Title: Light-cone formulation and Carrollian Physics

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Abstract:

Motivated by recent advances in "non-Lorentzian" physics, I will revisit the light-cone formulation of QFT, which has played a central role in the quantization of gauge theories. Much of its success can be traced to the presence of an underlying non-relativistic (Galilean) structure within the light-cone Poincaré algebra. I will show that light-cone Minkowski space belongs to a broader class of non-Lorentzian geometries, whose isometry algebra accommodates several interesting kinematical subalgebras—particularly of the Galilean and Carrollian types. These kinematical Lie algebras encode the symmetry transformations that govern spacetime kinematics and the evolution of physical systems. Finally, I will discuss certain aspects of light-cone field theories, emphasizing their connection to the hypersurface deformation algebra associated with Carrollian geometries, as well as their possible implications.

Light-cone Formulation and Carrollian Physics

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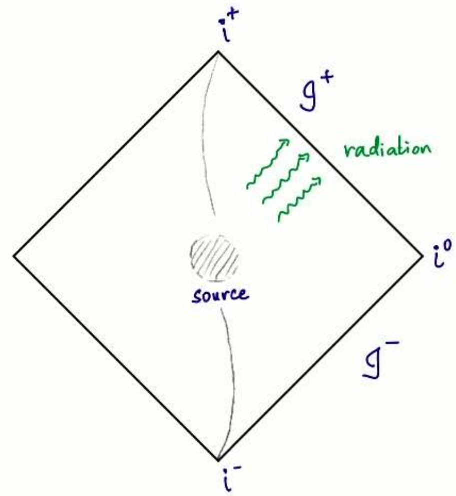
Motivation

Intersecting Null Hypersurfaces

- Asymptotically flat spacetimes
- Large-distance or IR behavior of gauge systems
- Asymptotic symmetries, Black hole physics
- Characteristic IVP, Double-null foliation
- Null quantization in GR
- Flat space holography:

$$(\mathrm{Grav})_d \longleftrightarrow (\mathrm{CFT})_{\mathrm{lower}\ d}$$

(i) Celestial CFT_2
(ii) Carrollian CFT_3 : $\mathrm{BMS}_4 \cong Ccarr_3$
"Carroll revolution" [Duval, Gibbons, Hovarthy (2014)]



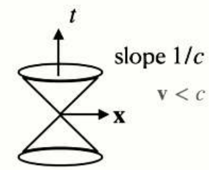
What is the Carroll group? [Bacry, Lévy-Leblond ’62][Sengupta ’62]

Poincaré group: It’s all relative

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$$

Lorentz boosts do not commute

$$[K_i, K_j] \sim M_{ij}$$



non-relativistic, $c \to \infty$

ultra-relativistic, $c \to 0$

Galilei group: Absolute time

$$t \to t'(t)\,,\; x^i \to R^i_j x^j - v^i t + a^i$$

Galilean boosts commute

$$[B_i, B_j] = 0\,, \quad [H, B_j] = -P_i$$

Carroll group: Absolute space

$$t \to t - b_i x^i + a^0\,,\; x^i \to x'^i(x^j)$$

Carroll boosts commute

$$[C_i, C_j] = 0\,, \quad [C_i, P_j] = \delta_{ij} H$$

Spacetimes with Galilei or Carroll as isometries: Non-Lorentzian spacetimes

(more details later)

Different routes to Carroll

- Group contraction method: Ultrarelativistic limit ($c \to 0$) of d-dim Poincaré

  [Levy-Leblond, Gomis, Hartong, Obers, Kleinschmidt, Duval, Gibbons, Horvathy, ...]

  Hamiltonian actions for Carroll field theories: *necessary and sufficient conditions*
  [Henneaux, Salgado-Rebolledo, 2021]

  Electric and Magnetic Carroll sectors

  Electric: $\mathcal{H}_E$ involve π or velocities $\dot{\phi}$

  Magnetic: $\mathcal{H}_M$ involve spatial gradients $\partial_i \phi$

- Intrinsic null geometry: Degenerate metric g, null vector $\mathfrak{n}^\mu$, etc.

  [Bergshoeff, Ciambelli, Gomis, Friedel, Hartong, Obers, Oling Vandoren, Petropoulous,, ...]

