Title: BPS lines, CoHA bimodules and Cluster Categorification

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Abstract:

In four-dimensional N=2 supersymmetric quantum field theories, half-BPS line defects survive the HT twist and are expected to form a monoidal category equipped with renormalized r-matrices. The Grothendieck group of this category is anticipated to coincide with the cluster algebra associated to the BPS quiver Q. Starting from such a quiver, we propose a candidate for such a category - roughly speaking, to each possible framing we associate a bimodule over the Cohomological Hall algebra of Q, and consider the subcategory they generate. We conjecture that this construction satisfies the expected properties. In this talk, I will present examples where this approach works and explain why we expect the RG functors to play the role of "categorified quantum torus charts".

BPS Lines, CoHA Bimodules, and Cluster Categorification

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TL;DR

Let

- $\mathcal{T}$ — some 4d $\mathcal{N} = 2$ SQFT,
- Q — a BPS quiver of $\mathcal{T}$,
- $\mathcal{H}_Q$ — CoHA of Q (some associative algebra in $\mathcal{C}^{IR}$),
- $\mathcal{C}_Q$ — subcategory of $\mathcal{H}_Q - \mathrm{Bimod}(\mathcal{C}^{IR})$ generated by $\{\mathcal{B}_f | f \in \mathbb{Z}^{Q_0}\}$.

We speculate that

The category of $1/2$-BPS lines in $\mathcal{T}$ is described by $\mathcal{C}_Q$.

Wish List

What do we want from $\mathcal{C}_Q$?

I. 1/2-BPS lines survive the HT-twist, hence $\mathcal{C}_Q \subset \mathrm{Line}(\mathcal{T}^{HT})$ and

- 2 top directions $\Rightarrow \mathcal{C}_Q$ is a rigid monoidal category.
- hol directions $\Rightarrow \mathcal{C}_Q$ has renormalized r-matrices.
  moral: $\mathcal{C}_Q = i_0^* \mathcal{C}^{ch}_{\mathbb{A}^1}$*; in chiral setting there is meromorphic braiding*

  $$R_{12}(z) : l_1 \otimes l_2^{(z)} \xrightarrow{\sim} l_2^{(z)} \otimes l_1$$

  but at a fiber we only feel the leading order

  $$R_{12}(z) = z^{\Lambda_{12}}(r_{12} + O(z))$$

  but r_{12} *may fail to be an iso.*
- counting spin in $\mathbb{C}$-directions $\Rightarrow \mathcal{C}_Q$ is $\mathbb{Z}$-graded

  $$\mathfrak{q} \simeq \_\otimes \mathfrak{q}(\mathbf{1}) \in End(\mathcal{C}_Q)$$

Wish list

II. We know the decategorification

$$K_0(\mathcal{C}_Q) \simeq \mathcal{O}_{\mathrm{q}}(\mathcal{M}_B(\mathcal{T})) \simeq \text{quantum Cluster algebra of } Q$$

Moreover, there's an injection *(cf. [GMN])* into

$$K_0(\mathcal{C}_Q^{IR}) \simeq \mathcal{O}_{\mathrm{q}}(\mathcal{M}_B(\mathcal{T}^{IR})) \simeq \text{Quantum torus of } Q$$

One may hope, there's a *(fiber?)* functor

$$RG_Q : \mathcal{C}_Q \to \mathcal{C}_Q^{IR}$$

III. If $Q \sim^{\mu} Q'$, then $\mathcal{C}_Q \simeq \mathcal{C}_{Q'}$
IV. If $\mathcal{T} = \mathcal{T}_{G,R}$, then $\mathcal{C}_Q \simeq \mathcal{KP}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathcal{R}_{G,N})$ *(cf. [CW])*

