Title: Complex Matrix Models for Protected Correlators of N=4 SYM

Speakers: Vladimir Kazakov

Collection/Series: Quantum Fields and Strings

Subject: Quantum Fields and Strings

Date: October 07, 2025 - 6:00 PM

URL: https://pirsa.org/25100142

Abstract:

I will present a complex matrix model (CMM) for the computation of protected two-point and three-point correlation functions of 1/2-BPS operators. The model can be efficiently studied in large N limit for huge operators (with dimensions ~N^2). An advantage of this approach is the direct identification of the distribution of eigenvalues of complex matrix with the shapes of droplets in Lin-Lunin-Maldacena (LLM) bubbling geometry. For correlators of two huge operators with a small SO(4) symmetric "probe" operator (with dimension N^0) we established a perfect correspondence with old computations of Skenderis-Taylor in dual LLM gravity. CMM allows efficient analytic study of certain two- and three-point correlators of huge operators of exponential and coherent state types and relates these problems to the Laplacian growth and 2d quantum gravity. We also compute explicitly the correlators of two huge operators with giant magnon operator. In a much more difficult case of 1/4- and 1/8-BPS operators we make a curious observation of equivalence of a pair correlator of such coherent state operators to the quenched Eguchi-Kawai reduction of the principal chiral model.

Perimeter Institute Seminar

October 7 2025

Complex Matrix Models for Protected Correlators of N=4 SYM

Vladimir Kazakov

Collaboration with Harish Murali and Pedro Vieira arXiv:2406.01798

Collaboration with Prokopii Anempodistov, Adolfo Holguin and Harish Murali arXiv:2508.20094

LPENS LABORATOIRE DE PHYSIQUE DE L'ÉCOLE NORMALE SUPÉRIEURE

ENS PSL★ cnrs SORBONNE UNIVERSITÉ Université Paris Cité

$\mathcal{N}$=4 SYM: small and huge operators, AdS/CFT and LLM/Matrix dualities

$\mathcal{S}_{SYM} = \frac{N}{\lambda}\int d^4x \,\mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi_j)^2 + [\Phi_j,\Phi_k]^2\right)$+fermions - dual string theory on $AdS_5 \times S^5$ background

Maldacena
Gubser,Klebanov,Polyakov
Witten

super-conformal theory with PSU(2,2|4) symmetry

Local operators – words of N × N matrices

$\mathcal{O}(x) = \mathrm{Function}\left[\mathrm{Tr}(\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots)\right](x)$

Anomalous dimensions $\Delta_{\mathcal{O}}$

$$\mathcal{O}(\xi x) \to \xi^{\Delta_{\mathcal{O}}(\lambda)}\mathcal{O}(x)$$

Operator product expansion,
structure constants, correlators, …

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k |x|^{-\Delta_i-\Delta_j+\Delta_k}\, C_{i,j}^k\, \mathcal{O}_k(0) + \ldots$$

At N → ∞ :
Little operators: $\Delta \sim 1$ - AdS/CFT, integrability (Quantum Spectral Curve, hexagons, etc)
Giant operators: $\Delta \sim N$ - still AdS, (e.g. giant gravitons)
Huge operators: $\Delta \sim N^2$ -non-AdS, black holes, etc

In general, huge operators are difficult to study. No integrability.

Fortunately, due to SUSY there exists sectors of operators (½,1/4,1/8,1/16BPS) with protected (coupling independent) dimensions - integers or ½-integers, and protected correlators
Dual to interesting geometries. E.g. ½-BPS operators are dual to LLM geometry

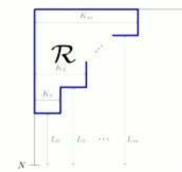
Lin, Lunin, Maldacena '04

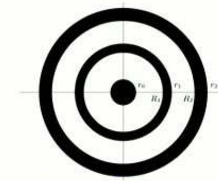
Can be studied by (sometimes) solvable matrix models

1/2BPS vs LLM metric

- ½-BPS operators: any invariant function of a chiral scalar field $Z = (y \cdot \Phi(x))\,, \quad y \cdot y = 0$

- GL(N) Character : $O_{\Delta,\mathcal{R}}(x,y) = \chi_{\mathcal{R}}\left(y \cdot \Phi(x)\right)$



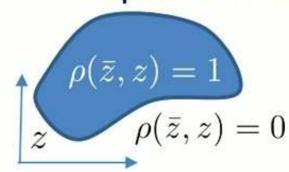


- Coherent state : $O_{CS}(Z,\Lambda) = \int [dU]_{U(N)} e^{\mathrm{Tr}[UZU^\dagger \Lambda]}$

Droplet of e.v.s

parameters

- Exponential : $O_{\boldsymbol{t}}(Z) = \exp\left[\sum_{k>0} \frac{t_k}{k} \mathrm{tr}[Z^k]\right] = e^{\mathrm{tr}[V_{\boldsymbol{t}}(Z)]}$

