices

The Distillability Problem

Open Quantum Problems

Search ...

Show 50 entries

Search:

Nr	Title	Contact	Date (Y/M/D)	Last Progress (Y/M/D)	Categories
1	All the Bell Inequalities	R.F. Werner	1999/10/25	2010	Quantum foundations
2	Undistillability implies ppt?	D. Bruß	2000/03/02	2006/08/16	Entanglement theory
5	Maximally entangled mixed states	K. Audenaert	2001/11/08	-	Entanglement theory

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Five open problems in theory of quantum information

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(Dated: December 21, 2020)

Entanglement

Definition (Quantum state)

A state $\rho \in L(\mathscr{H})$ is a positive semidefinite matrix with $\operatorname{tr} \rho = 1$ and where $\mathscr{H} = \mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$ is a Hilbert space.

Definition (Entanglement)

ρ is called separable if it can be written as a convex sum of tensor product of quantum states, i.e,

$$\rho = \sum_{i=1}^{k} \lambda_i \rho_i^1 \otimes \ldots \otimes \rho_i^n \tag{1}$$

Otherwise it is called entangled.

Question: How can we know if a state ρ is entangled?

PPT states

Definition (PPT)

A state $\rho \in L(\mathscr{H}_1 \otimes \mathscr{H}_2)$ is called PPT if the partial transpose

$$\rho^{T_1} := (T \otimes I)\rho \geq 0\,, \tag{2}$$

where T is the transpose map in $\mathscr{H}_1$.

Note: Separable $\Rightarrow$ PPT. Is the converse true? In general, NO!

Theorem

If ρ is a state in $\mathscr{H}$ with $\dim \mathscr{H} \leq 6$, then ρ is separable $\Leftrightarrow$ ρ is PPT.

[1] Peres, A. (1996). Separability criterion for density matrices. Physical Review Letters, 77(8), 1413.

Distillation of Entanglement

Definition (Distillation of Entanglement)

A state $\rho \in \mathcal{S}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ is distillable if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and an LOCC operation $\Lambda : (\mathbb{C}^d)^{\otimes n} \to (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes m}$ such that

$$\|\Lambda(\rho^{\otimes n}) - |\Phi\rangle\langle\Phi|^m\|_1 \leq \varepsilon \,, \tag{3}$$

where Φ is a maximally entangled state. If this does not hold, we say that ρ is undistillable.

Property: Separable $\Rightarrow$ PPT $\Rightarrow$ Undistillable.

Question: is in this case PPT $\Leftrightarrow$ Undistillability?

No idea... But in my opinion no.′

Proposition

ρ is distillable $\Leftrightarrow$ there exists $n \in \mathbb{N}$ and operators $A, B \in (\mathbb{C}^d)^{\otimes n} \to \mathbb{C}^2$ such that

$$\rho' = \frac{(A \otimes B)\rho^{\otimes n}(A \otimes B)^*}{\mathrm{tr}[(A \otimes B)\rho^{\otimes n}(A \otimes B)^*]} \tag{4}$$

is entangled.

[2] Nielsen, M. A. (1999). Conditions for a class of entanglement transformations. Physical Review Letters, 83(2), 436.

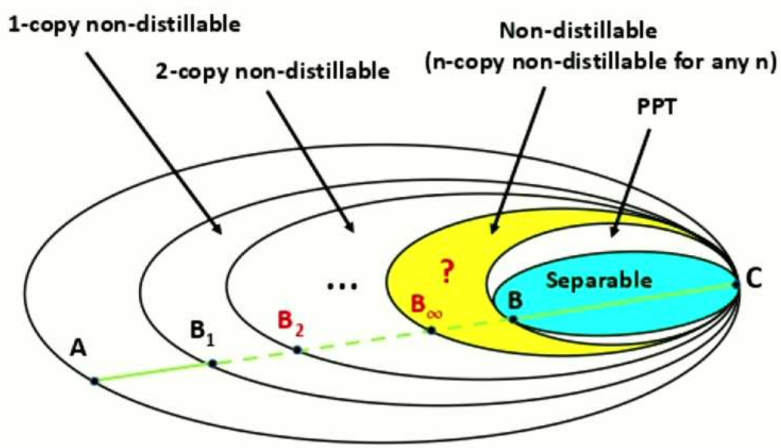
[3] Bennett, C. H., Brassard, G., Popescu, S., Schumacher, B., Smolin, J. A., & Wootters, W. K. (1996). Purification of noisy entanglement and faithful teleportation via noisy channels. Physical review letters, 76(5), 722..

