Title: Abelianization of Virasoro blocks at c=1

Speakers: Andrew Neitzke

Collection/Series: Holomorphic-topological field theories and representation theory

Subject: Mathematical physics

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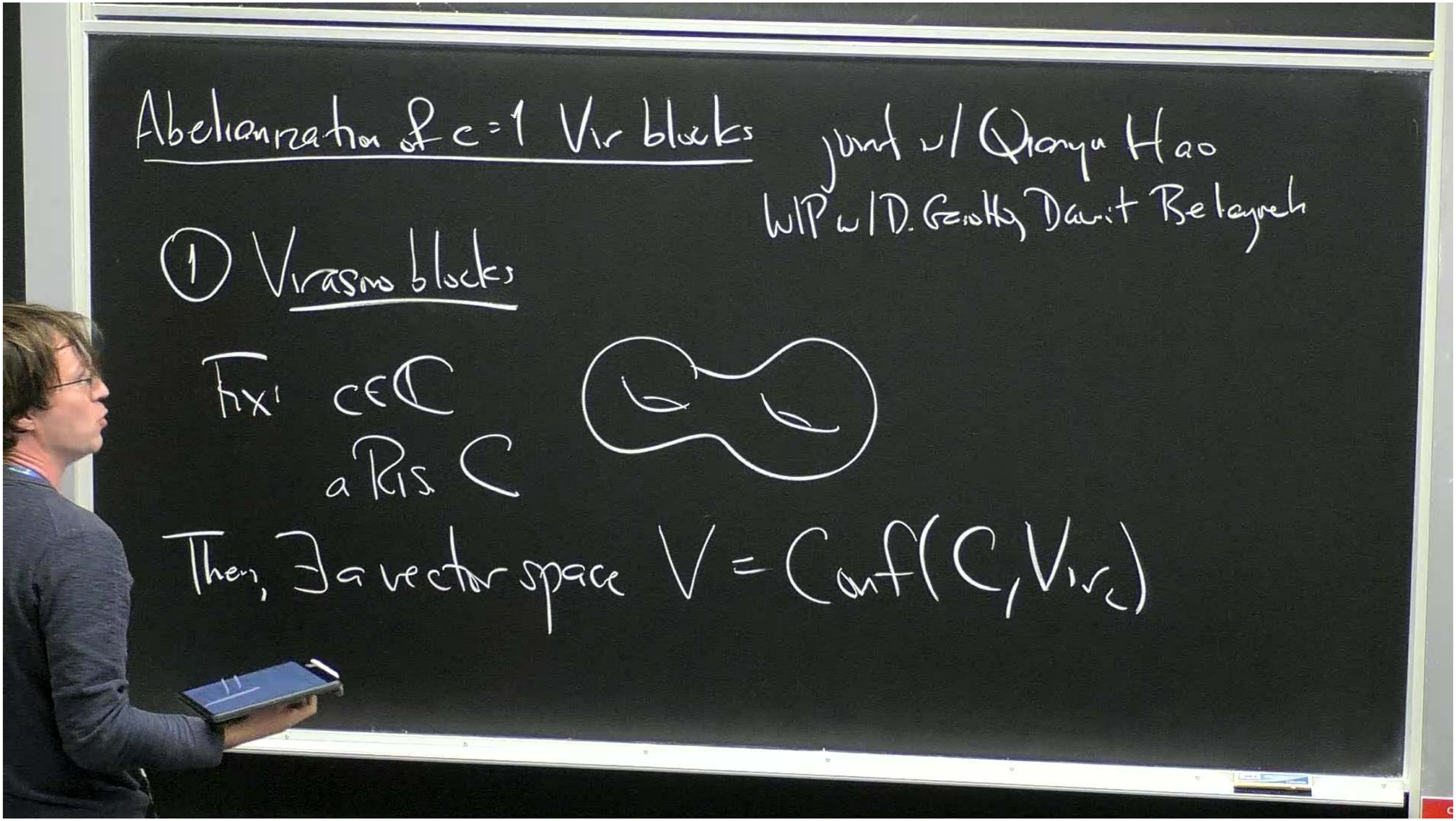
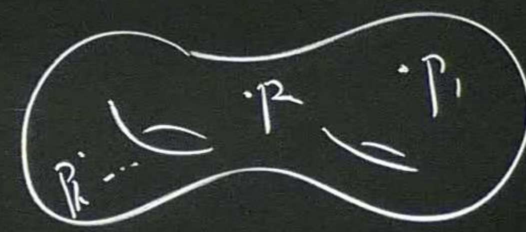
Abstract:

I will describe a new method for constructing conformal blocks for the Virasoro vertex algebra with central charge c=1, by relating them to conformal blocks for the Heisenberg algebra on a branched double cover. The construction is joint work with Qianyu Hao. It fits into the "holomorphic-topological" theme of the conference only inasmuch as vertex algebras are holomorphic objects and nonabelianization involves topological line defects. One concrete output is new formulas for tau functions in the sense of integrable systems -- most concretely, for Painleve equations -- as I will briefly explain; this part is joint work in progress with Dawit Belayneh and Davide Gaiotto.