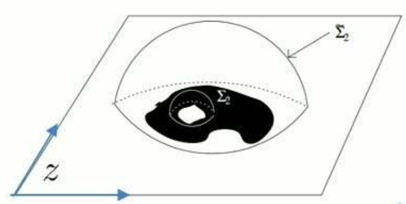


- 1/2BPS states in IIB SUGRA described by LLM metric: $R \times SO(4) \times SO(4)$

$$ds^2 = -\frac{y}{\sqrt{\frac{1}{4}-\zeta^2}}(dt + \bar{V}dz + Vd\bar{z})^2 + \frac{\sqrt{\frac{1}{4}-\zeta^2}}{y}(dy^2 + d\bar{z}dz) + y\sqrt{\frac{1+2\zeta}{1-2\zeta}}d\Omega_3 + y\sqrt{\frac{1-2\zeta}{1+2\zeta}}d\tilde{\Omega}_3$$

$$\zeta(\bar{z},z,y) = \frac{y^2}{\pi}\int_{\mathbb{C}} \frac{\left(\rho(\bar{z},z)-\frac{1}{2}\right)\, d^2z'}{[(\bar{z}-\bar{z}')(z-z')+y^2]^2}$$

$$V(\bar{z},z,y) = \frac{1}{2\pi}\int_{\mathcal{D}} \frac{\left(\rho(\bar{z},z)-\frac{1}{2}\right)(z-z')\, d^2z'}{[(\bar{z}-\bar{z}')(z-z')+y^2]^2}$$



y bubbling geometry

z

½-BPS character operators in N=4 SYM

Character operators – $O_i \equiv \mathcal{N}_{\mathcal{R}_i}\, \chi_{\mathcal{R}_i}(y_i \cdot \Phi(x_i))\,, \qquad y_i \cdot y^i = 0$

full orthonormal system of ½BPS primary operators

Two point correlators of such operators are protected

$$\langle O_i O_j \rangle = 1 \times \delta_{\mathcal{R}_i,\mathcal{R}_j} \left(\frac{y_i \cdot y_j}{(x_i - x_j)^2} \right)^{2L_i} \rightarrow \text{ dimension (Size of YT)}$$

Three point correlators of these ½BPS operators define protected structure constants

$$\left\langle \prod_{j=1}^{3} O_i \right\rangle = Z_3(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3) \times \prod_{1 \le i < j \le 3} \left(\frac{y_i \cdot y_j}{(x_i - x_j)^2} \right)^{\ell_{ij}}$$

"Bridge" lengths

$$\ell_{ij} = (L_i + L_j - L_k)/2$$

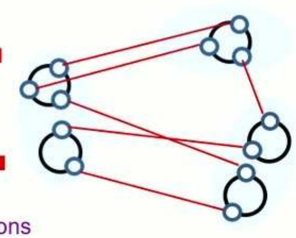
Counting of Wick contractions. Purely combinatorial problem! V.K., Murali, Vieira '24

Can be studied via interesting matrix integral

Eigenvalue representation:

$$Z(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3) = \int \prod_{i=1}^{N} dx_i\, e^{-x_i^2}\, \Delta^2(x)\; Q_1(x) Q_2(x) Q_3(x)$$

$$Q_n(x) = \frac{1}{\Delta(x)} \det_{1 \le i,j \le N} H_{h_{j,n}}(x_i)$$

$\sum$ 

Wick contractions

Reduces to solvable (1+1)d Riemann-Hopf fluid dynamics

Anempodistov, Holguin, V.K., Murali '25

BPS correlators via complex matrices

- Gaussian YM functional integral (no YM coupling dependence!)

$$\langle \mathcal{O}_1^{BPS}(x_1)\mathcal{O}_2^{BPS}(x_2)\,\ldots\rangle = \frac{1}{\mathcal{Z}}\int \prod_{I=1}^{6} D\Phi_I \; e^{-N\int \frac{d^4x}{(2\pi)^4}\mathrm{tr}\partial_\mu\Phi^I(x)\partial^\mu\Phi_I(x)}\Big(\mathcal{O}_1^{BPS}(x_1)\mathcal{O}_2^{BPS}(x_2)\,\ldots\Big)$$

replaced by gaussian complex matrix measure

$$\Longrightarrow \quad \int d^2X\, d^2Y\, d^2Z\; e^{-N\mathrm{tr}\left[X\bar{X}+Y\bar{Y}+Z\bar{Z}\right]}\mathcal{O}_1(X,Y,Z)\mathcal{O}_2(\bar{X},\bar{Y},\bar{Z})\,\ldots$$

$$X = \Phi_1 + i\Phi_2$$
$$Y = \Phi_3 + i\Phi_4$$
$$Z = \Phi_5 + i\Phi_6$$

- 1/2BPS 2-point correlator

$$\langle O\ \bar{O}\rangle = \frac{1}{\mathcal{Z}_\bullet}\int dZ d\bar{Z}\; e^{-N\,\mathrm{tr}[Z\bar{Z}]}\, O(Z)\ \bar{O}(\bar{Z})$$

