Title: Lecture - Relativity (Core), PHYS 604

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Subject: Cosmology, Strong Gravity

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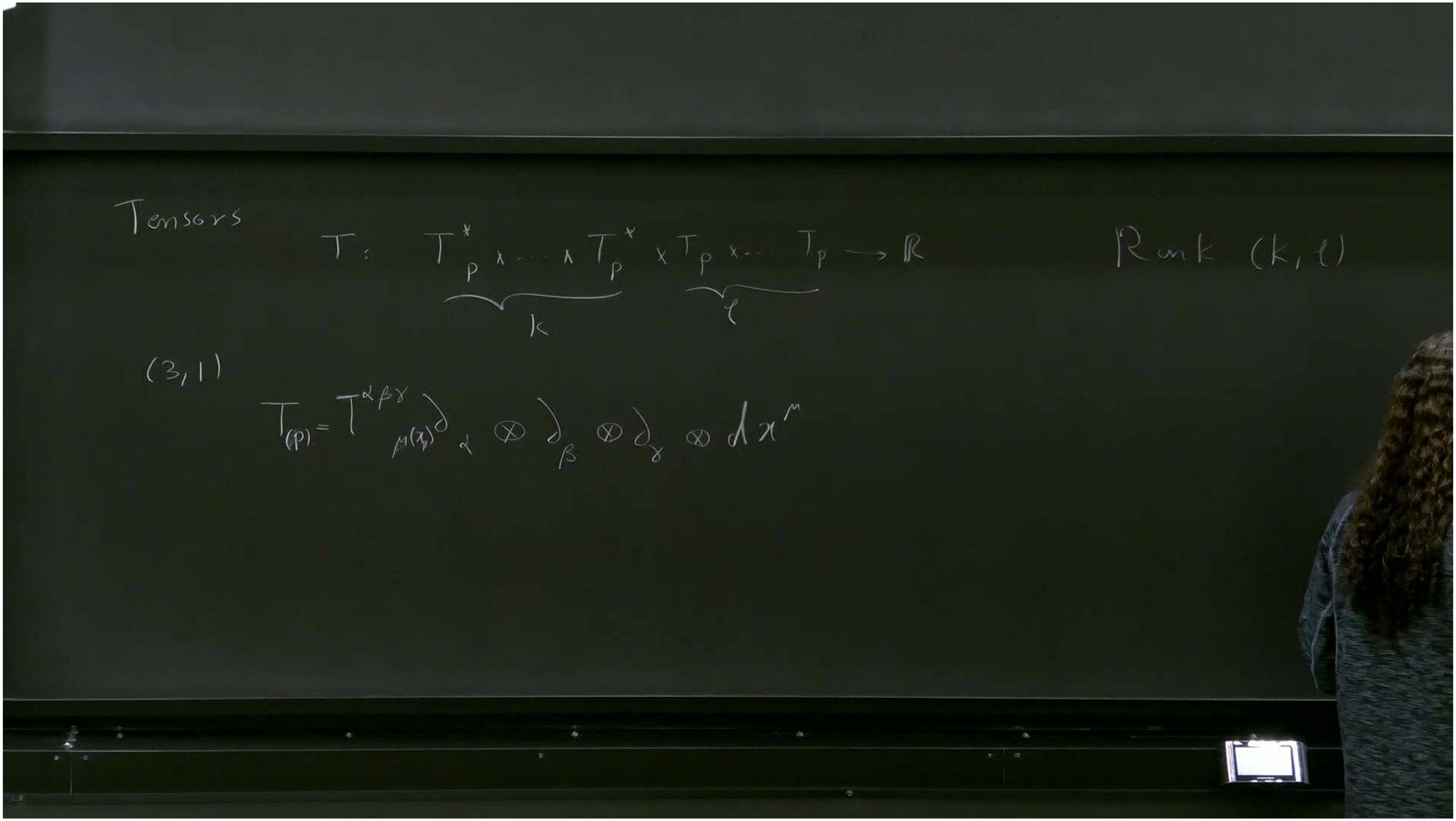