Abelianization of c=1 Vir blocks
joint w/ Qianyu Hao
WIP w/ D. Gaiotto, Dawit Belayneh
1 Virasoro blocks
Fix: c∈ℂ
a RS C
Then, ∃ a vector space V = Conf(C, Vir_c)

An element $\Psi \in Vir_c$ means a system of correlation functions

$$\langle T(p_1)\cdots T(p_k)\rangle \quad \forall k \geq 0$$



- S_k-invariant
- meromorphic in p_i, sing at $p_i = p_j$
- singularities dictated by Vir OPE

An element $\Psi \in \mathrm{Vir}_c$ means a system of <u>correlation functions</u>

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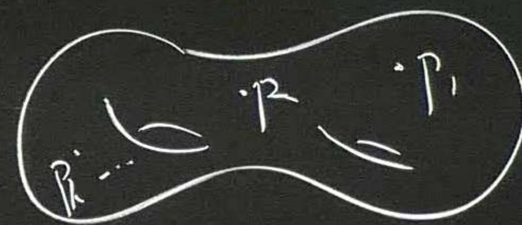


- S_k-invariant
- meromorphic in p_i, sing. at $p_i = p_j$
- singularities dictated by Vir OPE

$$T(p)\,T(p') = \frac{\frac{1}{2}c}{(z(p)-z(p'))^4} + \frac{2T(p')}{(z(p)-z(p'))^2} + \frac{\partial_{z(p')}T(p')}{z(p)-z(p')} + \text{reg.}$$

as $p \to$

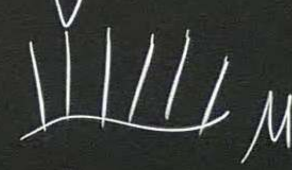
$$\left(\text{so e.g. } \langle \underbrace{T(p)\,T(p')\cdots}_{k+2 \text{ points}} \rangle = \frac{\frac{1}{2}c\langle\cdots\rangle}{(z(p)-z(p'))} + \cdots\right)$$

– coord xform law $\quad T(p)^w = T(p)^z\left(\frac{dz}{dw}\right)^2 + \frac{c}{12}\{z,w\}$

Basic Q: how to <u>produce</u> these objects?

Expectations:

- If we fix a pants decomposition of C and
  $a = (a_1, \dots, a_\ell) \in \mathbb{C}^\ell$ ($\ell = \#$ pant legs)
  then $\Psi_a \in V$. Ψ_a used Liouville theory, AGT
- V carries a (twisted) flat connection



$q = q(c)$

- V carries a representation of skein algebra $Sk_q(C)$

At $c=1$:

- $q=-1$, $Sk_{-1}(C, SL_2)$ is <u>commutative</u>,

  Spec $Sk_{-1}(C, SL_2)$ = (twisted) <u>character variety</u> $\mathcal{M}_B(C, SL_2)$

- For (generic) $X \in \mathcal{M}_B(C, SL_2)$ have a 1-d eigenspace

  $\mathcal{L}_X \subset V \longrightarrow$ line bundle $\mathcal{L} \to \mathcal{M}_B(C, SL_2)$

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$\mathcal{L}_X \subset V \longrightarrow$ line bundle $\mathcal{L} \to \mathcal{M}_B(C, SL_2)$

- If $\Psi_X \in \mathcal{L}_X \subset V$ is covariantly constant over $\mathcal{M}_q$ then $\langle 1 \rangle$

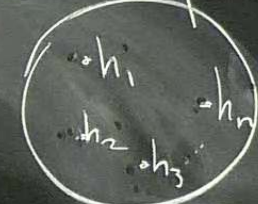
- If $\Psi_X \in \mathcal{L}_X \subset V$ is covariantly constant over M_g then $\langle \mathbb{1} \rangle : M_g \to \mathbb{C}$