- Null reduction method from $(d+1)$-dim Bargmann spacetimes

$$dS^2 = -2dtds + U(x,t)dt^2 + V_A(x,t)dx^A dt + g_{AB}dx^A dx^B\,, \quad \mathfrak{n} = \partial_s$$

$\longrightarrow$ Galilean spacetime: Quotient by $\mathfrak{n}$ (KK reduction)

$\longrightarrow$ Carrollian spacetime: Restrict to constant t surface (Embedding)

[Duval, Gibbons, Horvathy, Zhang, 2014] [Duval, Burdet, Künzle, Perrin (1985)] [Julia-Nicolai '95]

Bargmann Spacetimes

Example: pp-waves

$$dS^2 = -2dudv - F(u, x^A)du^2 + \delta_{IJ}dx^I dx^J\,, \quad \mathfrak{n} = \partial_v$$

This talk: $(d+1)$-dimensional light-cone Minkowski

Light-cone coordinates:

$$x^+ = \frac{x^0 + x^{d+1}}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^{d+1}}{\sqrt{2}}, \quad x^i \; (i = 1, 2, \ldots, d-1)$$

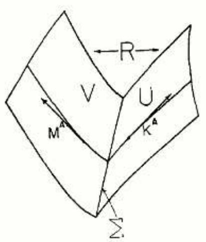
Metric: A special case of flat Bargmann

$$dS^2_{lc} = -2dx^+dx^- + \delta_{ij}dx^i dx^j$$

Vanilla version of double-null geometries

→ Minkowski: flat geometry

→ Field theories without gravity



[Gambini-Restuccia '78]
[d'Inverno-Smallwood '79]

Light-cone formulation

Light-cone formulation: Formulating QFTs using light-cone time x^+ or x^-

Why light-cone?

- One possible choice of 'time' in relativistic dynamics

  $x^+ = 0$ surface: Largest possible stability group [Dirac '49]

- Gauge constraints solvable $\rightarrow$ helps eliminate redundant d.o.f.

- Non-relativistic or Galilean aspects [Weinberg '66; Susskind'68]

  Light-cone Physics $\longleftrightarrow$ non-relativistic features

  $\longrightarrow$ 3D Galilei subgroup within 4D LC Poincaré

Many successes: QCD computations, String quantization, Scattering amplitudes, Self-dual actions, Higher spin theories...

Light-cone formulation and Carroll Physics

Contrails forming a light cone!

CPT Marseille, June 2024

Outline

- Kinematical Lie algebras
- Light-cone Minkowski and its isometries
- Subgroups of LC Poincaré
- Carroll theories from relativistic LC actions

Kinematical Lie algebras [Figueroa O'Farrill, arXiv: 2204.13609, arXiv:2204.13924]

Key ingredients:

Say, for a $(n+1)$-dim algebra

$$\{L_{ab}, P_a, B_b, H, Z\}\,, \quad a, b, = 1, 2, \dots n\,,$$

- $so(n)$ subalgebra: $[L, L] \sim L$
- P_a and B_b vectors: $[L_{ab}, P_c] = \delta_{a[c}P_{b]}\,, \quad [L_{ab}, B_c] = \delta_{a[c}B_{b]}$
- H scalar: $[L_{ab}, H] = 0$
- Z central element: $[Z, \cdot\,] = 0$ (optional)

The rest defines the type of Kin. Lie alg.

- Galilei: $[H, B_a] = -P_a\,, \quad [B_a, B_b] = 0$
- Bargmann: Central extension of Galilei - add $[P_a, B_b] = \delta_{ab}Z$
- Carroll: $[P_a, B_b] = \delta_{ab}H \quad [B_a, B_b] = 0$

Remarks

- Bargmann unifies Carroll and Galilei
- Non-Lorentzian spacetimes: these algebras as isometry $\rightarrow$ Boosts commute!

Light-cone Minkowski and its isometries

Cartesian coords in $\mathbb{R}^{d,1}$: $x^\mu = (x^0, x^1, \ldots, x^d)$

2 null vectors:

$$\mathfrak{m}^\mu = \frac{1}{\sqrt{2}}(-1, 0, ..., 1)\,, \quad \mathfrak{n}^\mu = \frac{1}{\sqrt{2}}(-1, 0, ..., -1)$$

$$\mathfrak{m}.\mathfrak{m} = \mathfrak{n}.\mathfrak{n} = 0\,, \quad \mathfrak{m}.\mathfrak{n} = -1$$

Light fronts:

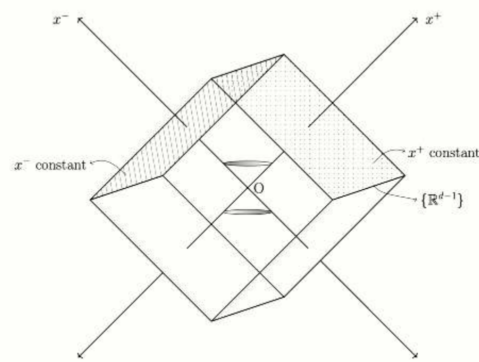
$$x^+ = \mathfrak{n}^\mu x_\mu = \frac{1}{\sqrt{2}}(x^0 + x^d)\,, \quad x^+ = \mathfrak{m}^\mu x_\mu = \frac{1}{\sqrt{2}}(x^0 - x^d)$$

$$dS^2 = \eta^{lc}_{\mu\nu} dx^\mu dx^\nu = -2dx^+ dx^- + dx^i dx_i$$

Double-null nature: $\underbrace{\mathbb{R}^{d-1}} \times \mathbb{R} \times \mathbb{R}$

If unhappy, do this:

$$y^+ = -\frac{1}{x^+}\,, \quad y^- = x^- - \frac{x^i x_i}{x^+}\,, \quad y^i = \frac{x^i}{x^+}$$



Isometry: $\mathbb{L}_\xi \eta^{lc} = 0$

Poincaré $\quad \xi^\mu = \omega^\mu_\nu x^\nu + a^\mu \qquad$ (duh!)

