Title: Coulomb branches without tears (or affine Grassmannians)

Speakers: Ben Webster

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Coulomb branches without tears (or affine Grassmannians)

Ben Webster

University of Waterloo
Perimeter Institute

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Context and definition

Very important 3d theory: given G a reductive connected group and $\mathbf{E}$ a symplectic representation of G, we have a theory $\mathcal{T}(G, \mathbf{E})$ given by gauging the 3d σ-model into $\mathbf{E}$. Most often $\mathbf{E} = T^*\mathbf{N}$ for a representation $\mathbf{N}$, but I don't want to assume this right away.

We have two natural topological twists Q_A, Q_B. We call theories **3-d mirror dual** if they are equivalent in a way that swaps $Q_A \leftrightarrow Q_B$.

Since theories are 3d, their local operators are commutative rings equipped with Poisson brackets. We think of these as functions on algebraic varieties called the **Higgs** and **Coulomb** branches $\mathfrak{M}_H$ and $\mathfrak{M}_C$.

The moduli of vacua $\mathfrak{M}_H(G, \mathbf{E}) = \operatorname{Spec} Z_B(S^2)$ for the B-twist (the **Higgs branch**) was studied by Hitchin, Karlhede, Lindström and Roček and interpreted as a hyperkähler quotient $\mathbf{E}////G$.

We interpret $||(\mu_I(x), \mu_J(x), \mu_K)||^2$ as an energy functional, so points of $\mathfrak{M}_H$ are G-orbits of points in $T^*\mathbf{N}$ where μ_I, μ_J, μ_K vanish.

Of course, not all orbits are the same. Some are free (complete symmetry breaking), some are not. The stabilizer G_x of a point x gives an invariant of the orbit (up to conjugacy).

"Higgs mechanism"

Physically speaking, we see this stabilizer from the fact that its Lie algebra gives the massless gauge fields.

1. For $G \subset U(1)^m \circlearrowright \mathbb{C}^n$, just depends on which coordinates in x are non-zero.
2. For quiver gauge theory, think of matrices as representation of doubled quiver; stabilizer captures dimension vectors of simple subrepresentations (in particular, orbit is free iff representation is simple).

The Higgs branch as a whole is singular, but the set of points with fixed stabilizer are smooth (and in fact, hyperkähler). They form a "symplectic leaf."

These leaves aren't a special feature of Higgs branches; any well-behaved singular hyperkähler (symplectic singularity) will have finitely many leaves.

In physics, we can think of these leaves as phases: each point is a choice of vacuum, and moving around the leaf, the theory changes in a continuous way, whereas if we move from one leaf to another, the changes will be discontinuous.

What are the symplectic leaves of the Coulomb branch?

The moduli of vacua for the A-twist (the **Coulomb branch**) proved much more mysterious. Denote this $\mathfrak{M}_C(G, \mathbf{N}) = \operatorname{Spec} Z_A(S^2)$ (I'm assuming $\mathbf{E} = T^*\mathbf{N}$).

The space $\mathfrak{M}_C(G, \mathbf{N})$ is birational to $T^*T^\vee/W = (\mathfrak{t} \times T^\vee)/W$, but with a modification of the multiplication of functions on this space.

Work of Cremonesi–Hanany–Zaffaroni, Bullimore–Dimofte–Gaiotto, and Braverman–Finkelberg–Nakajima gradually brought this picture into clearer view:

- $\mathbb{C}[\mathfrak{M}_C(G, \mathbf{N})]$ is a subalgebra of the fraction field $\mathbb{C}(T^*T^\vee)^W$,
- obtained by replacing m_λ, a character on $T^\vee$, by multiplying it by certain rational functions (roughly with denominator depending on G and numerator depending on $\mathbf{N}$).

We can think of this as an affine version of Springer theory:

Taylor series $\mathsf{C} = \mathbb{C}[[t]]$	$\mathsf{G} = G[[t]]$	$\mathsf{N} = \mathbf{N}[[t]]$
Laurent series $\mathcal{C} = \mathbb{C}((t))$	$\mathcal{G} = G((t))$	$\mathcal{N} = \mathbf{N}((t))$

Relevant spaces:

$$\mathsf{Y} = \mathsf{N}/\mathsf{G} = \mathrm{Map}(D = \mathrm{Spec}\,\mathsf{C} \to N/G)$$

$$\mathcal{Y} = \mathcal{N}/\mathcal{G} = \mathrm{Map}(D^* = \mathrm{Spec}\,\mathcal{C} \to N/G)$$

These can be interpreted as spaces of principal G bundles with a section of the associated N-bundle on D and D^*.