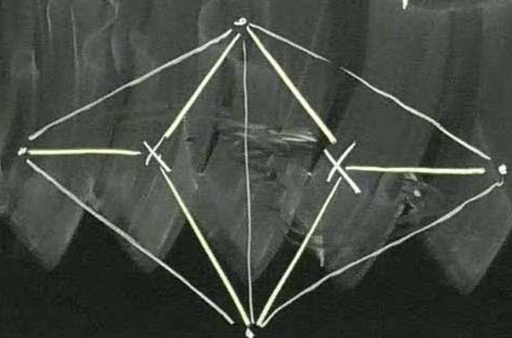
is a "τ-function with initial condition $\underline{\underline{X}}$"

From work of [I, IM, CLPT, FG, GS, ...]
we extract one more prediction: fix

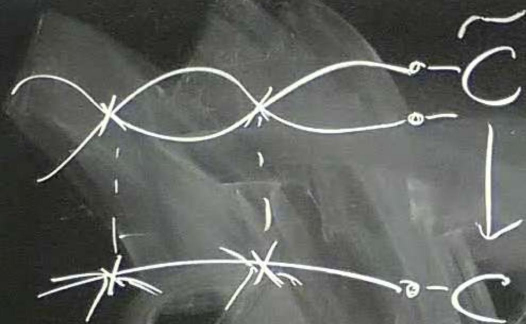
- a branched double cover $\tilde{C} \to C$
- and associated spectral network ($\leftrightarrow$ ideal Δ)

C with punctures:





x = branch pt of $\tilde{C} \to C$
• = punctures



— a Lagrangian decomposition

$$H_1(\tilde{C}, \mathbb{Z})^{odd} = \langle A_1, A_2, \ldots, A_k \rangle \oplus \langle B_1, \ldots, B_k \rangle$$

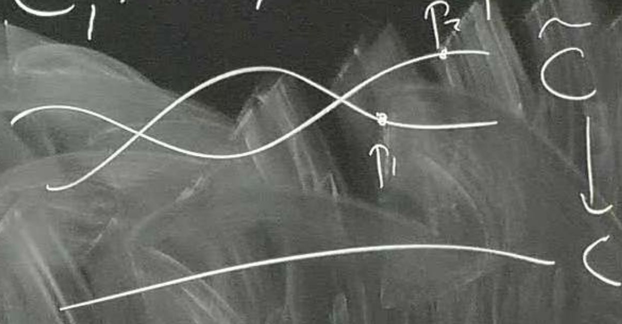
$$\Rightarrow a \in \mathbb{C}^k$$

Then, expect a block $\Psi_a \in V$ (cf. [CK, KT])

and if (ν, ρ) are cluster coords of $X \in \mathcal{M}(C, SL_2)$

then $\Psi_X = \sum_n \Psi_{a=\nu+2\pi i n}\, e^{in\rho} \in \mathcal{L}_X \subset V.$

A Heisenberg block $\Phi \in \mathrm{Conf}(\tilde{C}, \mathrm{Heis})$ is a sys. of correlation func.



$$\langle J(p_1) \cdots J(p_n) \rangle_\Phi$$

obeying (in c.f.) $$J(p)J(p') = \frac{1}{(z(p)-z(p'))^2} + \text{reg.}$$

$$J(p)^z = J(p)^w \frac{dw}{dz}$$

$\tilde{\Psi}_a \in \mathrm{Conf}(\tilde{C}, \mathrm{Heis})$ constructed explicitly:

$\langle 1 \rangle = 1$, $\eta \in \Gamma(K_{\tilde{C}})$ $\oint_{A_i} \eta = a_i$

$\langle J(p) \rangle = \eta(p)$

$\langle J(p) J(q) \rangle = \eta(p)\eta(q) + B(p,q)$

Bergman kernel: section of $K \boxtimes K$

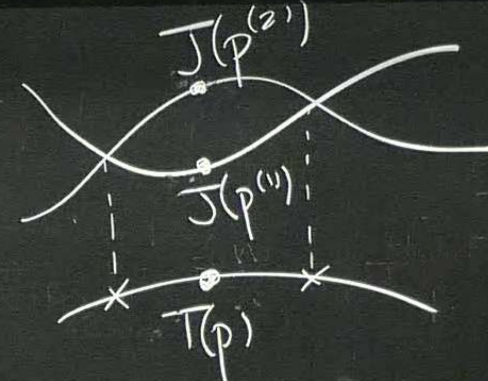
$\oint_{p \in A} B(p,q) = 0$. $B(p,q) \sim \dfrac{dz(p)\, dz(q)}{(z(p)-z(q))^2}$ as $p \to q$

How to build Ψ_a from $\tilde{\Psi}_a$?