- 1/2BPS 3-point correlator: two BPS operators and a BPS Probe $(Z+\bar{Z}+i(Y-\bar{Y}))$

$$\langle O\ \bar{O}\ \text{Probe}\rangle = \frac{1}{\mathcal{Z}_\bullet}\int dZ d\bar{Z}\; e^{-N\,\mathrm{tr}[Z\bar{Z}]}\, O(Z)\ \bar{O}(\bar{Z})\ F_{\text{Probe}}(Z+\bar{Z})$$

where $\quad F_{\text{Probe}}(X) = \dfrac{1}{\mathcal{Z}_M}\displaystyle\int dM\; e^{-\frac{N}{2}\mathrm{tr}M^2}\,\text{Probe}\left(X+\frac{iM}{2}\right)$

- Advantage of complex matrices: the e.v. distribution is exactly dual to the LLM droplet

Anempodistov, Holguin, V.K., Murali '25

Eigenvalue representation for ½-BPS

- Schur decomposition for 2-point correlator of exponential operators $Z = U(z+T)U^\dagger$

$$\langle O_{V_t}\, O_{\bar{V}_{\bar{t}}} \rangle \;=\; \int_{\mathbb{C}} \prod_{j=1}^{N} d^2z\, e^{-|z_j|^2 + V_{\boldsymbol{t}}(z_j) + \bar{V}_{\bar{\boldsymbol{t}}}(\bar{z}_j)} |\Delta(z)|^2$$

- Saddle point equation (Laplacian growth problem!)

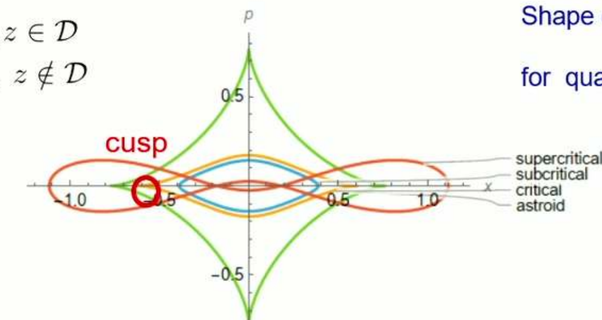
Kostov, Krichever, Mineev, Wigmann, Zabrodin '00
Krichever, Mineev-Weinstein, Wiegmann, Zabrodin
V.K., Marshakov '02

$$(\partial_z\, \partial_{\bar{z}})^{-1} \rho(z,\bar{z}) = z\bar{z} - V_{\boldsymbol{t}}(z) - \bar{V}_{\boldsymbol{t}}(z)$$

Vazquez '06

- Solution – constant density of e.v.'s in a spot defined by potentials: dual to LLM droplet!

$$\rho(\bar{z},z) = \begin{cases} 1 & , z \in \mathcal{D} \\ 0 & , z \notin \mathcal{D} \end{cases}$$



Shape of boundary (algebraic curve)

for quartic potential $V_{\boldsymbol{t}}(z) = t_2 z^2 + t_4 z^4$

$$z(w) = rw + \frac{a}{w} + \frac{b}{w^3},$$

where $w = e^{i\phi}$, $(0 < \phi < 2\pi)$

- 2d QG interpretation:

pure 2d gravity: $\log\langle O\,O\rangle \sim (t_* - t)^{5/2}$ V.K. '85 David '85

Ising + 2d gravity: $\log\langle O\,O\rangle \sim (t_* - t)^{7/3}$ V.K. '86

- Double scaling limit (Painleve I, mKdV, …): 2d QG → 10 SUGRA quantum effects !

1/2BPS coherent state correlator

Anempodistov, Holguin, V.K., Murali

- 1/2BPS CS correlator reduces to Charish-Chandra-Itsykson-Zuber (CHIZ) integral:

$$\langle O_\Lambda O_{\bar\Lambda}\rangle = \int d^2Z\, dU\, dV\, e^{N\,\mathrm{tr}[-Z\bar Z + \bar\Lambda U Z U^\dagger + \Lambda V \bar Z V^\dagger]} = \int dU e^{N\,\mathrm{tr} U\Lambda U^\dagger \bar\Lambda} = \frac{\det_{ij} e^{N\lambda_i \bar\lambda_j}}{\Delta(\lambda)\Delta(\bar\lambda)}$$