Cohomological Hall Algebra

Stack of objects in Rep_Q:

$$\mathfrak{M} = \coprod_{\gamma \in \mathbb{N}^{Q_0}} \left[\bigoplus_{i \to j \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j}) / \bigtimes_{i \in Q_0} GL(\gamma_i) \right]$$

a convolution diagram

$$\mathfrak{M} \times \mathfrak{M} \xleftarrow{p} \mathfrak{E} \xrightarrow{q} \mathfrak{M}$$

$$([C],[A]) \leftarrow\!\shortmid [A \to B \to C] \mapsto [B]$$

Cohomological Hall algebra

$$\mathcal{H}_Q := (H^*(\mathfrak{M}), m = q_* \circ p^*)$$

Bimodules

Let $f \in \mathbb{Z}^{Q_0}$. Stack of f-framed objects in Rep_Q: $f = [1, -1]$

$$\mathfrak{M}^f = \coprod_{\gamma \in \mathbb{N}^{Q_0}} \left[\frac{\bigoplus_{i \to j \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j})}{\bigoplus_{\substack{i \in Q_0 \\ f_i > 0}} (\mathbb{C}^{\gamma_i})^{f_i} \bigoplus_{\substack{i \in Q_0 \\ f_i < 0}} ((\mathbb{C}^{\gamma_i})^{-f_i})^*} \Big/ \bigtimes_{i \in Q_0} GL(\gamma_i) \right]$$

convolution diagrams

$$\mathfrak{M}^f \times \mathfrak{M} \xleftarrow{p_R^f} \mathfrak{E}^{f,R} \xrightarrow{q_R^f} \mathfrak{M}^f \qquad \mathfrak{M} \times \mathfrak{M}^f \xleftarrow{p_L^f} \mathfrak{E}^{f,L} \xrightarrow{q_L^f} \mathfrak{M}^f$$

$$([C], [A]) \leftarrow [A \to B \to C] \mapsto [B]$$

Resulting CoHA bimodule

$$\mathbf{B}_f := \left(H^*(\mathfrak{M}^f), act_{R/L}^f = (q_{R/L}^f)_* \circ (p_{R/L}^f)^* \right)$$

Explicit formulas

CoHA has some nice elements $\{e^i_n \in H^*(\mathfrak{M}|_{\gamma=\delta_i}) | i \in Q_0, n \in \mathbb{N}\}$;
let $e^i(z) := \sum_{n\geq 0} \frac{e^i_n}{z^{n+1}}$, then[1]

$$e^i(z)e^i(w) = -e^i(w)e^i(z); \qquad e^i(z)e^j(w) = (z-w)^{|i\to j|}e^j(w)e^i(z)|_{pol}.$$

For bimodules $\mathbf{B}_f|_{\gamma=0} \simeq \mathbb{C}b_f$ that satisfies

$$e^i_{n+f_i}b_f = \pm b_f e^i_n, \quad n \geq \max[0, -f_i]$$

Prop (WCF): as a right module, $\mathbf{B}_f \simeq H^*(\mathfrak{M}^f_{ss}) \otimes \mathcal{H}_Q$.

Define $\mathcal{B}_f := \mathcal{H}_Q b_f \mathcal{H}_Q$. We expect similarly $\mathcal{B}_f \simeq V_f \otimes \mathcal{H}_Q$

[1] Iff Q is acyclic it's a presentation!

IR category $\mathcal{C}^{IR}$

Look at a Hopf algebra $\Theta := \mathbb{C}[\alpha^i_n] \rtimes \mathbb{C}[J]$, with

$$[J, \alpha^i_n] = 2n\, \alpha^i_n, \qquad \Delta(J) = J \otimes 1 + 1 \otimes J + \sum_{i \to j} (\alpha^i_0 \otimes \alpha^j_0 - \alpha^j_0 \otimes \alpha^i_0)$$

Define $\mathcal{C}^{IR}$ to be a category of Θ-modules s.t.