Generator $\quad G = a^\mu P_\mu + \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu}$

Bargmann structure of LC Minkowski

Poincaré preserves $\eta^{lc}_{\mu\nu}$, but

what if we **also** wanted to preserve one of the null vectors?

$$\mathfrak{n} = \partial_+ \,, \quad \mathfrak{m} = \partial_-$$

Bargmann structure of LC Minkowski

Bargmann structure: $(\mathcal{M}^{d+1}, G_{\mu\nu}, \mathfrak{n})$ such that $G(\mathfrak{n}, \mathfrak{n}) = 0 \longrightarrow$ Isometry: Bargmann algebra

2 null vectors in light-cone frame:

$$\mathfrak{m} = \partial_-, \quad \mathfrak{n} = \partial_+$$

2 Bargmann algebras within Poincaré $(P_\mu, M_{\mu\nu})$

- $\mathbb{L}_\xi \eta^{lc} = 0\,, \quad \mathbb{L}_\xi \mathfrak{m} = 0 \quad \longrightarrow \quad \mathfrak{b}_+ = \{P_+, P_-, P_i, M_{ij}, M_{-i}\}$
- $\mathbb{L}_\xi \eta^{lc} = 0\,, \quad \mathbb{L}_\xi \mathfrak{n} = 0 \quad \longrightarrow \quad \mathfrak{b}_- = \{P_+, P_-, P_i, M_{ij}, M_{+i}\}$

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Recall: Bargmann unifies Carroll and Galilei

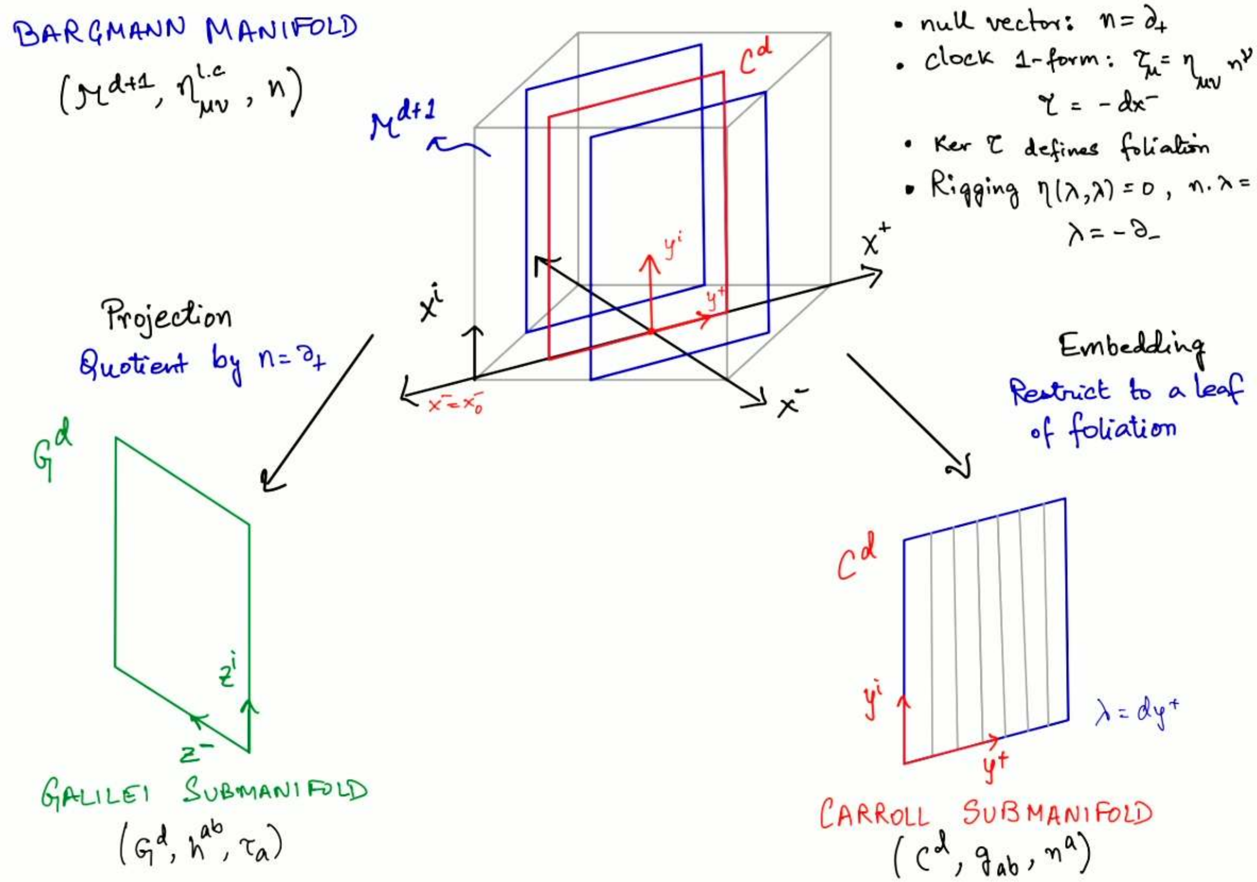
Both $\mathfrak{b}_\pm$ contain a pair of Carroll and Galilei subalgebra