[4] Horodecki, M., Horodecki, P., & Horodecki, R. (1998). Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature?. Physical Review Letters, 80(24), 5239.

We will say that ρ is n-copy undistillable if

$$\rho' = \frac{(A \otimes B)\rho^{\otimes n}(A \otimes B)^*}{\mathrm{tr}[(A \otimes B)\rho^{\otimes n}(A \otimes B)^*]} \tag{5}$$

is separable for such n and for every A, B.



[5] Horodecki, P., Rudnicki, Ł., & Życzkowski, K. (2022). Five open problems in quantum information theory. PRX Quantum, 3(1), 010101.

Distillation of Entanglement

A very special LOCC operation is the twirling map:

$$\mathscr{T}(\rho) = \int_{U(d)} (U \otimes U)\rho(U \otimes U)^* \mathrm{d}U \tag{6}$$

where $\mathrm{d}U$ denotes the Haar measure on the unitary group of $d \times d$ matrices $U(d)$. The twirling map satisfies:

1. It preserves separability and PPT.
2. It projects $\mathcal{S}(\mathbb{C}^d \otimes \mathbb{C}^d)$ onto a very special family: Werner states.

Werner states

The Werner states in a bipartite space $L(\mathbb{C}^d \otimes \mathbb{C}^d)$ are of the form

$$\rho_\alpha = \frac{\mathbb{1} + \alpha\mathbb{F}}{d^2 + \alpha d}, \tag{7}$$

where $\mathbb{F}$ is the flip operator $\mathbb{F}(x \otimes y) = y \otimes x$, for $x, y \in \mathbb{C}^d$, and $\alpha \in [-1, 1]$.

Proposition

The Werner states ρ_α for $\alpha \in [-1, 1]$, have the following properties:

1. Local symmetry: $\rho = (U \otimes U)\rho(U \otimes U)^*$, for every unitary U.
2. ρ_α is separable $\Leftrightarrow$ ρ_α is PPT $\Leftrightarrow$ $\alpha \geq -\frac{1}{d}$.
3. ρ_α is 1-copy undistiallable $\Leftrightarrow$ $\alpha \geq -\frac{1}{2}$.

Partial trace inequalities

Definition

For $n \in \mathbb{N}$ and $C \in L(\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n)$, we define the quadratic form

$$q^{(n)}(C) = \sum_{J \subseteq P(\{1,\ldots,n\})} \alpha^{|J|} \| \operatorname{tr}_J C\|_2^2 \tag{8}$$

For example,

$$\begin{aligned} q^{(1)}(C) &= \|C\|_2^2 + \alpha |\operatorname{tr} C|^2 \\ q^{(2)}(C) &= \|C\|_2^2 + \alpha(\|\operatorname{tr}_1 C\|_2^2 + \|\operatorname{tr}_2 C\|_2^2) + \alpha^2 |\operatorname{tr} C|^2 \end{aligned} \tag{9}$$

Theorem

Let $\alpha \in [-1, 1]$. A Werner state ρ_α is n-copy undistillable if and only if $q^{(n)}(\alpha, C) \geq 0$ for every $C \in L((\mathbb{C}^d)^{\otimes n})$ with rank ≤ 2.

Werner states

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A new bound

<u>**Note:**</u> By making use of

$$\|\operatorname{tr}_i C\|_1 \leq \|C\|_1 \quad \text{and} \quad \|C\|_2 \leq \|C\|_1 \leq \sqrt{r}\|C\|_2\,, \tag{10}$$

we obtain for $\alpha \in [-1, 0]$

$$\begin{aligned} q^{(2)}(\alpha, C) &= \|C\|_2^2 + \alpha\left[\|\operatorname{tr}_2 C\|_2^2 + \|\operatorname{tr}_1 C\|_2^2\right] + \alpha^2|\operatorname{tr} C|^2 \\ &\geq \|C\|_2^2 + \alpha\left[\|\operatorname{tr}_2 C\|_1^2 + \|\operatorname{tr}_1 C\|_1^2\right] + \alpha^2|\operatorname{tr} C|^2 \\ &\geq \|C\|_2^2 + 2\alpha\|C\|_1^2 + \alpha^2|\operatorname{tr} C|^2 \\ &\geq \left(\frac{1}{2} + 2\alpha\right)\|C\|_1^2 + \alpha^2|\operatorname{tr} C|^2. \end{aligned} \tag{11}$$