First try: free boson construction

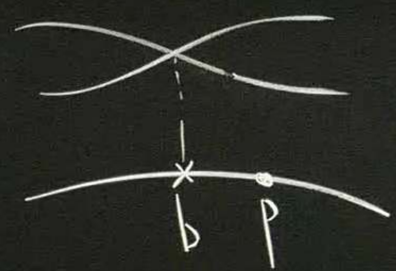
$$T(p) \overset{ab}{\rightsquigarrow} \frac{1}{4} :\left(J(p^{(1)}) - J(p^{(2)})\right)^2:$$

$$\langle \cdots \rangle_{\Psi_a} = \langle ab(\cdots) \rangle_{\tilde{\Psi}_a}$$



(★) almost works.

Singularities of $\langle T(p)\,T(p')\cdots\rangle$ as $p \to p'$ ✓ with $c=1$



But, as $p \to b$ branch point

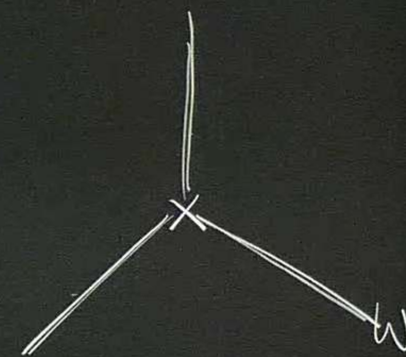
$\langle T(p)\cdots\rangle$ has a singularity $\sim \dfrac{\frac{1}{16}}{\left(z(p)-z(b)\right)^2}$

(like an insertion of $W_{\frac{1}{16}}(b)$)

