Title: Towards a Dolbeault AGT correspondence

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Abstract:

The AGT correspondence and its extensions posit geometric constructions of vertex algebras and their modules from cohomology of variants of moduli of sheaves on surfaces. Physically, the correspondence has found an explanation through the holomorphic-topological twist of the six dimensional N=(2,0) superconformal field theories. In this talk, I'll propose a variant of the AGT correspondence coming from the so-called minimal twist of these theories. Instead of vertex algebras, the natural algebras appearing will be holomorphic factorization algebras in three complex dimensions. From this data, I will explain how one extracts an associative algebra and a module which conjecturally agrees with a quantization of moduli of Higgs sheaves on surfaces. In examples, the pair will admit a Hodge-deRham deformation to the Heisenberg algebra and its action on cohomology of Hilbert schemes of surfaces, constructed in work of Grojnowski-Nakajima.

Towards a Dolbeault AGT correspondence

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Based on WIP with B. Williams

THE AGT CORRESPONDENCE

Let X be a smooth, projective surface and let $\mathcal{H}$ denote the Heisenberg algebra on $H_\bullet(X)$.

Theorem (Grojnowski-Nakajima)

The Vacuum module of $\mathcal{H}$ is $\bigoplus_n H_\bullet(\mathrm{Hilb}_n(X))$.

Many generalizations [Schiffmann and Vasserot, 2012], [Maulik and Okounkov, 2018], [Creutzig et al., 2023], [Gaiotto and Rapčák, 2018], [Rapčák et al., 2019], [Butson, 2023]$\cdots$

- Action extends to $\mathrm{Vir} \ltimes \mathcal{H}$.
- Replacing $\mathrm{Hilb}_n(X)$ with moduli of rank n instantons, affine Laumon spaces, spiked/folded/$\cdots$ instantons $\rightsquigarrow$ replacing $\mathcal{H}$ with $\mathcal{W}_n$ algebras, iterated $\mathcal{W}_n$-algebras, Gaiotto-Rapcak algebras $\cdots$

CONTEXT

These results can be contextualized in terms of the *6d* $\mathcal{N} = (2,0)$ *theory*.

There is an Ω*-deformed nonminimal twist*

- Defined on $M^4 \times \Sigma$ where M^4 has a torus action with isolated fixed points. Described by a 2d chiral CFT on Σ.
- space of local operators of 2d chiral CFT is the Hilbert space for 5d theory gotten by compactifying on $S^1 \subset \Sigma = \mathbb{C}$.

The 6d $\mathcal{N} = (2,0)$ theory also has a *minimal twist*. This is a holomorphic field theory in 3 complex dimensions.The Ω-deformed nonminimal twist arises as a further deformation.

Question

Is there a version of the AGT correspondence in the minimal twist?

I will try and motivate a statement in the language of *factorization algebras* [Costello and Gwilliam, 2017, 2021].

The abelian theory

Let Z be a complex 3-fold with $W_1, W_2 \to Z$ such that $W_1 \otimes W_2 \cong K_Z$.

Proposition ([Saberi and Williams, 2020])

The local observables Obs of the minimal twist of the abelian 6d $\mathcal{N} = (2,0)$ on Z is given by a twisted factorization envelope $\mathbb{U}_\kappa(\mathcal{L})$ where

- $\mathcal{L}$ is the $\mathbb{Z}/2$-graded abelian dgla

$$\begin{array}{cc} \underline{\textit{even}} & \underline{\textit{odd}} \\ \Omega_Z^{0,\bullet}(W_1 \oplus W_2) & \\ \Omega_Z^{0,\bullet} & \xrightarrow{\partial} \Omega_Z^{1,\bullet} \end{array}$$

- κ is the local cocycle $\kappa(\alpha_1, \alpha_2) = \int_Z \alpha_1 \partial \alpha_2$

Obs as a higher Heisenberg algebra

This factorization algebra has the expected symmetries on flat space.

Proposition

Let $Z = \mathbb{C}^3$, $W_1 = W_2 = K_Z^{1/2}$. There is an action of $\mathfrak{osp}(6|2)$ on Obs.

Given $(Q, S) \in \mathfrak{osp}(6|2)_1$ such that $[Q, Q] = [S, S] = 0$ and $[Q, S] = \mathcal{L}_V$ where V is a rotation $\rightsquigarrow$ the invariants Obs^S is a lower dimensional factorization algebra supported at the fixed points of V.