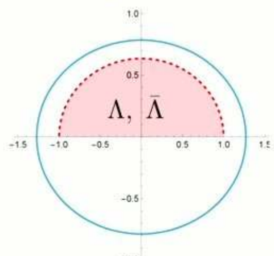
$$\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$$

- But what is the distribution of e.v.'s z_j (droplet shape!) ?
  It can be written in terms of integrals over z_j and solved via saddle point:

$$\langle O_\Lambda O_{\bar\Lambda}\rangle = \int \prod_j d^2 z_j\, e^{-\bar z_j z_j} \frac{|\det_{ij} e^{\lambda_i z_j}|^2}{\Delta(\lambda)\Delta(\bar\lambda)}$$

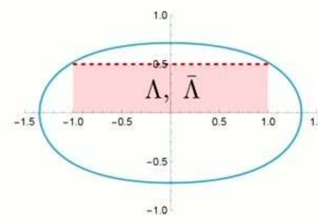
- We can immediately see that for holomorphic moments $\langle \mathrm{tr} Z^n\rangle = \mathrm{tr}\Lambda^n$
  But this is not enough to fix the shape of droplet, but it helps to solve the saddle point eqs.
- Examples: circle and ellipse distributions:

$$\rho(\lambda) = \frac{2\sqrt{c^2-\lambda^2}}{\pi c^2}$$



$$S(z) = \left(\frac{2a^2}{c^2} - 1\right) z - \frac{2}{c^2}\sqrt{z^2 - c^2}$$

$$\lambda_j = \alpha \frac{j-1}{N}$$



$$z(\theta) = \frac{1}{\alpha}\log\left(e^{\alpha^2} + e^{i\theta}\sqrt{e^{2\alpha^2} - e^{\alpha^2}}\right)$$

Droplet shapes for coherent states

- Mathematicians solved the problem of restoring shape of droplet from moments $\Lambda,\ \bar{\Lambda}$ for so called "point quadrature domains" obeying

Gustafsson '83

$$\langle \mathrm{Tr}\, f(Z)\rangle = \frac{1}{\pi}\int_{\mathcal{D}} d^2z\ f(z) = \sum_{k=1}^{n} a_k f(z_k) \quad = \frac{1}{2\pi i}\oint_{\partial\mathcal{D}} dz\ S(z) f(z)$$

$$S(z) = \sum_{k=1}^{n} \frac{a_k}{z - z_k} + (\texttt{analytic inside } \mathcal{D})$$

Swartz function

- Example: $\lambda_i = \begin{cases} \alpha_1 & \text{for } 1 \le i \le p \\ \alpha_2 & \text{for } p+1 \le i \le N \end{cases} \quad \Longrightarrow \quad \langle \mathrm{Tr} Z^n \rangle = \frac{p}{N}\alpha_1^n + \left(1 - \frac{p}{N}\right)\alpha_2^n$

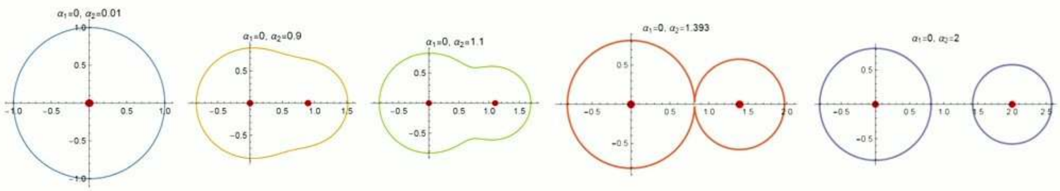
- Conformal map defining the shape of z-droplet $\Phi(w) = \mathsf{N}\left(\frac{w}{\xi_1 w - 1} + \frac{\gamma w}{\xi_2 w - 1}\right)$

  with parameters

$$a_1 = \mathsf{N}^2\left(\frac{\gamma}{(\xi_1\xi_2 - 1)^2} + \frac{1}{(\xi_1^2 - 1)^2}\right) = \frac{p}{N}, \qquad z_1 = \mathsf{N}\xi_1\left(\frac{\gamma}{\xi_1\xi_2 - 1} + \frac{1}{\xi_1^2 - 1}\right) = \alpha_1$$

$$a_2 = \mathsf{N}^2\left(\frac{\gamma^2}{(\xi_2^2 - 1)^2} + \frac{\gamma}{(\xi_1\xi_2 - 1)^2}\right) = 1 - \frac{p}{N}, \qquad z_2 = \mathsf{N}\xi_2\left(\frac{\gamma}{\xi_2^2 - 1} + \frac{1}{\xi_1\xi_2 - 1}\right) = \alpha_2$$

- Change of shapes of z- droplet when centers move away from each other:



- Can be generalized to any number of centers – in principle describes any droplet

