Title: Lecture - Combinatorial QFT, CO 739-002

Speakers: Michael Borinsky

Collection/Series: Combinatorial QFT, CO 739-002, September 4 - December 2, 2025

Subject: Mathematical physics, Quantum Fields and Strings

Date: November 04, 2025 - 3:00 PM

URL: https://pirsa.org/25110001

Suriyah, Gabi

Rajbir, Matteus

Zeus, Jeronimo

Kledion, Shreyas

Jury, Shiyue

Harper, Erica

Assignment due Nov 18

Thu 20

Suriyah, Jury, Shiyue

Nov 25

Jeronimo, Matteus, Erica

Nov 27

Kledion, Shreyas, Harper

Dec 2

Zeus, Gabi, Rajbir

Project guidelines → Course website

Fix $\alpha, \beta > 0$

Recall $\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t}$ $\quad (\Gamma(n+1) = n!)$

$$\Gamma^\alpha_\beta(z) = \alpha^{-z}\,\Gamma(z+\beta)$$

Definition: Let $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{R}[[x]]$.

We all $f(x)$ (α, β)-factorially divergent

if there is a sequence of numbers $c_0^f, c_1^f, c_2^f, \ldots \in \mathbb{R}$

s.t.

$$f_n \sim \sum_{k \geq 0} c_k^f \, \Gamma_\beta^\alpha(n-k)$$

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Example: $\varphi(x) = \sum_{n=0}^{\infty} n! \, x^n \rightarrow (1,1)$ factorially div.

$$c_0^\varphi = 1, \; c_1^\varphi = 0 = c_2^\varphi = \dots$$

Observation: If f, g are (α,β)-fac div

then also $h(x) = f(x) + g(x)$ is fac div.

(also scalar multiplication)

$\rightsquigarrow$ Let $\mathbb{R}^\alpha_\beta[[x]] \subset \mathbb{R}[[x]]$ be the <u>subspace</u>

of (α,β) fac. div. powerseries

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Lemma: If $f(x) = \sum_{n=0}^{\infty} f_n x^n$ with $|f_n| \leq C^n$ for some $C > 0$

Let $\mathbb{R}_\beta[[x]] \subset \mathbb{R}[[x]]$ be the subspace

of (q,β) fac. div. powerseries

Lemma: If $f(x) = \sum_{n=0}^{\infty} f_n x^n$ with $|f_n| \leq C^n$ for some $C > 0$

then $f(x) \in \mathbb{R}^q_\beta[[x]]$.

Lemma: If $f(x) = \sum_{n=0} f_n x^n$ with $|f_n| \leq C$ for some $C > 0$

then $f(x) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$.

Proof: $C^n \in \Theta(\Gamma^{\alpha}_{\beta}(n-R))$ for all α, β and $R \geq 0$.

then $f(x) \in \mathbb{R}_\beta[[x]]$.

Proof: $c^n \in \Theta(\Gamma_\beta^\alpha(n-R))$ for all α, β and $R \geq 0$.

Γ

$$f_n \sim 0 \cdot \Gamma_\beta^\alpha(n) + 0 \cdot \Gamma_\beta^\alpha(n-1) + \dots + \Theta(\Gamma_\beta^\alpha(n-R))$$

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Proposition (*): Let $f, g \in \mathbb{R}_\beta[[x]]$

then $h(x) = f(x)g(x) \in \mathbb{R}^a_\beta[[x]]$.

$\Rightarrow \mathbb{R}^a_\beta[[x]]$ is a subring of $\mathbb{R}[[x]]$.

Proof: $h(x) = \sum_{n=0} h_n x^n$; $h_n = \sum_{k=0}^{n} f_{n-k}\, g_k =$ (assume $2R \leq n$)

$$\sum_{k=0}^{R-1} f_{n-k}\, g_k + \sum_{k=0}^{R-1} f_k\, g_{n-k}$$

We still need to show that

$$\sum_{k=R}^{n-R} f_k g_{n-k} \in \Theta\left(\Gamma_\beta^\alpha(n-R)\right)$$

Lemma:

$$\Theta\left(\sum_{k=R}^{n-R} \Gamma_\beta^\alpha(k)\Gamma_\beta^\alpha(n-k)\right) \subseteq \Theta\left(\Gamma_\beta^\alpha(n-R)\right)$$

Proof. Uses log-convexity of Γ and $\Gamma(z+1) = z\Gamma(z)$.