E.g. Consider $\mathfrak{b}_-$ transformations

$$x^- \to \tilde{x}^-(x^-)\,, \quad x^+ \to \tilde{x}^+(x^+, x^i)\,, \quad x^i \to \tilde{x}^i(x^-, x^j)$$

- Carroll: Restrict to constant x^- surface
  $$x^+ \to \tilde{x}^+(x^+, x^i)\,, \quad x^i \to \tilde{x}^i(x^j) \quad \text{Absolute space}$$
- Galilei: Quotient by $\mathfrak{n} = \partial_+$
  $$x^- \to \tilde{x}^-(x^-)\,, \quad x^i \to \tilde{x}^i(x^-, x^j) \quad \text{Absolute time}$$

LC Minkwoski to Light front



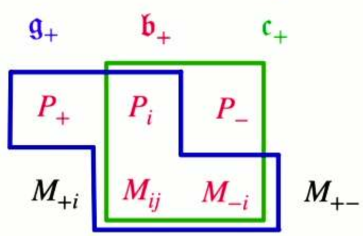
- null vector: $n = \partial_+$
- clock 1-form: $\tau_\mu = \eta_{\mu\nu} n^\nu$

  $\tau = -dx^-$
- Ker τ defines foliation
- Rigging $\eta(\lambda,\lambda) = 0$, $n \cdot \lambda = 1$

  $\lambda = -\partial_-$

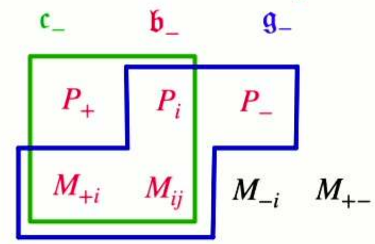
Kinematical subgroups within LC Poincaré

$\mathfrak{b}_+ = \{P_+, P_-, P_i, M_{ij}, M_{-i}\}$



x^+ Newtonian, x^- Carrollian

$\mathfrak{b}_- = \{P_+, P_-, P_i, M_{ij}, M_{+i}\}$



x^- Newtonian, x^+ Carrollian

- **Two copies** of d-dim Carroll, Bargmann, Galilei:

$$(\mathfrak{g}_+, \mathfrak{b}_+, \mathfrak{c}_+) \xleftrightarrow{x^+ \leftrightarrow x^-} (\mathfrak{g}_-, \mathfrak{b}_-, \mathfrak{c}_-)$$

- $\mathfrak{g}_+$: Galilei subgroup with light-cone Poincaré [Susskind '68] [Weinberg '66]

  $\rightarrow$ d-dim non-relativistic system in transverse x^i plane with dynamics along time x^+

- Carroll subgroups $\mathfrak{c}_\pm$: Stability group of light fronts at constant $x^\pm$

  $\rightarrow$ Preserves ultra-relativistic motion on lightlike planes

[**SM**, arXiv: 2406.10353] [Bagchi, Nachiketh, Soni (2024)]

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Outline

- Kinematical Lie algebras
- Light-cone Minkowski and its isometries
- Subgroups of LC Poincaré
- Carroll theories from relativistic LC actions

Field theories on light fronts

Consider x^+ (Carrollian) time and constant x^- hypersurface

Relevant subgroups: $\mathfrak{b}_- = \{P_+, P_-, P_i, M_{+i}, M_{ij}\}$ and $\mathfrak{c}_- = \{P_+, P_i, M_{+i}, M_{ij}\}$

Lorentzian LC action in $(d+1)$ dimensions

- Scalar field action:

$$\mathcal{S}[\phi,\dot{\phi}] \;=\; -\frac{1}{2}\int d^{d+1}x\,\eta^{lc}_{\mu\nu}\partial^{\mu}\phi\partial^{\nu}\phi \;=\; \int dx^{+}dx^{-}d^{d-1}x\,\left(\partial_{+}\phi\partial_{-}\phi-\frac{1}{2}\partial_{i}\phi\partial^{i}\phi\right)$$