Thus, the fiber product $\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}$ is the space of such bundles on the "raviolo" gluing two copies of D along D^*.

Previous experience tells us it would be fun to consider

$$A = H_*^{BM}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}) \qquad A^K = K_0(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}).$$

Using factorization arguments, we can see that A, A^K is a commutative $\mathbb{C}$-algebra of finite type.

Definition

The **(K-theoretic) Coulomb branch** *is the spectrum* $\mathcal{M} = \operatorname{Spec} A$ *(*$\mathcal{M}^K = \operatorname{Spec} A^K$*).*

This geometric definition has advantages, but it's very scary when you open the paper.

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Coulomb branches | Algebraic descriptions

Small examples

In the case of $G = \mathbb{C}^\times$, we obtain well-known varieties:

- $\mathfrak{M}_C(\mathbb{C}^\times, 0) \cong T^*\mathbb{C}^\times = \mathrm{Spec}(\mathbb{C}[t, m, m^{-1}])$
- If $\mathbf{N}$ is a representation of $\mathbb{C}^\times$ and $q_\pm$ is the sum of the positive and negative weights, $\mathbb{C}[\mathfrak{M}_C(\mathbb{C}^\times, \mathbf{N})]$ is the subalgebra generated by $t, m_+ = t^{q_+}m, m_- = t^{-q_-}m^{-1}$ with the relations

$$m_+m_- = t^{q_+-q_-}$$

so $\mathfrak{M}_C(\mathbb{C}^\times, \mathbf{N}) \cong \mathbb{C}^2/\mathbb{Z}_Q$ for $Q = q_+ - q_-$ via the map

$$m_+ \mapsto x^Q \qquad m_- \mapsto y^Q \qquad t \mapsto xy.$$

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In K-theoretic Coulomb branches, the formulas are always the same, except that we have to use K-theoretic formulae for the Euler classes and integration over $\mathbb{P}^1$.

- $\mathfrak{M}^K_C(\mathbb{C}^\times, 0) \cong \mathbb{C}^\times \times \mathbb{C}^\times = \mathrm{Spec}(\mathbb{C}[u^{\pm 1}, m^{\pm 1}])$
- If $\mathbf{N}$ is a representation of $\mathbb{C}^\times$ and $q^{(r)}_\pm$ are the positive and negative weights, $\mathbb{C}[\mathfrak{M}^K_C(\mathbb{C}^\times, \mathbf{N})]$ is the subalgebra generated by

$$u^{\pm 1} \qquad m_+ = \prod (1 - u^{-q^{(r)}_+}) m \qquad m_- = \prod (1 - u^{q^{(r)}_-}) m^{-1}$$

with the relations

$$m_+ m_- = \prod (1 - u^{-q^{(r)}_+}) \prod (1 - u^{q^{(r)}_-})$$

If $G = PGL(2)$ or $SL(2)$, then $\mathfrak{M}_C(G, \mathbf{N}) \approx \mathfrak{M}_C(\mathbb{C}^\times, \mathbf{N})/W$ for $\mathbb{C}^\times \subset G$. Note that in this case we have $q = q_+ = -q_-$. The group $W = \{1, s\}$ acts by $s \cdot m_\pm = (-1)^q m_\mp$ and $s \cdot t = -t$.

- If $G = PGL(2)$, the ring $\mathbb{C}[\mathfrak{M}_C(PGL(2), \mathbf{N})]$ is generated by

$$t^2, m_\bullet = \frac{m_+ - s \cdot m_+}{t}, m_\circ = m_+ + s \cdot m_+$$

modulo the relation $m_\circ^2 = t^2 m_\bullet^2 + (-1)^q \cdot 4t^{2q}$.

More geometrically, this means a W-invariant rational function can have a pole of order minumum of $|\langle \alpha^\vee, \lambda \rangle|$ for all monopole operators m_λ appearing.

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- If $G = SL(2)$ ($q > 0$), then the ring $\mathbb{C}[\mathfrak{M}_C(SL(2), \mathbf{N})]$ is generated by

$$t^2, m_\bullet = \frac{m_+ + s \cdot m_+}{t^2}, m_\circ = \frac{m_+ - s \cdot m_+}{t}$$

modulo the relation $m_\circ^2 = t^2 m_\bullet^2 - (-1)^{q-1} \cdot 4t^{2q-2}$.