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Lemma:

$$\Theta\left(\sum_{k=R}^{n-R} \Gamma^{\alpha}_{\beta}(k)\,\Gamma^{\alpha}_{\beta}(n-k)\right) \subseteq \Theta\left(\Gamma^{\alpha}_{\beta}(n-R)\right)$$

Proof: Uses log-convexity of Γ and $\Gamma(z+1) = z\Gamma(z)$.

To prove $(*)$ Use $|f_n| \leq C\,\Gamma^{\alpha}_{\beta}(n)$ (follows from $f_n \in \Theta(\Gamma^{\alpha}_{\beta}(n))$)

Definition: Let $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{R}[[x]]$.

We call $f(x)$ (α, β)-factorially divergent

if there is a sequence of numbers $c_0^f, c_1^f, c_2^f, \ldots \in \mathbb{R}$

s.t. $$f_n \sim \sum_{k \geq 0} c_k^f \, \Gamma_\beta^\alpha(n-k) \quad \text{for } n \to \infty$$

Example: $\varphi(x) = \sum_{n=0}^{\infty} n! \, x^n \to (1,1)$ factorially div.

$c_0^\varphi = 1, \; c_1^\varphi = 0 = c_2^\varphi = \ldots$

Example: $\varphi(x)=\sum_{n=0}^{\infty} n!\,x^n \to (1,1)$ factorially div.

$c_0^\varphi = 1,\ c_1^\varphi = 0 = c_2^\varphi = \dots$

Definition:

Let $A_\beta^\alpha : \mathbb{R}_\beta^\alpha[[x]] \to \mathbb{R}[[x]]$

be the operator that maps $\sum_{n=0}^{\infty} f_n x^n$, where

$f_n \sim \sum_{k=0} c_k^f \, \Gamma_\beta^\alpha(n-k)$

to $(A_\beta^\alpha f)(x) = \sum_{n=0} c_n^f x^n$

$$\ldots = \sum_{n=0} f_n x^n, \text{ where}$$

to $(A^\alpha_\beta f)(x) = \sum_{n=0} c^f_n x^n \qquad f_n \sim \sum_{k\geq 0} c^f_k \Gamma^\alpha_\beta(n-k)$

Idea: A^α_β maps a powerseries $\sum_{n=0}^{\infty} f_n x^n$ to the power series formed from the coefficients of the asy. exp. of f_n.

Definition: Let $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{R}[[x]]$.

We call $f(x)$ (α,β)-factorially divergent if there is a sequence of numbers $c^f_0, c^f_1, c^f_2, \ldots \in \mathbb{R}$

s.t. $f_n \sim \sum c^f \Gamma^\alpha (\ldots)$

then $f(x) \in \mathbb{R}_\beta[[x]]$.

Proof: $c^n \in \Theta(\Gamma_\beta^\alpha(n-R))$ for all α, β and $R \geq 0$.

* $\varphi(x) = \sum_{n \geq 0} n!\, x^n \Rightarrow (\mathcal{A}_\beta^\alpha \varphi)(x) = 1 + 0 \cdot x + 0 \cdot x^2 + \dots$

$\varphi(x) \in \mathbb{R}_1^1[[x]]$

* $f(x) = \sum_{n \geq 0} f_n x^n$ with $|f_n| \leq C^n$ for some $C > 0$

then $(\mathcal{A}_\beta^\alpha f)(x) = 0$

Quantitative version of Prop $(*)$:

If $h(x) = f(x)g(x)$ and $f, g \in \mathbb{R}^{\alpha}_{\beta}[[x]]$

then $(\dot{\Delta}^{\alpha}_{\beta} h)(x) = f(x)(\dot{\Delta}^{\alpha}_{\beta} g)(x) + (\dot{\Delta}^{\alpha}_{\beta} f)(x)\, g(x)$

$\rightsquigarrow$ $\dot{\Delta}^{\alpha}_{\beta}$ is a <u>derivation</u> on $\mathbb{R}^{\alpha}_{\beta}[[x]]$

also called alien derivative.

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$$f_n = \sqrt{n!} \rightsquigarrow (\mathcal{A} f)(x) = 0 \qquad \sum_{n=0}^{\infty} f_n x^n = \infty \quad \text{if } x \neq 0$$