- Hamiltonian formulation:

Conjugate momenta: $\pi=\dfrac{\delta\mathcal{L}}{\delta(\partial_{+}\phi)}=\partial_{-}\phi \quad\longrightarrow$ not related to velocities $\partial_{+}\phi$

$\Rightarrow$ Primary SCC : $\chi=\pi-\partial_{-}\phi$ typical in LC theories

Canonical Hamiltonian $\mathcal{H}\;=\;\pi\partial_{+}\phi-\mathcal{L}\;=\;\dfrac{1}{2}\partial_{i}\phi\partial^{i}\phi \quad\rightarrow$ No ∂_{+} or π terms in $\mathcal{H}$

Light-cone Hamiltonian action for scalars

$$\boxed{\mathcal{S}[\phi,\pi,\lambda]=\int dx^{+}dx^{-}d^{d-1}x\,\Big\{\pi\partial_{+}\phi-\mathcal{H}-\lambda(\pi-\partial_{-}\phi)\Big\}}$$

$${}^{(d+1)}\Omega=\int dx^{-}d^{d-1}x\;d_{V}\pi\wedge d_{V}\phi\,,\quad \{\pi(x),\phi(y)\}_{PB}=\delta(x^{-}-y^{-})\delta^{d-1}(\mathbf{x}-\mathbf{y})$$

If π eliminated, we go back to $\mathcal{S}[\phi,\dot{\phi}]\rightarrow$ reduced phase space

$$\{\phi(x),\phi(y)\}_{DB}\sim\theta(x^{-}-y^{-})\delta^{d-1}(\mathbf{x}-\mathbf{y})$$

Null reduction to d-dim Carroll theories

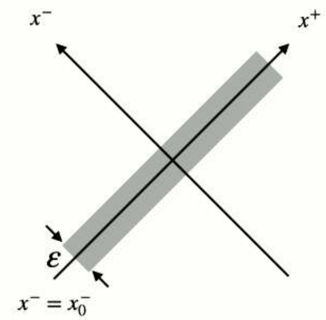
- Smearing function to restrict to constant x^- surface [Chen, Liu, Zheng, 2023]

  (Evaluate $\mathcal{S}$ around $x^- = x_0^- + \varepsilon$, then take $\varepsilon \to 0$)

$$\delta_\varepsilon(x - x_0^-) = \begin{cases} \frac{1}{\varepsilon}\,, & x_0^- - \frac{\varepsilon}{2} < x^- < x_0^- + \frac{\varepsilon}{2} \\ 0\,, & \text{otherwise} \end{cases}$$

  Assume x^- behaviour of fields (Rescaling in $c \to 0$ method)

$$\phi|_{x^-=x_0^-} = \phi_m(x^+, x^i)\,, \quad \pi|_{x^-=x_0^-} = p_m(x^+, x^i)$$



- Carroll action:

$$\lim_{\varepsilon\to 0} {}^{(d+1)}\mathcal{S}_\varepsilon[\phi, \pi] = {}^{(d)}\mathcal{S}_{Carr}[\phi_m, p_m]$$

- **Only** magnetic Carroll scalars obtained

$$\boxed{{}^{(d)}\mathcal{S}_{Carr}[\phi_m, p_m] = \int dx^+ d^{d-1}x \left\{ p_m \partial_+ \phi_m - \frac{1}{2}\partial_i \phi_m \partial^i \phi_m \right\}}$$

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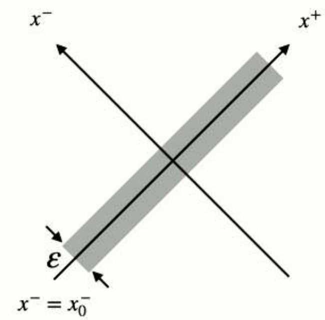

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- No way to get electric Carroll sector for *flat spacetimes* $\longrightarrow$ no π terms in $\mathcal{H}$
  Similar story for Galilean theories from null reduction
  [Julia, Nicolai '95] [Bergshoeff, Figueroa O'Farrrill, Gomis]

Resolution [SM, Arxiv: 2507.03081]

- Deform $\mathcal{L}^{lc}[\phi, \dot{\phi}]$ to break Poincaré invariance by hand

  Pick a null vector: $\mathfrak{n} = \partial_+ \quad \Rightarrow \quad$ Relevant subgroup $\mathfrak{b}_- = \{P_+, P_-, P_i, M_{+i}, M_{ij}\}$

$$\mathcal{L}^{Barg}[\mathfrak{n}^\mu, \phi, \dot{\phi}] = \frac{1}{2}\alpha(\partial_+\phi)^2 + \partial_+\phi\partial_-\phi - \frac{1}{2}\partial_i\phi\partial^i\phi$$