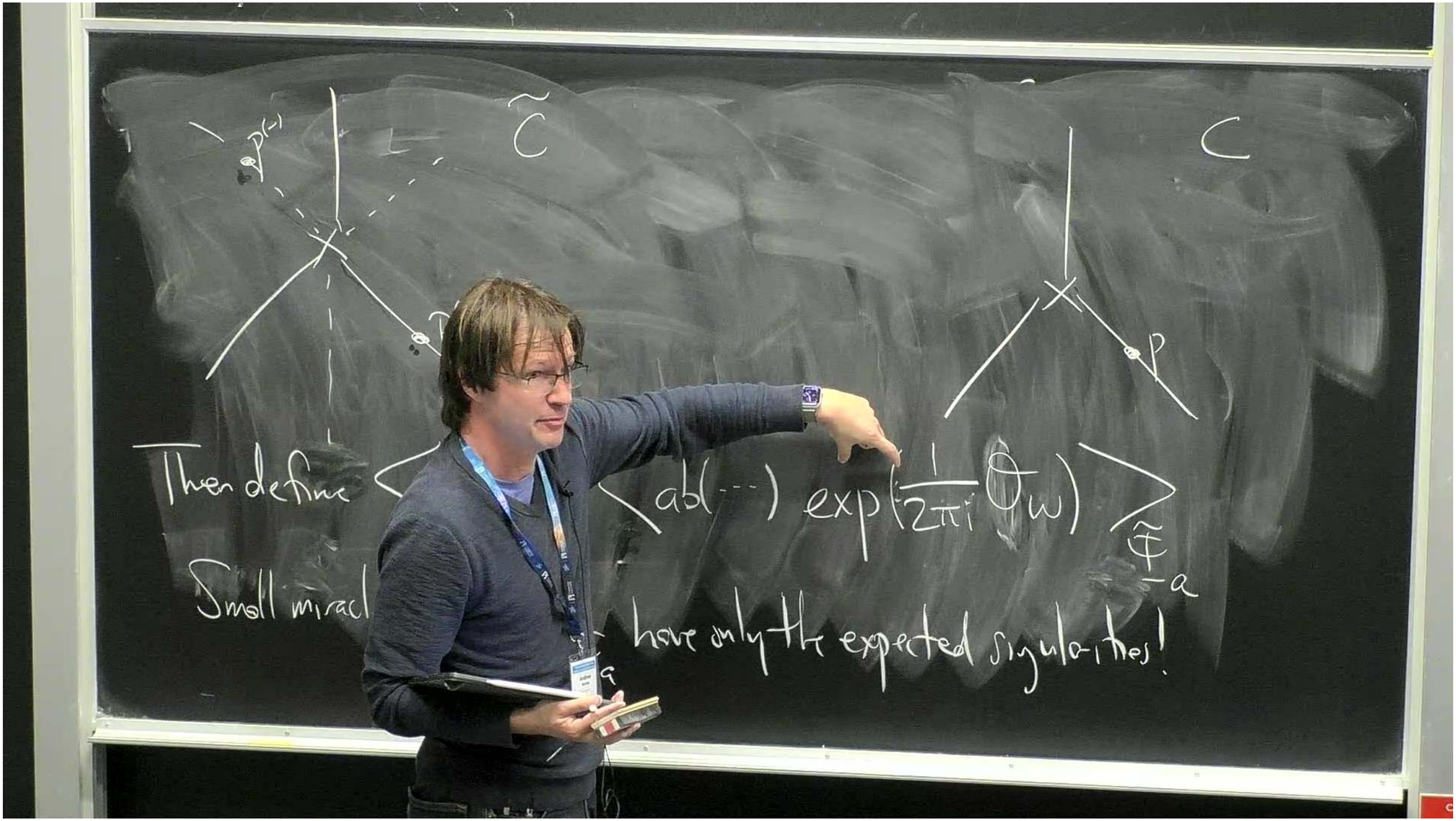
$$J(p)^2 = J(p)^w \frac{dw}{dz}$$

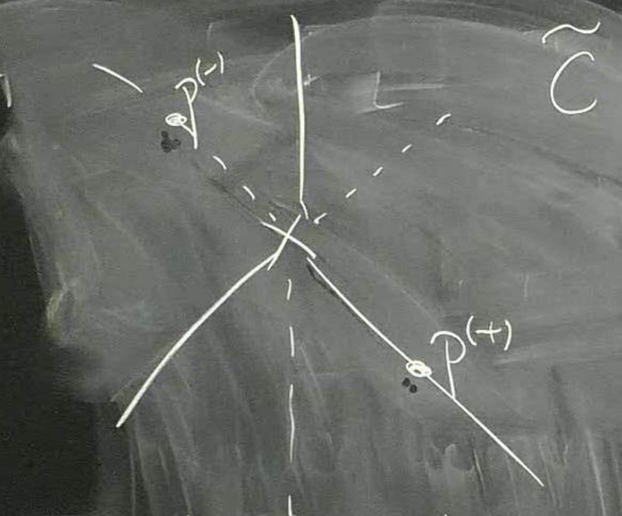
How to kill the singularities at branch pts?

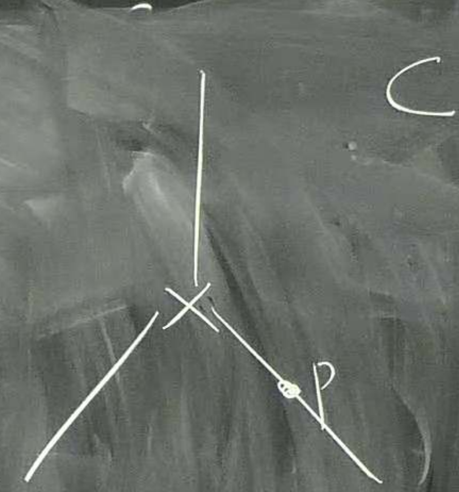
Define $\psi_+(p)\psi_-(q) = \dfrac{1}{z(p)-z(q)} :\exp \int_p^q J:$



$$\mathcal{O}_w = \int_w \psi_+(p^{(+)})\psi_-(p^{(-)})$$


C̃
C
p
Then define
exp
Small miracl
have only the expected singularities!





Then define $\langle \cdots \rangle_{\Phi_a} = \left\langle ab(\cdots)\, \exp\left(\frac{1}{2\pi i}\Theta_W\right) \right\rangle_{\tilde{\Phi}_a}$