Corollary

The Werner states are 2-copy undistillable for $\alpha \geq -\frac{1}{4}$.

Partial trace inequalities

For $\alpha = -1/2$, $q^{(2)}(C) \geq 0$ can be rewritten as

$$\|\operatorname{tr}_1 C\|_2^2 + \|\operatorname{tr}_2 C\|_2^2 \leq 2\|C\|_2^2 + \frac{1}{2}|\operatorname{tr} C|^2 \tag{12}$$

This problem can be generalized to a general matrix C with rank r:

$$\|\operatorname{tr}_1 C\|_2^2 + \|\operatorname{tr}_2 C\|_2^2 \leq r\|C\|_2^2 + \frac{1}{r}|\operatorname{tr} C|^2 . \tag{13}$$

We show this inequality for C **rank 1** and C **normal**. We also show for **any** matrix

$$|\|\operatorname{tr}_1 C\|_2^2 - \|\operatorname{tr}_2 C\|_2^2| \leq r\|C\|_2^2 - \frac{1}{r}|\operatorname{tr} C|^2 , \tag{14}$$

$$\|I_1 \otimes \operatorname{tr}_1 C - \operatorname{tr}_2 C \otimes I_2\|_\infty \leq \|C\|_1 , \tag{15}$$

$$\sum_{J \subseteq P(\{1,\ldots,n\})} \left(-\frac{1}{d}\right)^{|J|} \|\operatorname{tr}_J C\|_2^2 \geq 0 . \tag{16}$$

Dilations

Definition (Dilation)

Given $A \in L(\mathbb{C}^d)$, we say that $C \in L(\mathbb{C}^d \otimes \mathbb{C}^r)$ is a dliation of A if $\mathrm{tr}_2[C] = A$

Proposition (Structured dilations)

For every $A \in L(\mathbb{C}^d)$ we can find under specific assumption on r a matrix C such that $\mathrm{tr}_2[C] = A$ and

1. C is a normal matrix.
2. C is a unitary matrix.
3. C is a nilpotent matrix.
4. C is an idempotent matrix.

Flanders-similarity

Definition

We say that two matrices $A \in L(\mathbb{C}^{d_A})$, $B \in L(\mathbb{C}^{d_B})$ are Flanders-similar if

1. They have the same Jordan block structure for the non-zero eigenvalues.
2. If $a_1 \geq a_2 \geq \ldots$ are the sizes of the Jordan blocks of A that correspond to eigenvalue 0 and $b_1 \geq b_2 \geq \ldots$ are the sizes of the respective Jordan blocks of B and the shorter sequence is padded with zeros, then $|a_i - b_i| \leq 1$ for all i.

[6] Flanders, H. (1951). Elementary divisors of AB and BA. Proceedings of the American Mathematical Society, 2(6), 871-874.

Joint dilations

Theorem (Simultaneous rank-one dilation)

If $C \in L(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is rank 1, then $B = \operatorname{tr}_1 C$ and $A = \operatorname{tr}_2 C$ are Flander similar.
Conversely, if two matrices A, B are Flanders-similar, then there is a rank 1 joint dilation C such that $\operatorname{tr}_1 C = B$ and $\operatorname{tr}_2 C = A$

.

Proposition (Simultaneous rank-two dilation)

For any two matrices $A, B \in L(\mathbb{C}^d \otimes \mathbb{C}^d)$ there exists a rank-two matrix $C \in L(\mathbb{C}^d \otimes \mathbb{C}^d)$ s.t. $\operatorname{tr}_1 C = B$ and $\operatorname{tr}_2 C = A$ iff $\operatorname{tr} A = \operatorname{tr} B$.