- Going to a general Bargmann spacetime

$$\eta^{lc}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix} \quad \longrightarrow \quad G_{\mu\nu} \begin{pmatrix} 0 & -1 & 0 \\ -1 & -\alpha & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}$$

  Now, only $\mathfrak{n} = \partial_+$ lies in the kernel of G, not $\mathfrak{m} = \partial_-$

- Change of coordinates to 'Bargmann LC coordinates' $(x^+_\alpha, x^-_\alpha, x^i_\alpha)$

$$\begin{aligned} x^+_\alpha &= x^+ + \frac{\alpha}{2} x^- \, , \\ x^-_\alpha &= x^- \, , \\ x^i_\alpha &= x^i \, , \end{aligned}$$

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Already known in LC quantization literature!!!
"Near light-front coordinates" [Lenz-Thies (1991)]

Resolution [SM, Arxiv: 2507.03081]

- Go to the Hamiltonian formulation

$$\mathcal{L}^{Barg} = \frac{1}{2}\alpha(\partial_+\phi)^2 + \partial_+\phi\partial_-\phi - \frac{1}{2}\partial_i\phi\partial^i\phi$$

$$\pi = \alpha\partial_+\phi + \partial_-\phi\,, \quad \mathcal{H}_C^{Barg} = \frac{1}{2\alpha}(\pi - \partial_-\phi)^2 + \frac{1}{2}\partial_i\phi\partial^i\phi \quad \longrightarrow \quad \text{No SCCs}$$

$$\boxed{\mathcal{S}^{Barg} = \int dx^+ dx^- d^{d-1}x \left\{\pi\partial_+\phi - \mathcal{H}_C^{Barg}\right\}, \quad \Omega^{Barg} = \int dx^- d^{d-1}x\, d_V\pi \wedge d_V\phi}$$

- Two possibilities that preserve canonical structure

Magnetic Carroll sector

$$\phi \to \phi_m, \quad \pi = \partial_-\phi \to p_m$$

$$\boxed{\mathcal{S}_M^{Carr} = \int dx^+ d^{d-1}x \left(p_m\partial_+\phi_m - \frac{1}{2}\partial_i\phi_m\partial^i\phi_m\right)}$$

EOM: $\partial_+\phi_m = 0\,, \quad \partial_+ p_m = \partial^i\partial_i\phi_m$

$\rightarrow$ Momenta not related to velocities

Electric Carroll sector

$$\phi \to \varepsilon\phi_e, \quad \pi \to p_e/\varepsilon, \quad \alpha \to \alpha/\varepsilon^2$$

$$\boxed{\mathcal{S}_E^{Carr} = \int dx^+ d^{d-1}x \left(p_m\partial_+\phi_m - \frac{1}{2\alpha}p_e^2\right)}$$

EOM: $\alpha\partial_+\phi_m = p_e\,, \quad \partial_+ p_m = 0$

$\rightarrow$ Momenta related to velocities

Resolution [SM, Arxiv: 2507.03081]

- Go to the Hamiltonian formulation

$$\mathcal{L}^{Barg} = \frac{1}{2}\alpha(\partial_+\phi)^2 + \partial_+\phi\partial_-\phi - \frac{1}{2}\partial_i\phi\partial^i\phi$$

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$\to$ Momenta not related to velocities

Electric Carroll sector

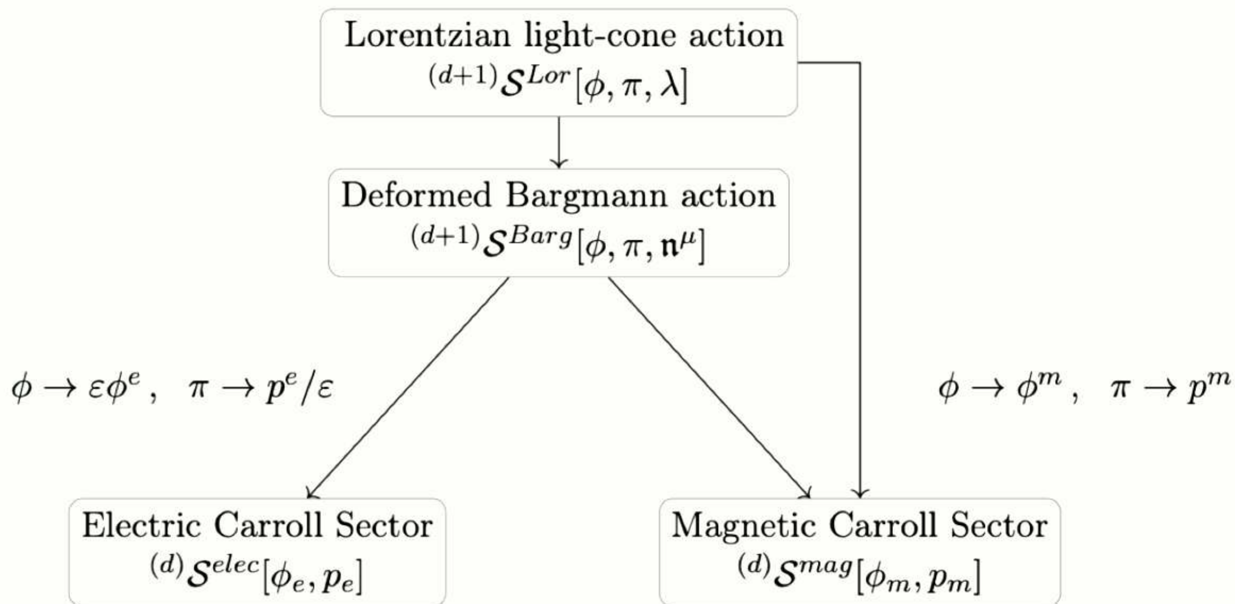
$$\phi \to \varepsilon\phi_e, \quad \pi \to p_e/\varepsilon, \quad \alpha \to \alpha/\varepsilon^2$$