$\mathbb{R}[[x]]$

* $f(x) = \sum_{n \geq 0} f_n x^n$ with $|f_n| \leq C^n$ for some $C > 0$

  then $(\mathcal{A}^{\alpha}_{\beta} f)(x) = 0$

To prove $(*)$ use $|f_n| \le C\,\Gamma^{\gamma}_{\beta}(n)$ (follows from $f_n \in \mathcal{O}(\Gamma^{\gamma}_{\beta}(n))$)

$$h_n = \sum_{k=0}^{n} k!(n-k)! \qquad \varphi(x) = \sum_{n=0}^{\infty} n!\,x^n$$

$$\rightsquigarrow h(x) = \varphi(x)^2 \rightsquigarrow (\mathcal{A}'_1 h) = 2\varphi(x) = 2(1 + x + 2x^2 + 6x^3 + \dots$$

$$\rightsquigarrow h_n = c_0^h \Gamma^1_1(n) + c_1^h \Gamma^1_1(n-1) + \dots$$

$$= 2n! + 2(n-1)! + 4(n-2)! + \dots$$

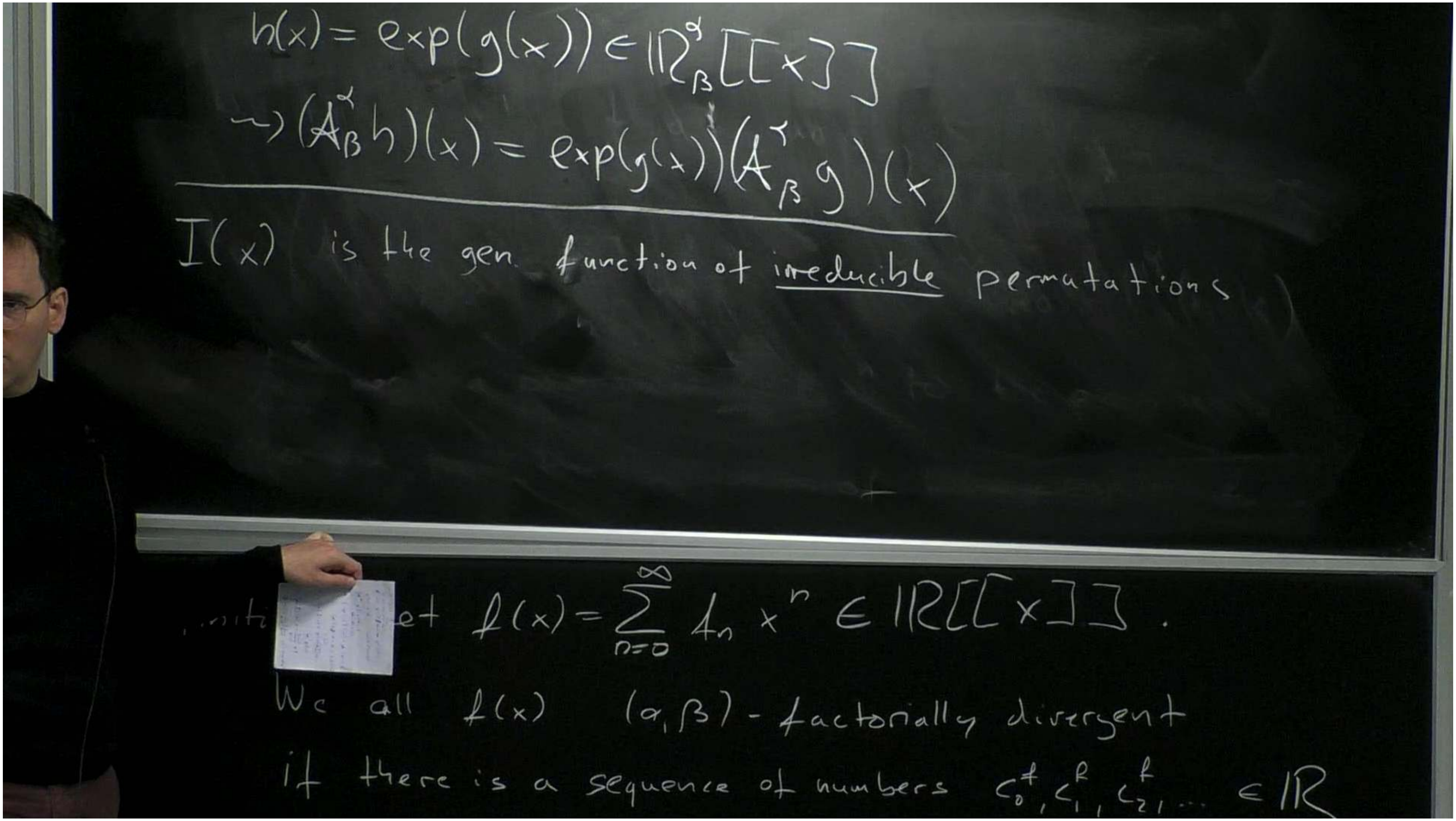
More properties of $\mathbb{R}^{\alpha}_{\beta}[[x]]$ and $\mathcal{A}^{\alpha}_{\beta}$.