Proposition

- Obs^S has no sections away from $\mathbb{C} \subset \mathbb{C}^3$ and $\mathrm{Obs}^S|_{\mathbb{C}}$ is holomorphic.
- The vertex algebra associated to $\mathrm{Obs}^S|_{\mathbb{C}}$ is isomorphic to the Heisenberg vertex algebra.

Dolbeault-AGT

Let X be a projective surface, $L \to X$ a line bundle, and suppose $Z = \mathrm{Tot}\, L$. Choose a hermitian metric on L.

$$\begin{array}{ccc} & & Z \\ & \overset{\tilde{\pi}}{\swarrow} & \downarrow \pi \\ X \times \mathbb{R}_{\geq 0} & \longrightarrow & \mathbb{R}_{\geq 0} \end{array} \quad \rightsquigarrow \quad \begin{array}{rl} \mathcal{A}(X; L, W_1, W_2) = & (\pi_* \mathrm{Obs})(a, b), \text{ assoc. algebra} \\ & (\pi_* \mathrm{Obs})_0, \text{ module} \end{array}$$

Quasi-conjecture

$(\pi_* \mathrm{Obs})_0$ is the Hilbert space of the minimal twist of abelian 5d $\mathcal{N} = 2$ gauge theory in the presence of certain background fields.

Compared to previous instances of AGT - the "vertex algebra plane" can be fibered over the surface nontrivially! In the case where L is the trivial bundle, we can in fact extract vertex algebras

$$\begin{array}{ccc} \mathbb{C} \times X & \xrightarrow{p} & \mathbb{C} \\ \downarrow \pi & \swarrow |-| & \\ \mathbb{R}_{\geq 0} & & \end{array} \rightsquigarrow \begin{array}{l} \mathcal{V}(X; L, W_1, W_2) \text{ the vertex algebra associated to } p_* \mathrm{Obs} \\ (\pi_* \mathrm{Obs})_0 \text{ its vacuum module} \end{array}$$

We have descriptions of these vertex algebras in examples.

Recovering Göttsche's formula

Let $W_1 = \mathcal{O}_{\mathbb{C}} \boxtimes K_S$ and $W_2 = K_{\mathbb{C}} \boxtimes \mathcal{O}_S$.

Proposition (R.-Williams)

The vertex algebra $\mathcal{V}(X; L, W_1, W_2)$ is quasi-isomorphic to

$$\mathcal{H}[H^\bullet(X, \Omega^1_X)] \otimes \beta\gamma\,[H^\bullet(X, K_X)] \otimes \mathrm{Sym}(H^\bullet(X, K_X)[1])$$

where the last factor is equipped with the trivial vertex algebra structure. Its character is given by

$$\prod_{m\geq 1} \prod_{k,\ell=0}^{2} \left(1 - (-1)^{k+\ell} x^{k+m-1} y^{\ell+m-1} q^m\right)^{(-1)^{k+\ell+1} h^{k,\ell}(X)}$$

which recovers the formula of Göttsche's formula for the Hodge-Poincaré polynomial of the Hilbert schemes of X

$$\sum_{n\geq 0} \sum_{i,j=0}^{2n} q^n h^{i,j}(X^{[n]})(-x)^i(-y)^j.$$

The minimal twist in five dimensions

Suppose that $W_i = \pi_L^* L_i$ where L_i are line bundles on X. In the absence of background supergravity fields, the minimal twist of 5d $\mathcal{N} = 2$ takes the form of a holomorphic-topological BF theory. The perturbative fields around the trivial bundle consist of:

- BF fields

$$A \in \Omega^\bullet(\mathbb{R}) \otimes \Omega^{0,\bullet}(X) \otimes \mathfrak{g}[1], \qquad B \in \Omega^\bullet(\mathbb{R}) \otimes \Omega^{2,\bullet}(X) \otimes \mathfrak{g}[1]$$

- Hypermultiplet fields

$$\phi \in \Omega^\bullet(\mathbb{R}) \otimes \Omega^{0,\bullet}(X; L_1 \oplus L_2) \otimes \mathfrak{g}[1]$$

The action is

$$S(\phi, A, B) = \int_{\mathbb{R}\times X} (B, F_A) + \frac{1}{2}\int_{\mathbb{R}\times X} (\phi, d_A\phi).$$