$$\boxed{\mathcal{S}_E^{Carr} = \int dx^+ d^{d-1}x \left(p_m\partial_+\phi_m - \frac{1}{2\alpha}p_e^2\right)}$$

EOM: $\alpha\partial_+\phi_m = p_e\,, \quad \partial_+ p_m = 0$

$\to$ Momenta related to velocities

Deformation essential for electric Carroll sector



Proof of "Carroll-ness"

Hypersurface deformation algebra: [Teitelboim (1973)]

$$G[\xi^+,\xi^i] = \int d^\perp x\,(\xi^+\mathcal{H}^C + \xi^i\mathcal{P}_i^C)\,,\quad \xi^+ = b_i x^i + a^+\,,\quad \xi^i = \omega^i_j x^j + a^i$$

Carroll Hamiltonian actions: necessary and sufficient conditions

[Henneaux, Salgado-Rebolledo (2021)]

- Carroll commutation relations hold

$$\left[\mathcal{H}^C(x),\, \mathcal{H}^C(y)\right] = 0\,,\quad \left[\mathcal{H}^C(x),\, \mathcal{P}_i^C(y)\right] = \ldots\,,\quad \left[\mathcal{P}_i^C(x),\, \mathcal{P}_j^C(y)\right] = \ldots$$

  for both electric or magnetic cases

  $\longrightarrow$ Zero signature spacetimes [Henneaux (1978)] (Related to boosts commuting)

$$H = \int d^\perp x\mathcal{H}\,,\quad P_i = \int d^\perp x\mathcal{P}^i\,,\quad M^{ij} = x^{[i}P^{j]}\,,\quad B^i = x^i H$$

- Carrollian actions, $\mathcal{S}_M$ and $\mathcal{S}_E$, invariant under

$$\delta_{\xi^+,\xi^i}\phi^C = \{\phi^C, G[\xi^+,\xi^i]\}\,,\quad \delta_{\xi^+,\xi^i}p^C = \{p^C, G[\xi^+,\xi^i]\}$$

Proof of “Carroll-ness”

Hypersurface deformation algebra: [Teitelboim (1973)]

$$G[\xi^+,\xi^i] = \int d^\perp x \, (\xi^+ \mathcal{H}^C + \xi^i \mathcal{P}_i^C)\,, \quad \xi^+ = b_i x^i + a^+\,, \quad \xi^i = \omega^i_j x^j + a^i$$

Carroll Hamiltonian actions: necessary and sufficient conditions

[Henneaux, Salgado-Rebolledo (2021)]

- Carroll commutation relations hold

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- Additionally, under Carroll boosts b^i,

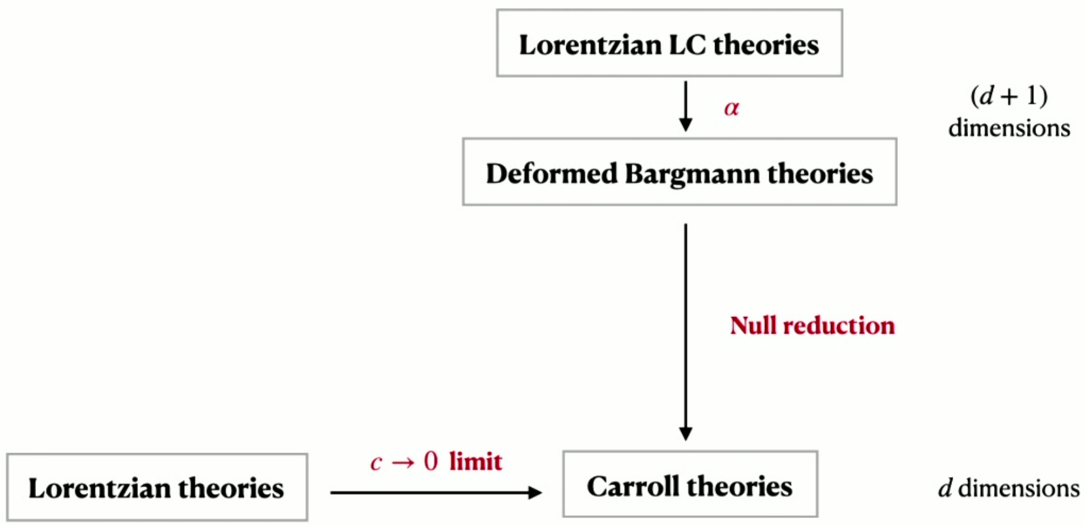
magnetic: $\delta_b \phi_m = 0 \,, \quad \delta_b p_m \neq 0$