* If $f(x) = \sum_{n=0}^{\infty} f_n x^n$ with $|f_n| < C^n$ for some $C > 0$

  and $g(x) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$ with $g(0) = 0$.

  then $f(g(x)) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$

  $$\left(\mathcal{A}^{\alpha}_{\beta} f(g(x))\right) = f'(g(x))\left(\mathcal{A}^{\alpha}_{\beta} g\right)(x)$$


$h(x) = \exp(g(x)) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$
$\rightsquigarrow (\mathcal{A}^{\alpha}_{\beta} h)(x) = \exp(g(x)) (\mathcal{A}^{\alpha}_{\beta} g)(x)$
$I(x)$ is the gen. function of irreducible permutations
et $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{R}[[x]]$.
We call $f(x)$ (α, β)-factorially divergent
if there is a sequence of numbers $c_0^f, c_1^f, c_2^f, \ldots \in \mathbb{R}$

$$h(x) = \exp(g(x)) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$$

$$\rightsquigarrow (\mathcal{A}^{\alpha}_{\beta} h)(x) = \exp(g(x)) (\mathcal{A}^{\alpha}_{\beta} g)(x)$$

$I(x)$ is the gen. function of irreducible permutations

$$I(x) = 1 - \frac{1}{\varphi(x)} \qquad (\mathcal{A}^{1}_{1} I)(x) = \frac{1}{\varphi(x)^2} (\mathcal{A}^{1}_{1} \varphi)(x) = \frac{1}{1-(1-\varphi(x))^2} = \frac{1}{\varphi(x)^2}$$

$$\rightsquigarrow I(x) \in \mathbb{R}^{1}_{1}[[x]]$$

Definition: Let $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{R}[[x]]$.

We call $f(x)$ (α, β)-factorially divergent

if there is a sequence of numbers $c_0^f, c_1^f, c_2^f, \ldots \in \mathbb{R}$

* Other derivative: $\partial_x : \mathbb{R}[[x]] \to \mathbb{R}[[x]]$

$$\sum_{n\geq 0} f_n x^n \longmapsto \sum_{n\geq 0} n f_n x^{n-1}$$

$$[A^\alpha_\beta, \partial_x] = \frac{\alpha}{x^2} - \frac{\beta}{x}$$

*

$f_n = n! \rightsquigarrow (\mathcal{A} f)(x) = 0 \quad \sum_{n=0} f_n x^n = \infty \quad \text{if } x \neq 0$

* $f, g \in \mathbb{R}^{\alpha}_{\beta}[[x]]$; $g(x) = x + \mathcal{O}(x^2)$

then $h(x) = f(g(x)) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$

and $(\mathcal{A}^{\alpha}_{\beta} h)(x) = f'(g(x))(\mathcal{A}^{\alpha}_{\beta} g)(x) + \left(\frac{x}{g(x)}\right)^{\beta} \exp\left(\alpha\left(\frac{1}{x} - \frac{1}{g(x)}\right)\right)(\mathcal{A}^{\alpha}_{\beta} f)(g(x))$

$f_n = \sqrt{n!}$ $\rightsquigarrow (\mathcal{A} f)(x) = 0$ $\sum_{n=0} f_n x^n = \infty$ if $x \neq 0$

* $f, g \in \mathbb{R}^{\alpha}_{\beta}[[x]]$; $g(x) = x + \mathcal{O}(x^2)$

then $h(x) = f(g(x)) \in \mathbb{R}^{\alpha}_{\beta}[[x]]$

and $(\mathcal{A}^{\alpha}_{\beta} h)(x) = f'(g(x))(\mathcal{A}^{\alpha}_{\beta} g)(x) + \left(\frac{x}{g(x)}\right)^{\beta} \exp\left(\alpha\left(\frac{1}{x} - \frac{1}{g(x)}\right)\right)(\mathcal{A}^{\alpha}_{\beta} f)(g(x))$

Remark: Define $(\tilde{\mathcal{A}}^{\alpha}_{\beta} f)(x) = x^{-\beta} \exp(-\frac{\alpha}{x})(\mathcal{A}^{\alpha}_{\beta} f)(x)$

then $\tilde{\mathcal{A}} h = f'(g(x))(\tilde{\mathcal{A}} g) + (\tilde{\mathcal{A}} f) \circ g$

$\rightsquigarrow$ trans-series $\mathbb{R}[[x, \exp(-\frac{\alpha}{x})]]$