Title: The Planimeter and Contact Transformation: A Perfect Embodiment of the Weyl-Heisenberg Group and Canonical Transformation's Lost Twin Sister

Speakers: Christopher Jackson

Collection/Series: 100 Years of Quantum: Perspectives on its Past, Present, and Future

Subject: Quantum Foundations

Date: October 21, 2025 - 7:40 PM

URL: https://pirsa.org/25100166

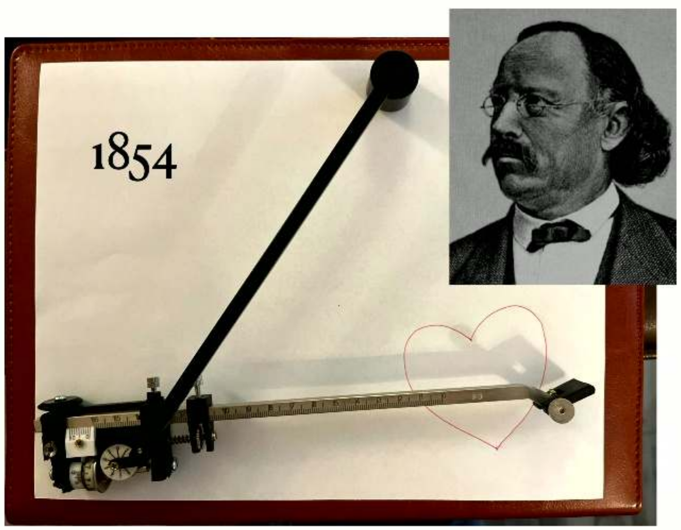
Abstract:

Once Heisenberg unlocked the Bohr frequency condition and the canonical commutation relation came out, Quantum Mechanics hit the ground running. In the rush of it all, it’s not clear to me who knew then (and who knows now) that the non-commutativity of phase space displacement is in fact an idea that has been around for at least as long as Jacobi, the father of canonical transformation theory. First conceived of in 1818 and then patented in 1854, the Amsler planimeter is a measuring instrument, known to even Maxwell, that in fact operates on exactly the same commutation relation, hiding in plain sight. Meanwhile, Lie's 1880 theory of transformation groups was also founded on exactly the same structure, what he called the contact element. Who knew? Why aren’t more physicists aware of this? Join me as we explore Poincare’s sudden death, the origins of the Gruppenpest, and Hilbert’s declining health when physicists like Wigner needed him more than ever.

$$[Q, P] = i\hbar 1$$

The Planimeter and Contact Transformations

A perfect embodiment of the Weyl-Heisenberg group and the lost twin sister of symplectic transformations.



by **Chris Jackson**
at the Perimeter Institute
on Oct 21, 2025
for **100 Years of Quantum:**
Perspectives on its Past, Present, and Future

1911

1870

$$\mathcal{L}_X Y \equiv [X, Y]$$

1925

Heisenberg's Breakthrough

Happy 100 Years of Quantum!

"virtual oscillators" 1922

$$\frac{d}{dt}\mathfrak{A}_{mn} = i\frac{E_m - E_n}{\hbar}\mathfrak{A}_{mn} \ldots$$

-Bohr, <u>Kramers</u>, Slater 1924

$$\omega_{mn} = \frac{E_m - E_n}{\hbar}$$

-Bohr 1913

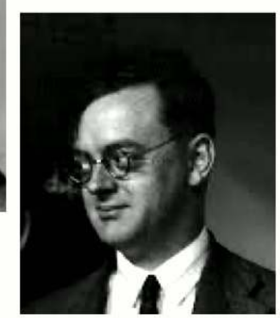
Electric Dipole: $\mathfrak{A} = -eQ$

$$\frac{d}{dt}\mathfrak{A} = \frac{i}{\hbar}[H, \mathfrak{A}] \ !$$

-Heisenberg 1925

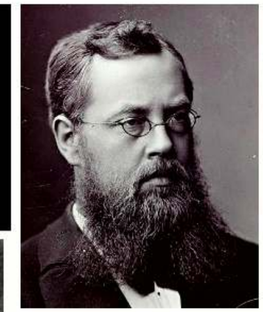
$$[Q, P] = i\hbar 1$$

-Born and Jordan 1925



Born, Weyl, and the Gruppenpest

Math at Gottingen and beyond


Weyl and Born were **Gottingen** classmates in 1905+
and attended Hilbert's lectures in Mathematical Physics

1925

Born: Bah!