Theorem

$\mathfrak{M}_C(SL(2), \mathbf{N}) \cong \mathfrak{M}_C(PGL(2), \mathbf{N})/C_2$ where $C_2 = \{1, c\}$ acts by $c \cdot m = -m, c \cdot t = t$.

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Coulomb branches | Algebraic descriptions

Bootstrap from rank 1

Consider any $\lambda \in \mathfrak{t}$. Let

- $\mathbf{N}_\lambda$ be the fixed locus of λ.
- G_λ be the centralizer of λ in G
- G^0_λ, G^1_λ be the kernel and image of G_λ acting on $\mathbf{N}_\lambda$.

Think about hyperplane arrangement on $\mathfrak{t}$ given by roots and weights of $\mathbf{N}$. $\mathbf{N}_\lambda, G^0_\lambda, G^1_\lambda$ are clear from intersection of hyperplanes λ lies in. Let $\mathfrak{t}^\circ$ be $\mathfrak{t}$ minus the hyperplanes that don't contain λ.

Theorem

Around the fiber over λ, the variety $\mathfrak{M}_C(G,\mathbf{N})$ looks like $\mathfrak{M}_C(G_\lambda,\mathbf{N}_\lambda)$. That is, we have a Cartesian diagram

$$\begin{array}{ccccc} \mathfrak{M}_C(G,\mathbf{N}) & \longleftarrow & \mathfrak{M}^\circ_C(G_\lambda,\mathbf{N}_\lambda) & \hookrightarrow & \mathfrak{M}_C(G_\lambda,\mathbf{N}_\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{t}/W & \longleftarrow & \mathfrak{t}^\circ/W_\lambda & \hookrightarrow & \mathfrak{t}/W_\lambda \end{array}$$

If we range over λ subgeneric (generic in a root or weight hyperplane), we reduce to the cases where $G_\lambda^1 = \mathbb{C}^\times, SL(2), PGL(2)$.

Theorem

The union of these open sets has complement of codimension 2.
The variety $\mathfrak{M}_C(G, \mathbf{N})$ is affine normal, so $\mathbb{C}[\mathfrak{M}_C(G, \mathbf{N})]$ precisely the intersection in $C(T^*T^\vee)^W$ of the functions which pull back to regular functions on $\mathfrak{M}_C^\circ(G_\lambda, \mathbf{N}_\lambda)$ for λ subgeneric.

Can also define explicitly as an intersection of DVRs (that is, specify all Weil divisors and what poles/zeros on them are.

Powerful idea. Lets us reduce lots of theorems to rank 1 case:

1. Key to many proofs that $\mathfrak{M}_C(G, \mathbf{N})$ is a known variety.
2. Can reduce Teleman's gluing to this case.
3. Prove functoriality (recent work w/ Gannon) without annoying hypotheses in arXiv version.

Small examples

In the case of $G = \mathbb{C}^\times$, we obtain well-known varieties:

- $\mathfrak{M}_C(\mathbb{C}^\times, 0) \cong T^*\mathbb{C}^\times = \mathrm{Spec}(\mathbb{C}[t, m, m^{-1}])$
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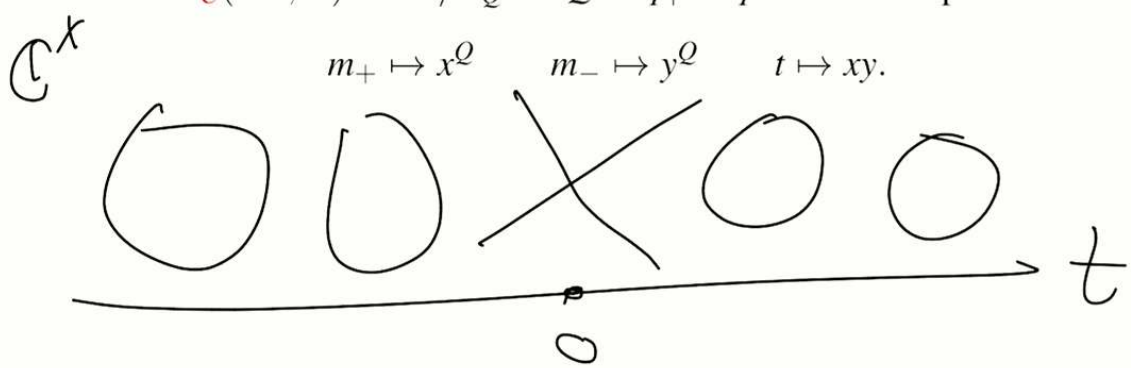