SYM/sugra correspondence for HHL correlators

- Skenderis and Taylor computed a few HHL correlators from LLM supergravity, for generic Little probe operators (up to dimension=4). They matched them successfully to some (extremal) Little probes on SYM side.
- We matched HHL for all such Little probes on SYM side to their SUGRA results

- The considered probes are the $SO(4)_R$ invariant single trace primaries $O_{\Delta,k}$ of dimension Δ and charge k, e.g.

$$O_{2,0}(x) = \sqrt{\frac{2}{3}}\mathrm{tr}[Z\bar{Z} - \frac{1}{2}\sum_{i=1}^{4}\Phi_i^2], \qquad O_{2,2}(x) = \frac{1}{\sqrt{2}}\mathrm{tr}[Z^2], \qquad O_{3,1}(x) = \frac{1}{\sqrt{2}}\mathrm{tr}[Z(Z\bar{Z} - \sum_{i=1}^{4}\Phi_i^2)]$$

Skenderis, Taylor '07

$\rho(\bar{z}, z)$ - LLM background droplet shape

$$\langle O_{\Delta,\Delta}\rangle = \frac{N^2}{\sqrt{\Delta}\pi^2}(\Delta-2)\sqrt{\Delta-1}\int dzd\bar{z}\rho(z,\bar{z})z^{\Delta},$$
$$\langle O_{2,0}\rangle = \frac{N^2}{\pi^2}\frac{\sqrt{2}}{\sqrt{3}}\int dzd\bar{z}\rho(z,\bar{z})\left(z\bar{z}-\frac{1}{2}\right),$$
$$\langle O_{3,1}\rangle = \frac{N^2}{\pi^2}\int dzd\bar{z}\rho(z,\bar{z})z^2\bar{z},$$
$$\langle O_{4,0}\rangle = \frac{N^2}{\pi^2}\frac{\sqrt{3}}{\sqrt{5}}\int dzd\bar{z}\rho(z,\bar{z})\left(3(z\bar{z})^2-4z\bar{z}+1\right),$$
$$\langle O_{4,2}\rangle = \frac{N^2}{\pi^2}\frac{4\sqrt{3}}{\sqrt{10}}\int dzd\bar{z}\rho(z,\bar{z})\,z^2\,(z\bar{z}-1)$$

Our computation

$\rho(\bar{z}, z)$ - e.v. distribution in complex matrix model

$$\tau_{\Delta,\Delta} = \frac{1}{\sqrt{\Delta}}\mathrm{tr}[z^{\Delta}] \to \frac{N}{\sqrt{\Delta}}\int dzd\bar{z}\,\rho(z,\bar{z})\,z^{\Delta},$$
$$\tau_{2,0} = \mathrm{N}_{2,0}\mathrm{tr}\left[z\bar{z} - \frac{1}{2}\left(1+\frac{1}{N}\right)\mathbb{I}\right] \to \sqrt{\frac{2}{3}}N\int dzd\bar{z}\rho(z,\bar{z})\left(z\bar{z}-\frac{1}{2}\right),$$
$$\tau_{3,1} = \mathrm{N}_{3,1}\mathrm{tr}\left[z^2\bar{z} - \left(1+\frac{1}{N}\right)z\right] \to \frac{N}{\sqrt{2}}\int dzd\bar{z}\rho(z,\bar{z})\left(z^2\bar{z}-z\right),$$
$$\tau_{4,0} = \mathrm{N}_{4,0}\mathrm{tr}\left[3(z\bar{z})^2 - 4\left(1+\frac{2}{3N}\right)z\bar{z} + \left(1+\frac{1}{N}\right)\left(1+\frac{3}{N}\right)\mathbb{I}\right] - \frac{4}{N}(\mathrm{tr}z)(\mathrm{tr}\bar{z}),$$
$$\to \frac{N}{2\sqrt{5}}\int dzd\bar{z}\,\rho(z,\bar{z})\left(3(z\bar{z})^2-4z\bar{z}+1\right)$$
$$\tau_{4,2} = \mathrm{N}_{4,2}\mathrm{tr}\left[z^3\bar{z} - \left(1+\frac{2}{3N}\right)z^2\right] - \frac{2}{N}(\mathrm{tr}z)^2 \to \sqrt{\frac{2}{5}}N\int dzd\bar{z}\,\rho(z,\bar{z})z^2\,(z\bar{z}-1)$$

- Perfect match if we fix normalization by comparing the coefficients of chiral primary at Δ=4

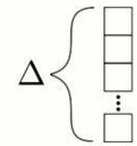
$$\langle O_{\Delta,k}\rangle_{\mathrm{gravity}} = \mathcal{N}_k^{\mathrm{sugra}} \times \langle\tau_{\Delta,k}\rangle_{MM}$$
$$\mathcal{N}_k^{\mathrm{sugra}} = \frac{N}{\pi^2}(k-2+\delta_{k,2})\sqrt{k-1}$$