Weyl:

$$[Q, P] = i\hbar 1$$

"This generates a Lie group!"

1926

Jordan: Bah!

Weyl:

$$e^{-ipx+iqy} \longmapsto e^{-iPx+iQy}$$

"This is how to quantize!"

Zee 2015

$$[Q, P] = i\hbar 1$$

Doesn't consider this a Lie algebra!

Today: Quasiprobability and Phase-Space Correspondence

$$[iQ, -iP] = i\hbar 1$$

Displacement Operator

$$D_\alpha = e^{-iPq+iQp}$$

After Wigner, Moyal & Groenewold, Klauder & Glauber, Gilmore & Perelomov ....

Coherent state = **Displaced** vacuum:

$$a = \frac{Q+iP}{\sqrt{2}}$$

$$|\alpha\rangle \equiv D_\alpha |0\rangle$$

$$a|0\rangle = 0$$

Covariant Frame Operators

$$D_\alpha D_z D_\alpha^{-1} = D_z e^{-ipu+iqv}$$

Wigner Function

~ Parity Operator

$$\Pi_0 = \int \frac{d^2z}{\pi} D_z$$

Coherent Measurement

~ Vacuum Projection Operator

$$|0\rangle\langle 0| = \int \frac{d^2z}{\pi} e^{-\frac{1}{2}|z|^2} D_z$$

Transformation Noncommutativity

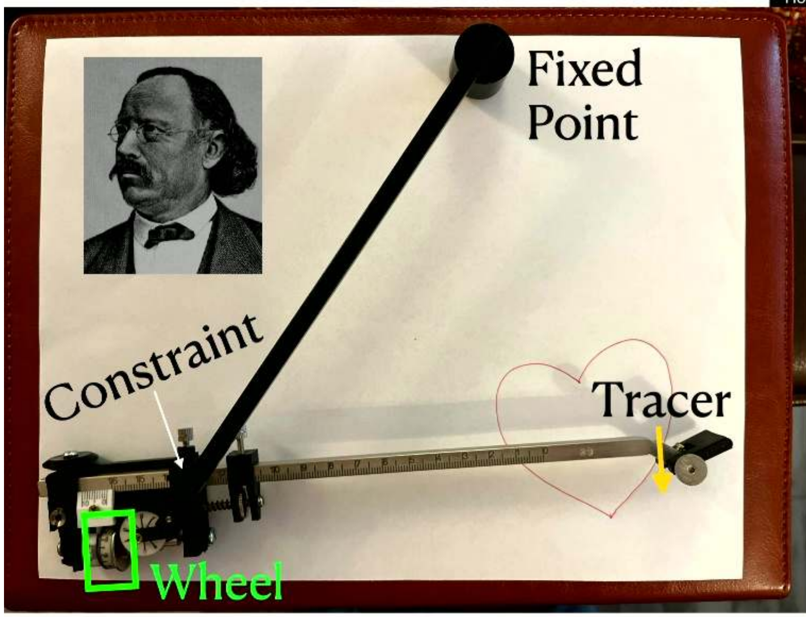
and Observable duality

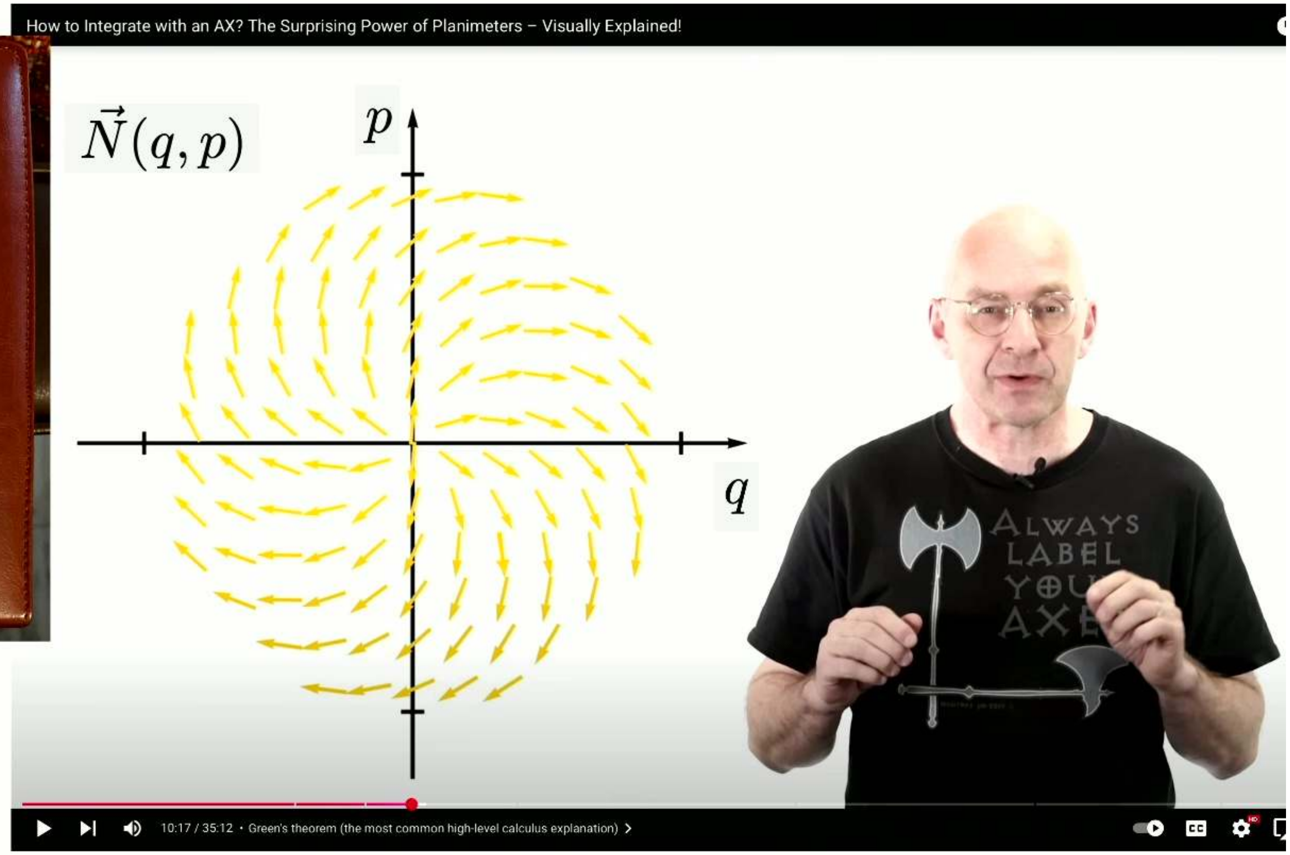
1870s | 1920s

Angular Momentum Component	**:Observables:**	Quadrature Components
Virtual Rotations	**:Transformations:**	Virtual ***Displacements***
$[-iJ_x, -iJ_y] = -iJ_z$	**(Infinitesimal) :Generators:**	$[iQ, -iP] = i\hbar 1$
The Sphere	**:Base Space:**	The Plane
Okay!		***But why?...***