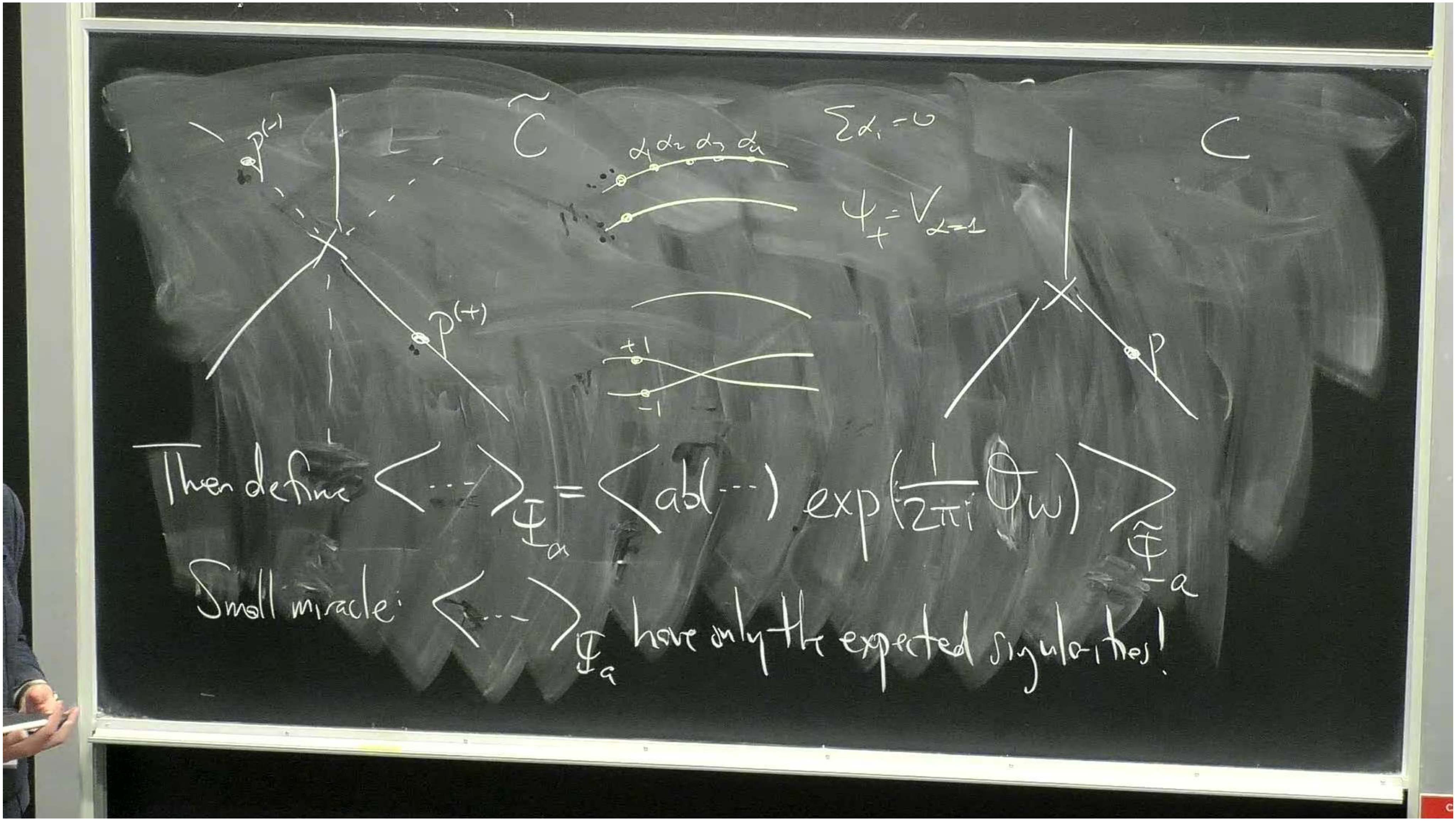
Small miracle: $\langle \cdots \rangle_{\Phi_a}$ have only the expected singularities!

$\Rightarrow$ τ-function

$$\langle 1 \rangle_{\Psi_X} = \frac{\Theta_{\tilde{C}}\left[\frac{\nu}{2\pi i}\middle|\frac{-\rho}{2\pi i}\right](0)}{\eta_{\tilde{C}}} \det\nolimits_{reg}(\mathbb{1} + I_{\nu,\rho})$$

$I_{\nu,\rho}$ is $\int$ op. acting on sections of $K^{1/2}$ over W

$K(p,q) = S_{\nu,\rho}(p^{(+)}, q^{(-)})$ (twisted Szegő kernel)


$\tilde{C}$
$p^{(-)}$
$\alpha_1 \alpha_2 \alpha_3 \alpha_4$
$\sum \alpha_i = 0$
$\Psi_+ = V_{\alpha=1}$
$p^{(+)}$
+1
-1
C
p
Then define $\langle \cdots \rangle_{\Phi_a} = \langle ab(\cdots) \exp(\frac{1}{2\pi i} \theta_w) \rangle_{\tilde{\Phi}_a}$
Small miracle: $\langle \cdots \rangle_{\Phi_a}$ have only the expected singularities!