HH-Giant graviton (HHGg) correlators

Anemapodistov, Holguin, V.K., Murali '25

- HHGg correlator : $\langle \chi_{\{h\}}(\bar{Z})\, \chi_{\{h\}}(\bar{Z})\, O_{Gg}(\Delta)(\bar{Z}+Z)\rangle$

$$O_{\rm Gg} = \chi_\Delta(Z+\bar{Z})$$

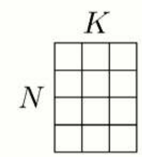


$$C_{R\,R'\,\Delta} = \overbrace{\frac{1}{\sqrt{\mathcal{N}_\Delta}}\frac{(\sqrt{N})^{N-\Delta}}{(N-\Delta)!}\frac{1}{\mathcal{Z}_\sigma}\int d\sigma\, e^{-\frac{N}{2}\sigma^2}\, \underbrace{\mathsf{He}_{N-\Delta}(\sqrt{N}\sigma)}_{\text{Hermite pol.}}}^{\text{Gg operator}}$$

$$\underbrace{\times \frac{1}{\sqrt{\mathcal{Z}_R \mathcal{Z}_{R'}}}\int \prod_k |dz_k|^2\, e^{-N z_k \bar{z}_k}\, \det\left(z_j^{h_l}\right)\det\left(\bar{z}_n^{h'_m}\right)}_{\text{HH background}}\prod_j \left(z_j+\bar{z}_j+\sigma\right)$$

- Explicit result for $K \times N$ rectangular YT

$$C_{K\,K\,\Delta} = (-1)^{\Delta/2}\sqrt{\frac{(N-\Delta)!}{N!}}\times\frac{(K+\Delta/2-1)!}{(K-1)!}\frac{(N-\Delta/2)!}{(N-\Delta)!(\Delta/2)!} \simeq (-1)^{\frac{\Delta}{2}}\left[\frac{\left(1-\frac{\delta}{2}\right)^{1-\frac{\delta}{2}}\left(\gamma+\frac{\delta}{2}\right)^{\gamma+\frac{\delta}{2}}}{(1-\delta)^{\frac{1-\delta}{2}}\,\gamma^\gamma\left(\frac{\delta}{2}\right)^{\frac{\delta}{2}}}\right]^N$$



$$\gamma = K/N, \qquad \delta = \Delta/N$$

- Saddle point calculation w.r.t. σ : Gg in general HH background looks like an instanton in matrix models

$$C_{\bar{H}H,\Delta} \sim \int d\sigma\, e^{-\frac{N}{2}\sigma^2}\, \mathrm{He}_{N-\Delta}(\sqrt{N}\sigma)\, \exp\left[N\int dz d\bar{z}\, \rho_O(z,\bar{z})\log(z+\bar{z}+\sigma)\right]$$

- Challenge: to reproduce it in SUGRA from configuration LLM metric + D-brane

Explicit results for Heavy-Heavy-Heavy 1/2-BPS correlator

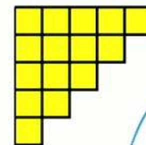
V.K., Murali, Vieira '24

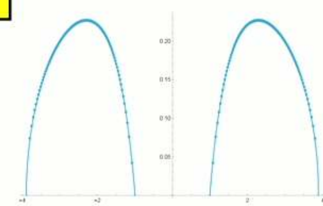
- General approach to HHH correlators of character operators through (1+1) fluid dynamics

Kostov '88

- Structure constant for 3 triangular Young tab. (similar to O(n=1) model on planar graphs)

$$Z^{\texttt{triangle}} \sim \int \prod_{j=1}^{K} dx_j \, e^{-\frac{N}{2}x_j^2} [\Delta_N(x)]^2 \prod_{j>k=1}^{N} (x_j + x_k)^3$$





density $\rho(x)$

- Exact saddle point solution in terms of an algebraic curve

$$\mathcal{S}(f,w) = f^3 - 3r(w)f - 2s(w) = 0$$

$$r(w) = \frac{c_2 w^2 + c_4 w^4 + c_6 w^6}{(w^2-1)^4}\,, \qquad s(w) = \frac{d_2 w^2 + d_4 w^4 + d_6 w^6}{(w^2-1)^4}$$

- HHH correlator of 3 exponential operators $\quad O(M) = e^{N\,\mathrm{tr}\,(-t_2 M^2 + t_3 M^3)}$
  (related to 2d gravity + 3-state Potts, c_m=4/5)

V.K. '88
V.K., Kostov '91
Daul '94
Eynard, Zinn-Justin '90
Eynard, Kristijansen '94