Original Idea by Hermann 1818
Perfected by Amsler 1854

The (Polar-)Planimeter





<u>Angle of wheel rotation</u>

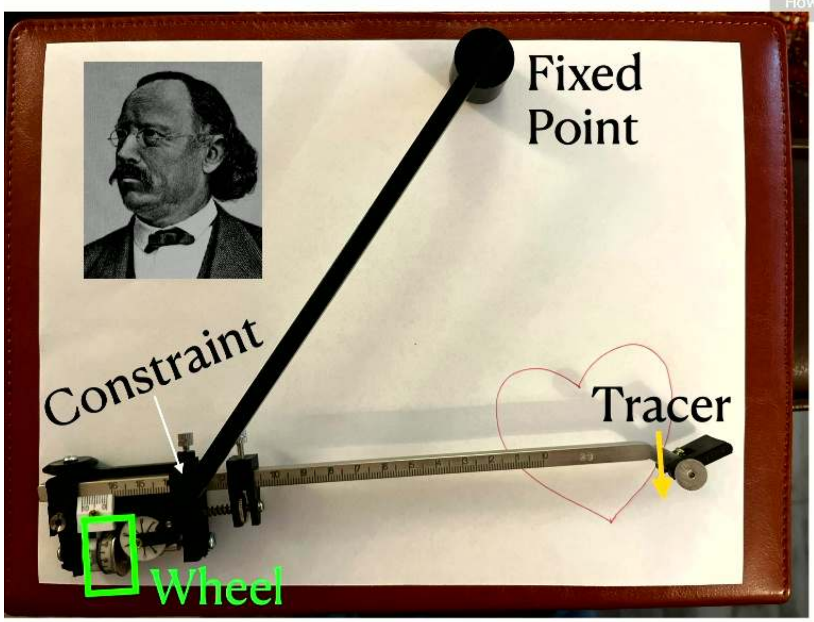
$$\phi[\gamma] = \frac{1}{h}\int_\gamma \vec{N}\cdot d\vec{r}$$

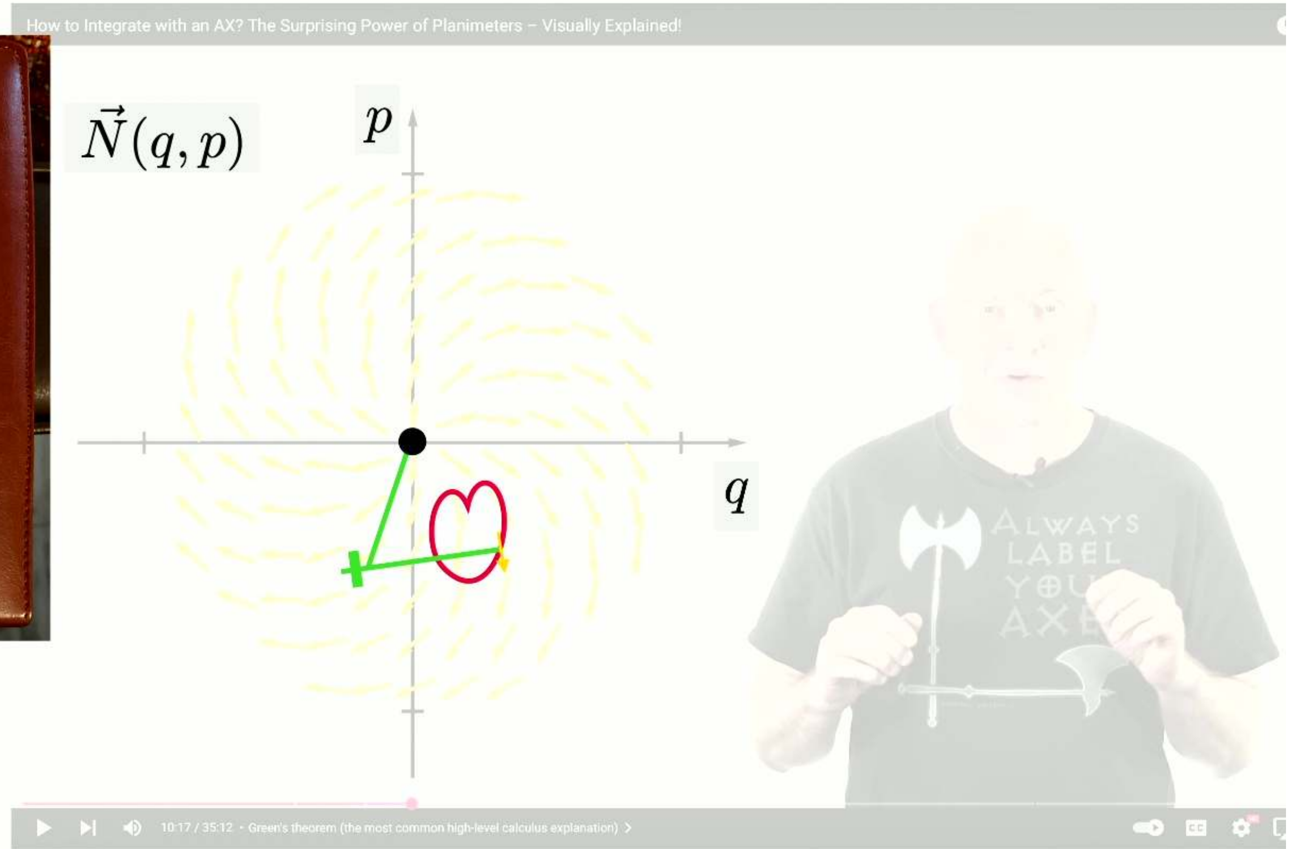
$h =$ (arm length)(wheel radius)