electric: $\delta_b \phi_e \neq 0 \,, \quad \delta_b p_e = 0$

Same theories, different means

Starting point: Two very different Hamiltonian actions for Lorentzian field theories

→ Constraints, IVP, Phase space, etc.



- Equivalence of Carroll theories obtained from two different methods
- Works for EM, Yang-Mills, p-form fields, etc. [SM, Arxiv: 2507.03081]

General p-form gauge fields

- Lorentzian action for p-form field on $Mink^{d+1}$

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \,, \quad F = dA \,, \quad \mathcal{S}^{lc} = \int_M F^2$$

- Deformed Bargmann action

$$\mathcal{S}^{Barg} = \int_M F^2 + \alpha \int_M \mathfrak{n}^{\mu_1} \mathfrak{n}^{\nu_1} \dots \eta^{\mu_{p+1} \nu_{p+1}} F_{\mu_1 \dots \mu_{p+1}} F_{\nu_1 \dots \nu_{p+1}}$$

- Null reduction to Carroll:

Restrict to $x^- = x_0^- + \varepsilon$, then take $\varepsilon \to 0$

Set ‘minus’ component to zero: $F_{-\mu_1 \dots \mu_p} = 0$ Light-cone gauge

$$\mathcal{S}^{Carr} = \lim_{\varepsilon \to 0} \mathcal{S}^{Barg} = \int_C F^2$$

Magnetic Carroll sector

$$\boxed{F_{\mu_1 \dots \mu_{p+1}} \longrightarrow F_{i_1 \dots i_{p+1}}}$$

Scalars: $F_\mu \longrightarrow F_i = \partial_i \phi$

EM: $F_{\mu\nu} \longrightarrow F_{ij} \sim \epsilon_{ijk} B_k$

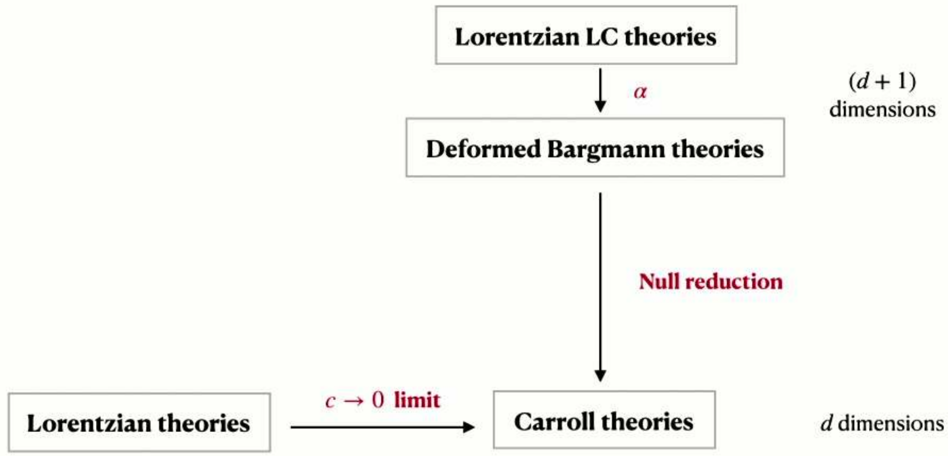
Electric Carroll sector

$$\boxed{F_{\mu_1 \dots \mu_{p+1}} \longrightarrow F_{+i_1 \dots i_p}}$$

Scalars: $F_\mu \longrightarrow F_+ = \partial_+ \phi$

EM: $F_{\mu\nu} \longrightarrow F_{+i} \sim E_i$

Summary and Outlook



- To-do list: Fermions [WIP], Curved geometries, Gravity (Double-null foliation)
- Lessons for Quantum Carroll theories?

  For Lorentzian QFTs: hard and cumbersome but possible

  (time t or x^0) Equal-time Quantization $=$ Light-front Quantization

  Do quantum properties from the two methods agree?
- Phenomenology: High-energy scattering, physics of Quark-Gluon plasma, etc.