- α^i_0 and J act semi-simply with integral eigenvalues.
- Eigenspaces are f.d. and eigenvalues are bounded from below*

Prop:

1. If $\det \chi^{skew}_Q = 1$,
   $\mathcal{C}^{IR,\, f.d.} \simeq \mathcal{P}^{G(\mathcal{O}) \rtimes \mathbb{C}^*}_{coh}(\mathrm{Gr}_{(\mathbb{C}^*)^{|Q_0|/2}}) \simeq$ Lines in IR
2. $K_0(\mathcal{C}^{IR,\, f.d.}) \simeq \mathbb{Z}[\mathfrak{q}^{\pm}]\langle X_\gamma \rangle / X_\gamma X_{\gamma'} = \mathfrak{q}^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}$
3. $K_0(\mathcal{C}^{IR}) \simeq \{\sum_{\gamma > \gamma_0} m_\gamma(\mathfrak{q}) X_\gamma | m_\gamma \in \mathbb{Z}[\mathfrak{q}^{-1}, \mathfrak{q}]]\}$
4. $\mathcal{H}_Q$ is an algebra in $\mathcal{C}^{IR}$ via
   $\alpha^i_n \cdot e^j_m = \delta_{ij} e^i_{n+m}$, $J \cdot e^i_n = (2n+1) e^i_n$

Chiralization

CoHA is acted on by $\mathbb{C}_+$, via $\tau_w : e^i(z) \mapsto e^i(z-w)$.

One can define new bimodules $\mathcal{B}_f^{(w)}$ by twisting both actions.

Conj: For $w \neq 0$, there's a rational iso

$$R(w) : \mathcal{B}_f^{(w)} \otimes \mathcal{B}_{f'} \xrightarrow{\sim} \mathcal{B}_{f'} \otimes \mathcal{B}_f^{(w)}$$

s.t. $r := w^{\#} R(w)|_{w=0}$ is a renormalized r-matrix.

Example: $Q = A_2$

There are five bimodules $\mathcal{B}_i := \mathcal{B}_{f_i}$ that should generate everything

$$f_0 = [0,-1],\ f_1 = [-1,0],\ f_2 = [0,1],\ f_3 = [1,0],\ f_4 = [1,-1].$$

They satisfy

$$\mathcal{B}_i \otimes \mathcal{B}_{i+1} \simeq \mathfrak{q}^{-2}\mathcal{B}_{i+1} \otimes \mathcal{B}_i$$
$$0 \to \mathfrak{q}\mathcal{B}_i \to \mathcal{B}_{i+1} \otimes \mathcal{B}_{i-1} \to \mathcal{H}_Q \to 0$$
$$0 \to \mathcal{H}_Q \to \mathcal{B}_{i-1} \otimes \mathcal{B}_{i+1} \to \mathfrak{q}^{-1}\mathcal{B}_i \to 0$$

Prop: There's an injective map $\mathcal{A}_{\mathfrak{q}}(A_2) \to K_0(\mathcal{C}_{A_2})$

Example: $Q = A_2$

$0 \to 0$

$B_i \quad B_{i+1}$

There are five bimodules $\mathcal{B}_i := \mathcal{B}_{f_i}$ that should generate everything

$$f_0 = [0, -1],\ f_1 = [-1, 0],\ f_2 = [0, 1],\ f_3 = [1, 0],\ f_4 = [1, -1].$$

They satisfy $\quad [B_{i-1}][B_{i+1}] = 1 + [B_i]$

$$\mathcal{B}_i \otimes \mathcal{B}_{i+1} \simeq \mathfrak{q}^{-2}\mathcal{B}_{i+1} \otimes \mathcal{B}_i$$

$$0 \to \mathfrak{q}\mathcal{B}_i \to \mathcal{B}_{i+1} \otimes \mathcal{B}_{i-1} \to \mathcal{H}_Q \to 0$$

$$0 \to \mathcal{H}_Q \to \mathcal{B}_{i-1} \otimes \mathcal{B}_{i+1} \to \mathfrak{q}^{-1}\mathcal{B}_i \to 0$$

Prop: There's an injective map $\mathcal{A}_{\mathfrak{q}}(A_2) \to K_0(\mathcal{C}_{A_2})$