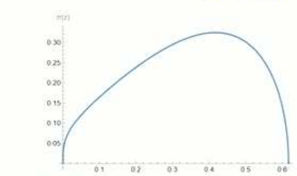
$$\langle O\,O\,O\rangle = \int \prod_{i=1}^{3} dM_i \; e^{N\,\mathrm{tr}\left[-\sum_i \frac{M_i^2}{2} + \sum_{i<j} M_i M_j\right]} \; O(M_1)\,O(M_2)\,O(M_3)$$

$$= \int dX \, e^{-\frac{N}{2}\mathrm{tr}X^2} \left(\int dM \; e^{N\,\mathrm{tr}\left[X\,M - \frac{4+t_2}{2} M^2 + t_3\,M^3\right]} \right)^3$$

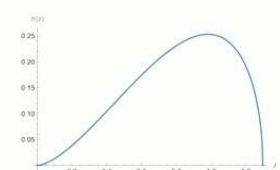
density $\quad \pi(z(x)) \equiv \rho(\sqrt{\alpha x + c}$

- Similar method of exact solution

- Critical regime: $\quad \log\langle O\,O\,O\rangle \sim (t_c - t)^{11/5}$



(a) The density $\pi(z)$ on the critical line outside of the critical point. Here $\alpha = 0.1$.



(b) The density $\pi(z)$ at the critical point. Here $\alpha = \alpha_c$.

¼- and 1/8-BPS coherent state operators

- 1/8-BPS coherent state operators are given by unitary integral

$$O_P = \int [dg]_{SU(N)} \, e^{\mathrm{tr}[g^\dagger P^\mu g \, X_\mu]} , \qquad \mu = 1,2,3$$

Berenstein, Wang '22

  - function of traces of "words" made of complex matrices $\{X_1, X_2, X_3\} = \{X, Y, Z\}$ and of traces of various products of diagonal parameter matrices P_μ

- Pair correlator is a (difficult!) unitary integral

$$\langle O_P \bar{O}_{\bar{P}} \rangle = \int [dg] \; e^{\mathrm{tr}[g^\dagger \bar{P}^\mu g P_\mu]} , \qquad g \in SU(N)$$

Donos '08

- May be possible to relate it to bubbling geometry description for ¼ and 1/8 BPS states

- 3-point correlator of one ¼-BPS with two ½-BPS is protected.

D'Hoker, Ryzhov '01
Bissi, Fardelli, Manenti '21

- For CS ¼-BPS with two exponential ½-BPS it is a (difficult!) complex matrix integral:

$$\langle O_V \, O_{\bar{V}} \, O_P \rangle = \int d^{2N^2} Z \, e^{\mathrm{tr}[-\bar{Z}Z + V(Z) + \bar{V}(\bar{Z}) + PZ + \bar{P}\bar{Z}]} \xrightarrow{V(Z) = \nu Z^2} \exp \mathrm{tr} \left[\frac{-\bar{t}P^2 - t\bar{P}^2 - 2P\bar{P}}{2\,(t\bar{t} - 1)} \right]$$

$$[P, \bar{P}] = 0$$

elliptic droplet in LLM

- The following operator in β-deformed SYM is protected at least at one loop

$$O_q = \int [dG]_{SU(N)} \, e^{N \, \mathrm{tr}(G^\dagger P_1 G X + G^\dagger P_2 G Z)} , \qquad [P_1, P_2]_q = 0, \qquad q = e^{i \frac{\pi k}{N}}$$

dilation operator $\mathcal{D}^q_{1-loop} \, O_q = 0$

planar 4-loop D^q is known
Minahan, Sieg '11

¼-BPS CS correlator as Eguchi-Kawai reduction of principal chiral model

Anempodistov, Holguin, V.K., Murali '25

- Two-point correlator of ¼,1/8-BPS coherent state operators coincides with partition function of 2D, 3D principal chiral model (PCM) after quenched Eguchi-Kawai (EK) dim. reduction

$$Z_{\mathrm{PCM}} = \int [\mathcal{D}g]_{SU(N)}\ e^{-\frac{1}{2\lambda}\int d^D x\, \mathrm{tr}[\partial_\mu g \partial^\mu g^\dagger]}$$

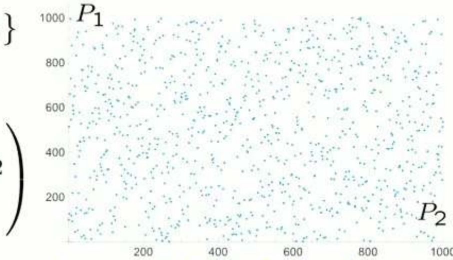
Gross, Kitazawa '78 $\quad g(x) \to e^{-iP\cdot x} g e^{iP\cdot x}$

$$Z_{\mathrm{red}} = \int [dg]\ e^{\frac{V_D}{2\lambda}\mathrm{tr}[P^\mu, g][P_\mu, g^\dagger]} = \int [dg]\ \exp\left(\frac{V_D}{2\lambda}\sum_{k,j=1}^{K}(p^k - p^j)^2 |g_{kj}|^2\right)$$

$$P^\mu = \mathrm{diag}\{p_1^\mu, \ldots, p_N^\mu\}$$

$$0 < p_j^\mu < \Lambda$$

uniform distribution of quenched momenta in a box



- Does it describe the PCM physics (asymptotic freedom, mass spectrum, etc)? Left to check
- PCM is integrable! Possibility to study ¼-BPS via PCM (and vice versa)
- Reduced PCM is also integrable, at least classically