Prof. Burkard Polster @ Monash University
a.k.a. the ``Mathologer'' (June 7th, 2025)

Original Idea by Hermann 1818
Perfected by Amsler 1854

The (Polar-)Planimeter





Angle of wheel rotation

$$\phi[\gamma] = \frac{1}{h}\int_\gamma \vec{N}\cdot d\vec{r}$$

$h =$ (arm length)(wheel radius)

Prof. Burkard Polster @ Monash University
a.k.a. the ``Mathologer'' (June 7th, 2025)

The Weyl-Heisenberg Group Within

The Canonical and Contact One-Forms

Reeb Vector

$$\partial_\phi$$

Contact One-Form

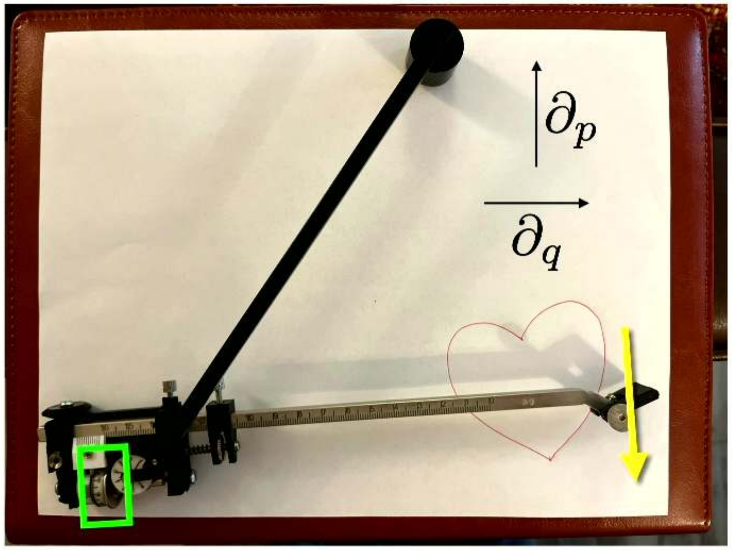
$$\alpha \equiv \vec{N} \cdot d\vec{r} - h\, d\phi$$

Canonical One-Form

$$\vec{N} \cdot d\vec{r} = p\, dq + dF$$

Kinematic Constraint

$$\alpha(\overline{X}) = 0$$



Angle of wheel rotation

$$\phi[\gamma] = \frac{1}{h} \int_\gamma \vec{N} \cdot d\vec{r}$$

h = (arm length)(wheel radius)

Physical Displacements

$$\overline{\partial_p} = \partial_p + \frac{1}{h} F_p \partial_\phi$$

$$\overline{\partial_q} = \partial_q + \frac{1}{h}(p + F_q)\partial_\phi$$

W-H Commutators!

$$\left[h\overline{\partial_p}, h\overline{\partial_q}\right] = h\partial_\phi$$

$$\left[\partial_\phi, h\overline{\partial_p}\right] = \left[\partial_\phi, h\overline{\partial_q}\right] = 0$$

Two Sides to the Weyl-Heisenberg Group....