Tensors
$T: \underbrace{T_p^* \times \cdots \times T_p^*}_{k} \times \underbrace{T_p \times \cdots T_p}_{\ell} \to \mathbb{R}$
Rank (k, ℓ)
$(3,1)$
$T_{(p)} = T^{\alpha\beta\gamma}{}_{\mu(x_p)} \partial_\alpha \otimes \partial_\beta \otimes \partial_\gamma \otimes dx^\mu$

Rank (k, l)

Let T & S be tensors of Rank t, s

i) $T + S \rightarrow$ if Same type

ii) tensor product : $T \otimes S$ is Rank $t + s$

Ex : $T^{(2,1)}$ $S^{(0,1)}$

$T = T^{\alpha\beta}{}_{\gamma}\, \partial_\alpha \otimes \partial_\beta \otimes dx^\gamma$ $S = S_\gamma dx^\gamma$

$T \otimes S = T^{\alpha\beta}{}_{\gamma} S_\sigma\, \partial_\alpha \otimes \partial_\beta \otimes dx^\gamma \otimes dx^\sigma$ $(2,2)$

iii) Contraction $\longrightarrow$ lowers the rank

$$T = T^{\alpha\beta}{}_{\mu}\, \partial_\alpha \otimes \partial_\beta \otimes dx^\mu \qquad T_{\text{contrac.}} = T^{\alpha\beta}{}_{\alpha}\, \partial_\beta = T(dx^\sigma, \cdot, \partial_\sigma)$$

$(2,1)$

$$= T'^{\gamma\beta}{}_{\sigma}\, \partial'_\gamma \otimes \partial_\beta \otimes dx'^\mu$$

$r + s - 2 \times$ # of contractions

$$(T \cdot S)^{\alpha\delta \ldots \mu \ldots}{}_{\gamma \ldots \nu \ldots \lambda} = T^{\alpha \ldots \delta \ldots}{}_{\gamma \ldots}\, S^{\mu \ldots}{}_{\nu \ldots \delta \ldots}$$

$(1,0)$

$$S_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right) = T_{(\mu,\nu)}$$

$$A_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right) = T_{[\mu,\nu]}$$

u^α $\qquad u = u^\alpha \partial_\alpha$

iii) Covariant derivative (∇_μ) $\longleftarrow \Gamma$ (needs connectio

Scalar functions: $\phi \longrightarrow d\phi := \partial_\mu \phi \, dx^\mu$ $\qquad d\phi(V)$

$\nabla \phi \rightarrow d\phi$ $\qquad \omega_\alpha = \partial$

$0 \longrightarrow (0,1)$

Vector V^α : $v^\alpha{}_{,\beta} = \frac{\partial v^\alpha}{\partial x^\beta}$

$(1,0) \xrightarrow{?} (1,1)$

$\cancel{v'^\alpha{}_{,\beta} \overset{?}{=} \left(\frac{\partial x'^\alpha}{\partial x^\mu}\right)\left(\frac{\partial x^\nu}{\partial x'^\beta}\right) v^\mu{}_{,\nu}}$

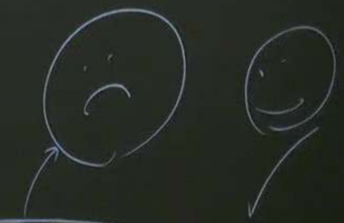
$x \rightarrow x'$

$(\;)^\nu{}_\mu$

(needs connection)

$$d\phi(V) = \partial_\mu \phi \, dx^\mu (v^\alpha \partial_\alpha) = v^\mu \partial_\mu \phi = V(\phi)$$

$$\omega_\alpha = \partial_\alpha \phi$$



$$v'^\alpha{}_{,\mu} = \frac{\partial v'^\alpha}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\alpha}{\partial x^\sigma} v^\sigma\right) = \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^\alpha}{\partial x^\sigma} v^\sigma_{(x)}\right) = \boxed{\left(\frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\sigma}\right) v^\sigma} + \boxed{\left(\frac{\partial x^\beta}{\partial x'^\mu}\right)\left(\frac{\partial x'^\alpha}{\partial x^\sigma}\right) v^\sigma{}_{,\beta}}$$

Covariant Derivative: $\nabla_\mu v^\alpha = v^\alpha_{,\mu} + \Gamma^\alpha_{\mu\sigma} v^\sigma$