Polyakov, Wiegmann '84
Wiegmann '85
Fateev, V.K., Wiegmann '94
V.K., Sobko, Zarembo '19, '23

Currents: $\quad j = g^\dagger[\mathcal{P}, g], \qquad \bar{j} = g^\dagger[\bar{\mathcal{P}}, g] \qquad \mathcal{P} = P_1 + iP_2$

Lax pair: $\quad J(u) = \dfrac{j}{1+u}, \qquad \bar{J}(u) = \dfrac{\bar{j}}{1-u}$

$$i[\bar{\mathcal{P}}, J(u)] - i[\mathcal{P}, \bar{J}(u)] - [J(u), \bar{J}(u)] = 0$$

Quenched EK reduced PCM in external fields

- External fields are chemical potentials of conserved charges of SU(N) × SU(N) symmetry

$$g(x,t) \to e^{iHt} g(x,t) e^{iHt}, \qquad H = \mathrm{diag}\{H_1, H_2, \ldots, H_N\}$$

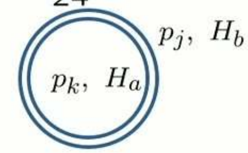
- In reduced version of PCM we take g → SU(NK) × SU(NK) and separate the K × K subspace of quenched momenta from N × N subspace of external fields:

$$g(x) \to e^{-iP_0^+ t - iP_1 x}\ g\ e^{iP_0^- t + iP_1 x}$$

$$P_0^\pm = \mathrm{diag}\{p_1^0, \ldots, p_K^0\} \otimes \mathbb{I}_{N\times N} \mp \mathbb{I}_{K\times K} \otimes \mathrm{diag}\{H_1, \ldots, H_N\}$$
$$P_1 = \mathrm{diag}\{p_1^1, \ldots, p_K^1\} \otimes \mathbb{I}_{N\times N}$$

- Perturbation theory for educed PCM action: $g = e^{iA} = 1 + iA - \frac{1}{2}A^2 - \frac{i}{6}A^3 + \frac{1}{24}A^4 \ldots$

$$E = -\frac{1}{\lambda N}\sum_{k=1}^{N} H_k^2 - \frac{1}{2\pi K N}\sum_{k,j=1}^{K}\sum_{a,b=1}^{N} \log[(p_k^{(1)} - p_j^{(1)})^2 + (p_k^{(0)} - p_j^{(0)})^2 + (H_a - H_b)^2]$$



Reproduces well known one-loop result: $E = -\frac{1}{\lambda}\sum_{k=1}^{\mathcal{N}} H_k^2 + \frac{1}{2}\sum_{a,b=1}^{N}(H_a - H_b)^2 \log(H_a - H_b)^2$

- PCM is all-loops integrable for any H! E.g., for particular profile of fields $H_k = h\frac{\cos\left(\frac{\pi(2k-1)}{2N}\right)}{\sin\left(\frac{\pi}{2N}\right)}$

$$\frac{E}{N2} = \frac{1}{8\pi}\mathrm{B}^2 I_1(\mathrm{B}) K_1(\mathrm{B}) \qquad \text{where} \quad \frac{m}{h} = \mathrm{B}K_1(\mathrm{B}), \qquad \mathrm{B} = \frac{4\pi}{\lambda}$$

Fateev, V.K., Wiegmann '94
V.K., Sobko, Zarembo '23

Discussion and prospects

- Structure constants of half-BPS of "huge" operators of N=4 SYM can be efficiently computed through the dedicated matrix integrals (saddle point at $N=\infty$, integrable (1+1)D fluid dynamics, exact solutions...)
- For particular huge operators (triangular, rectangular, trapezium Yts, cubic exponential operators, etc) it is often solvable via algebraic curve methods.
- Deformations of LLM geometry in the presence of a probe. Perfect match with matrix models for small probes. Nice instanton-type formula for large probe – giant graviton. For huge probes the geometry is not yet known.

Donos '06
Bin Chen et al '07

- Matrix models near criticalities and double scaling can give insight into highly quantum regime of supergravity.
- Generalization to ¼-BPS and 1/8-BPS operators? Some proposals exist for the bubbling solutions. Geometry solution for 2D and 3D SU(∞) principal chiral model?