Symplectic = Canonical = Strict Contact Transformations

Reeb Vector

$$\partial_\phi$$

Contact One-Form

$$\alpha \equiv \vec{N}\cdot d\vec{r} - h\, d\phi$$

The Lie Derivative

$$\mathcal{L}_X \lambda \equiv \lim_{\epsilon\to 0} \frac{\lambda\big(q+\epsilon X[q], p+\epsilon X[p]\big) - \lambda(q,p)}{\epsilon}$$

Canonical One-Form

$$\vec{N}\cdot d\vec{r} = p\, dq + dF$$

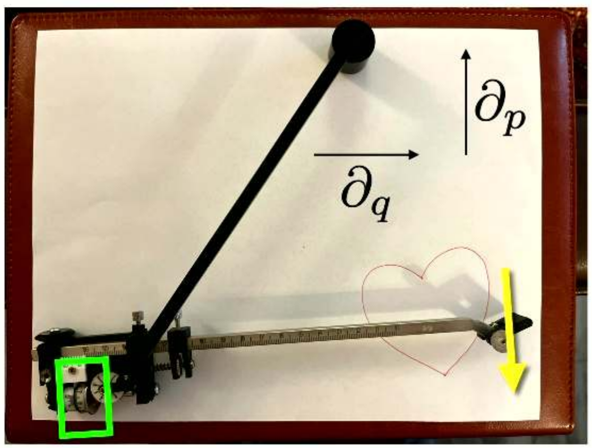
$$\omega = d\lambda \qquad \lambda \equiv \vec{N}\cdot d\vec{r}$$

Symplectic = Canonical Transformation

$$\mathcal{L}_X \omega = 0 \qquad \mathcal{L}_X \lambda = dL$$

= Contact Transformation

$$\mathcal{L}_{\widehat{X}} \alpha = 0$$



Canonical Displacements

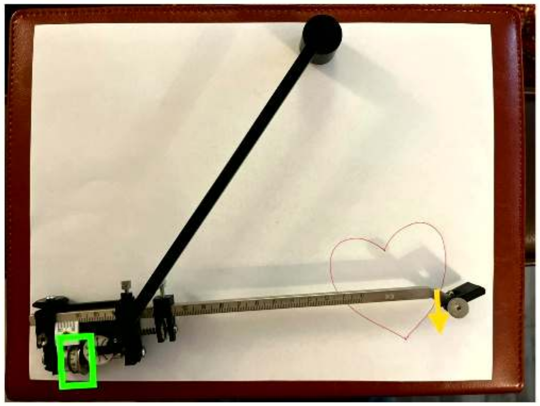
$$\widehat{\partial}_q = \overline{\partial_q} - \frac{1}{h} p\, \partial_\phi$$

$$\widehat{\partial}_p = \overline{\partial_p} + \frac{1}{h} q\, \partial_\phi$$

W-H Commutators again!

$$\left[h\widehat{\partial}_p, h\widehat{\partial}_q\right] = -h\partial_\phi$$

$$\left[\partial_\phi, h\widehat{\partial}_p\right] = \left[\partial_\phi, h\widehat{\partial}_q\right] = 0$$



The Missing Link!