$\hookrightarrow$ connection

$(0,1) \rightarrow (1,$

$$\star \quad \Gamma'^{\alpha}_{\beta\gamma} = \left(\frac{\partial x'}{\partial x^\kappa}\right)^\alpha \left(\frac{\partial x^\delta}{\partial x'^\beta}\right)\left(\frac{\partial x^\iota}{\partial x'^\gamma}\right) \Gamma^\kappa_{\delta\iota} - \frac{\partial^2 x'^\alpha}{\partial x^\kappa \partial x} \frac{\partial x^\kappa}{\partial x'^\beta} \frac{\partial x^\delta}{\partial x'^\gamma}$$

$$\nabla_\mu W_\alpha = \partial_\mu W_\alpha - \Gamma^\sigma_{\mu\alpha} W_\sigma$$

$$\nabla_\alpha T^{\gamma \ldots}{}_{\beta \ldots} = \partial_\alpha \left(T^{\gamma \ldots}{}_{\beta \ldots}\right) + \Gamma^\gamma_{\alpha\sigma} T^{\sigma \ldots}{}_{\beta \ldots} + \ldots - \Gamma^\sigma_{\alpha\beta} T^{\gamma \ldots}{}_{\sigma \ldots}$$

Covariant Derivative: $\nabla_\mu v^\alpha = v^\alpha{}_{,\mu} + \Gamma^\alpha_{\mu\sigma} v^\sigma$ $\qquad (0,1) \rightarrow ($

↳ connection

$$\star \quad \Gamma'^\alpha_{\beta\gamma} = \left(\frac{\partial x'}{\partial x^\kappa}\right)^\alpha \left(\frac{\partial x^\delta}{\partial x'^\beta}\right)\left(\frac{\partial x^\iota}{\partial x'^\gamma}\right)\Gamma^\kappa_{\delta\iota} - \frac{\partial^2 x'^\alpha}{\partial x^\kappa \partial x}\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}$$

$$\nabla_\mu W_\alpha = \partial_\mu W_\alpha - \Gamma^\sigma_{\mu\alpha} W_\sigma \qquad \Gamma^\sigma_{\mu\alpha}$$

$$\nabla_\alpha T^{\gamma\ldots}{}_{\beta\ldots} = \partial_\alpha\left(T^{\gamma\ldots}{}_{\beta\ldots}\right) + \Gamma^\gamma_{\alpha\sigma} T^{\sigma\ldots}{}_{\beta} + \ldots - \Gamma^\sigma_{\alpha\beta} T^{\gamma\ldots}{}_{\sigma\ldots}$$

$(0,1) \to (1,1)$

ection

Torsion Tensor : $T^{\alpha}{}_{\beta\gamma} \equiv -2\,\Gamma^{\alpha}{}_{[\gamma\beta]} = \Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma}$

GR: $T = 0 \to \Gamma^{\alpha}{}_{\gamma\beta} = \Gamma^{\alpha}{}_{\beta\gamma}$

$$\nabla_\mu \left(T^{\gamma\beta}{}_{\gamma} \right) = \mathrm{Cont}\left(\nabla_\mu T^{\gamma\beta}{}_{\sigma} \right)_{\sigma \to \gamma}$$

The metric tensor.

$$d(p,q) = \int_p^q d\lambda \sqrt{\left(\frac{ds}{d\lambda}\right)^2} = \int_p^q d\lambda \sqrt{\frac{d\vec{x}}{d\lambda}\cdot\frac{d\vec{x}}{d\lambda}}$$

g :

g: inner products $\quad g: T_p(M) \times T_p(M) \longrightarrow \mathbb{R} \quad (0,2)$

1) symmetric $\quad g(u,v) = g(v,u)$

2) non degenerate: $\quad g(\tilde{u}, v) = 0$ for all $v \in T_p(M) \Rightarrow u_p = 0$

$g(\tilde{u}, \cdot\,)$

g: inner products $\quad g : T_p(M) \times T_p(M) \longrightarrow \mathbb{R} \quad (0,2)$

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$\downarrow g^{-1}$

$$(g^{-1})^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}{}_{\nu}$$

$(gU)\vec{v} = \vec{0} \quad \forall \vec{v} \in \mathbb{R}^n$

$gU = 0$

$\underbrace{g^{-1} g}_{I} U = 0 \Rightarrow U = 0$