Suppose $L = \mathcal{O}_X, L_1 = K_X, L_2 = \mathcal{O}_X[2]$. Perturbatively, the phase space of the 5d theory is given by

-2	-1	0
	$\Omega_X^{0,\bullet} \otimes \mathfrak{g}$	
$\Omega_X^{0,\bullet} \otimes \mathfrak{g}$		$\Omega_X^{0,\bullet}(K_X) \otimes \mathfrak{g}$
	$\Omega_X^{0,\bullet}(K_X) \otimes \mathfrak{g}$	

The above describes the tangent complex at a point in $T^*(\mathrm{Bun}_G X)_{Dol}$. Geometric quantization is (roughly) $\mathcal{O}((\mathrm{Bun}_G X)_{Dol})$, which has a natural deformation to $H^\bullet_{\mathrm{dR}}(\mathrm{Bun}_G X)$

The general phase space

Now allow the line bundles to be arbitrary, but take $G = \mathbb{C}^\times$. The coupling to supergravity introduces a Chern-Simons term

$$\begin{array}{ccc}
\underline{-2} & \underline{-1} & \underline{0} \\
 & \Omega^{0,\bullet}_X & \\
\Omega^{0,\bullet}_X(L_2) & \Big\downarrow \partial\theta\wedge\partial & \Omega^{0,\bullet}_X(L \otimes L_1) \\
 & \Omega^{0,\bullet}_X(K_X) &
\end{array}$$

Where θ is a connection on L. Now letting

$$\mathcal{M}_G(X, L_2) = \{(P, \phi) : P \in \mathrm{Bun}_G X, \phi \in \Gamma(X, \mathrm{ad}\, P \otimes L_2)\}$$

the above describes the tangent complex at a point in a twisted cotangent bundle to $\mathcal{M}_G(X, L_2)$.

THE SYMPLECTIC FORM ON PHASE SPACE

Let's describe the twist. Letting $\kappa \in H^4(BG)$ be a level, can produce a symplectic form on $\mathcal{M}_G(X, L_2)$ via transgression

$$
\begin{array}{ccc}
X \times \mathcal{M}_G(X, L_2) & \xrightarrow{\int_X} & \mathcal{M}_G(X, L_2) \\
\downarrow q_2 & & \\
X \times \operatorname{Bun}_G X & \xrightarrow{\text{ev}} & BG \\
\downarrow q_1 & & \\
X & &
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{l}
\kappa_\theta = \int_X q_2^* \left(q_1^*(\partial\theta) \wedge \text{ev}^* \kappa \right) \\
\in \Omega^2 \left(\mathcal{M}_G(X, L_2) \right)
\end{array}
$$

The phase space is $T^*_{\kappa_\theta} \mathcal{M}_G(X, L_2)$. Very reminiscent of description of flat bundles on a curve as a twisted cotangent bundle to holomorphic bundles.

Conjecture (R-Williams)

There is an action of $\mathcal{A}$ on $H^\bullet\left(\widetilde{\mathcal{M}_G}(X, L_2), \mathcal{L}_\theta\right)$ where $\mathcal{L}_\theta$ is a particular line bundle.

A modest test when $X = \mathbb{P}^2, L = \mathcal{O}(-1), L_1 = \mathcal{O}(-2), L_2 = \mathcal{O}[2]$:

Proposition ([Raghavendran and Williams, 2022])

The character of $(\pi_* \mathrm{Obs})_0$ as an $\mathfrak{osp}(6|2)$-module is given by

$$\mathrm{PExp}\left(\frac{qy + q^2y^{-1} - q^2(y_1^{-1} + y_2^{-1} + y_3^{-1}) + q^3}{(1 - y_1q)(1 - y_2q)(1 - y_3q)}\right).$$

This matches exactly with the partition function of the relevant deformation of 5d $\mathcal{N} = 2$ gauge theory, computed via instanton-counting [Kim et al., 2013].

Further work

- Nonperturbative description of twisted abelian $6d\ \mathcal{N} = (2,0)$ theory via Deligne complex classifying holomorphic $\mathbb{C}^\times$-gerbes. Gives a strategy for constructing $\mathcal{L}_\kappa$ via transgression in differential cohomology
- Couple to twist of $6d\ \mathcal{N} = (2,0)$ conformal SUGRA - described in terms of a local version of $E(3|6)$, and localizes to Virasoro algebra.
- . . .

Thanks!