Between Transformation-Hamiltonian Equivalence

Contact Transformations

$$\mathcal{L}_{\widehat{X}}[\alpha] = 0 \quad \Longleftrightarrow \quad -i_X\omega = d\left(\alpha(\hat{X})\right)$$

Hamilton's Equations

Cartan's "Magic" Formula

<u>Contact Displacements</u>

$$\widehat{\partial}_q = \overline{\partial_q} - \frac{1}{h} p\,\partial_\phi$$

$$\widehat{\partial}_p = \overline{\partial_p} + \frac{1}{h} q\,\partial_\phi$$

$$\widehat{X_H} \longleftarrow X_H \equiv \{H, \_\} \longleftarrow H$$

Contact Transformation $\widehat{X} \longmapsto \alpha(\widehat{X})$ Hamiltonian

<u>Contact Hamiltonians!</u>

$$p = \alpha(-\widehat{\partial}_q)$$

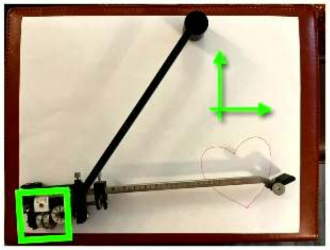
$$q = \alpha(\widehat{\partial}_p)$$

"Synthetic" Lie Bracket (Commutator)

$$\alpha\left(\widehat{[X, Y]}\right) \stackrel{?}{=} \left\{\alpha(\widehat{X}), \alpha(\widehat{Y})\right\}$$

$$[\widehat{X_F}, \widehat{X_G}] = [\widehat{X_F, Y_G}] = \widehat{X_{\{F,G\}}}$$

"Analytic" Poisson Bracket



$\left[h\widehat{\partial}_p, h\widehat{\partial}_q\right] = -h\partial_\phi$

Reflections

$\mathcal{L}_X Y \equiv [X, Y]$


What came first? The transformation or the function?

- **d'Alembert** (1743) and **Euler**: Virtual *Displacements**
- **Lagrange** (1788) and **Poisson** (1801): *Variation* of ***Constants***, which became Analysis $\{G, H\}$
- **Jacobi** (1837): Father of the Canonical *Transformation* $X_H \equiv \{H, \_\}$
- **Lie** (1873): The Fountainhead — Contact *Transformation* Groups and Lie Derivatives

- **Klein** (1873): The Erlangen Programm, which became Category Theory (**1945**)

- **Poincare** (1892): Physicist's Portal to the Canonical *Transformation*

- **Cartan** (1922): Geometer Supreme — significantly developed the Lie Derivative $\mathcal{L}_X \lambda = dL$*
- **Levi-Civita** (1925) and **Caratheodory** (1935): Einstein's Math Consultants


- **Reeb** and **Arnold** (**1960s**): Further develop Contact $H_X = \alpha(\widehat{X})$

∂_ϕ

$e^{i\phi}\psi(x)$

Quantum Measuring Instruments Course!

"Quantum Measurement and Continuous Markov Processes"

- **The Standard-Coherent POVM:**
  - Arthurs and Kelly (1965)
  - Heterodyne (Personick 1972)
  - Barchielli, et al. (1982)
  - Indirect Heterodyne (Wisemann et al. 1994, Goetsch et al. 1994)

For details and links, email

cjackson3@perimeterinstitute.ca

Begins Monday!
(October 27th)

- **The Spin-Coherent POVMs!**
  - D'Ariano (2002) [Spin analog of Arthurs-Kelly doesn't work universally....]
  - Jackson et al. (2018, 2021, 2023) [The simultaneous indirect measurement of the three spin components does work universally!]
- **General theory of continuous-in-time measuring instruments**
  - Instrumental Groups
  - Kraus-Operator Densities and their evolution
- **Transformation Group and Operator Algebra Goodies**

$$[iQ, -iP] = i\hbar 1$$

$$[-iJ_x, -iJ_y] = -iJ_z$$

Summary and Acknowledgments

- The planimeter is remarkable and perfectly embodies the Weyl-Heisenberg Group.
- Contact transformations are easier to understand than canonical transformations!
- Strict contact transformations are Lie-equivalent to classical Hamiltonians.

- I'll start teaching a class on quantum measuring instruments starting this Monday....

For details and links, email

cjackson3@perimeterinstitute.ca

- Deep Thanks to **Carl Caves** for his collaboration and devoted attention!
- Deep Thanks to **Lucien Hardy** and **Rob Spekkens** for their support!

and Thank All of You!