Norm inequality template

Theorem (Norm inequality template)

Let $C \in L(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n})$ have rank r and partial traces $\mathrm{tr}_{\{1,\ldots,n\}\setminus\{i\}} C$. For any $\gamma \geq 0$ the following holds:

$$\sum_{i=1}^{n} \left|\!\left|\!\left|\mathrm{tr}_{\{1,\ldots,n\}\setminus\{i\}} C\right|\!\right|\!\right| \quad \leq \quad \gamma \, \|C\|_1 + \kappa(\gamma) \, |\!|\!|C|\!|\!| \ , \tag{17}$$

$$\kappa(\gamma) \quad := \quad \sup \phi^*\big(\Lambda_1, \ldots, \Lambda_r\big), \tag{18}$$

where Λ is a vector with $\prod_{i=1}^{n} d_i$ decreasingly ordered components of the form $\big(-\gamma + \sum_{i=1}^{n} \lambda_{j_i}^{(i)}\big)_+$ and the supremum is taken over all n-tuples of vectors $\big(\lambda^{(i)} \in \mathbb{R}_+^{d_i}\big)_{i=1}^{n}$ satisfying $\phi^*(\lambda^{(i)}) \leq 1$.

Revisiting the distillability problem

For the p-norms and $n=2$,

$$\kappa(\gamma) = \sup\left\{ \sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(x_i + y_j - \gamma)_+^q \;\middle|\; \sum_{i=1}^{d_1}|x_i|^q = \sum_{j=1}^{d_2}|y_j|^q = 1 \right\}^{1/q},$$

Proposition (Extended Audenaert inequality)

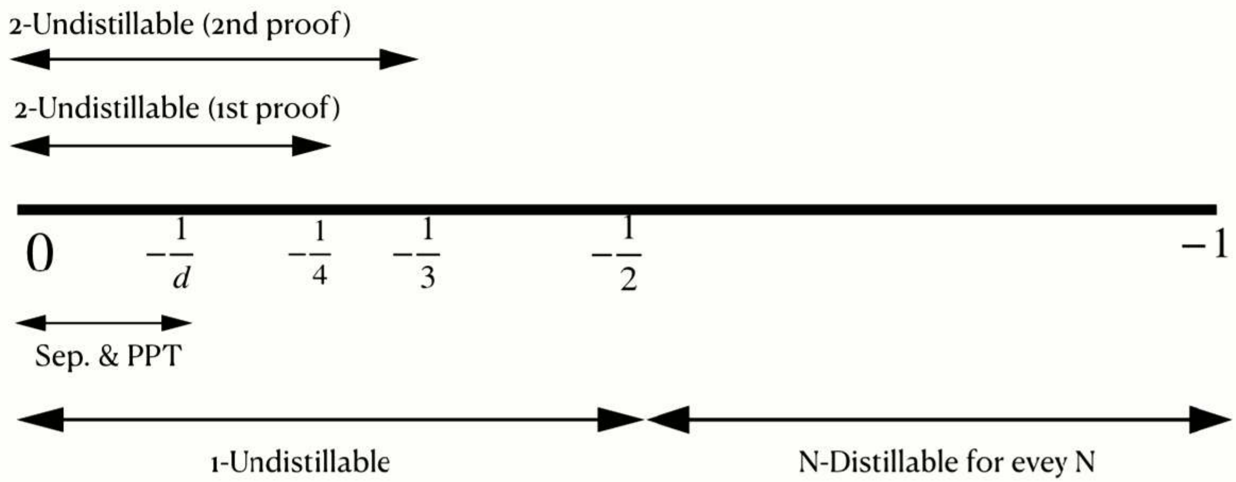
If $M \in L(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n})$, then for all $p, s \in [1,\infty]$:

$$\sum_{i=1}^{n} \left\|\mathrm{tr}_{\{1,\ldots,n\}\setminus\{i\}}\, C\right\|_p^s \le (n-1)\|C\|_1^s + \|C\|_p^s. \qquad (19)$$

Corollary

The Werner states are 2-copy undistillable for $\alpha \ge -\frac{1}{3}$.

Entanglement Properties of Werner states.



Thank you!