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Bootstrap from rank 1

Works without change for many generalizations:

- For some $\mathbf{E}$ not of cotangent type, there is a Coulomb branch. There are many choices of birational map $\mathfrak{M}_C(T, \mathbf{E}) \dashrightarrow T^*T^\vee$, and you have to modify the W-action on $T^*T^\vee$ to make it equivariant. If you can do this, you can use this as the definition (equivalent to Teleman, BDFRT, Bielawski-Foscolo).
- Same point applies to the K-theoretic case:

$$\begin{array}{ccccc} \mathfrak{M}_C^K(G, \mathbf{N}) & \longleftarrow & \mathfrak{M}_C^{K\circ}(G_\lambda, \mathbf{N}_\lambda) & \hookrightarrow & \mathfrak{M}_C^K(G_\lambda, \mathbf{N}_\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ T/W & \longleftarrow & T^\circ/W_\lambda & \hookrightarrow & T/W_\lambda \end{array}$$

- Similarly with "non-simply-laced" Coulomb branches of Nakajima-Weekes.
- There is a non-commutative version of this theorem, where you intersect subrings of a localization of differential operators $D(T^\vee)^W$.

Theorem (W.)

The symplectic leaves of $\mathfrak{M}_C(G, \mathbf{N})$ *are in bijection with pairs* $(\mathbf{N}_0, L)$ *modulo the action of* $\mathcal{Z}^\vee$ *where*

- $\mathbf{N}_0 \subset \mathbf{N}$ *is the fixed locus of a cocharacter* λ *(generic with this property)*
- L *is a zero-dimensional leaf of* $\mathfrak{M}_C(G^1_\lambda, \mathbf{N}_0)$.
- $\mathcal{Z}^\vee$ *is a finite group acting on* $\mathfrak{M}_C(G^1_\lambda, \mathbf{N}_0)$ *arising from failure of* $G_\lambda \to G^1_\lambda$ *to split.*

A neighborhood of the whole fiber over λ is a fiber bundle over $T^*\check{T}^0_\lambda$ with fiber $\mathfrak{M}_C(G^1_\lambda, \mathbf{N}_0)$, and $\mathcal{Z}^\vee$ captures the monodromy of this bundle.

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I could just stop here, but I want to think a little more carefully about what this construction says.

For each root α, we have a map $\mathbb{C}(T^*T^\vee) \to \mathbb{C}(T^*T^\vee)$ defined by

$$\partial_\alpha(f) = \frac{s_\alpha f - f}{2\alpha}$$

This is an isomorphism from the -1-eigenspace of s_α to the 1-eigenspace, which is inverse to multiplication by α. In particular, the projections to these eigenspaces are:

$$e_+ = \partial_\alpha \alpha \qquad e_- = \alpha\partial_\alpha$$

Definition

Let $A \subset \mathbb{C}(T^*T^\vee)$ be an s_α invariant subalgebra which contains $\mathbb{C}[\mathfrak{t}]$. Let $A^\natural_\alpha$ be the algebra generated by A and ∂_α.

This is actually a 2×2 matrix algebra of rank 2 over its center $Z = Z(A^\natural_\alpha) = e_+ A^\natural_\alpha e_+$.

Given $Z \to \mathbb{C}$, we have an induced A-module $A^\natural_\alpha e_+ \otimes_Z \mathbb{C}$. By acting on e_+, we can think of this as A/I where $I \subset A$ is an ideal. This quotient is 2-dimensional, and free over the group algebra $\mathbb{C}\{1, s_\alpha\}$

Let $\mathrm{Hilb}^{\mathrm{free}}(\mathrm{Spec}(A))$ be the Hilbert scheme of such ideals.

Theorem (Bielawski–Foscolo, Chan–Leung, W.)

If G has semisimple rank 1 and $G \not\cong SL(2) \times Z(G)$ (for example, $G = PGL(2)$ or $G = GL(2)$), then

$$A = \mathbb{C}[\mathfrak{M}_C(T, \mathbf{N})] \quad \Rightarrow \quad Z(A_\alpha^\natural) = \mathbb{C}[\mathfrak{M}_C(G, \mathbf{N})]$$

You can "fix" the $SL(2)$ case by taking $G' = GL(2) \times Z(G)$ acting on the same representation and then symplectic reduction by $\mathbb{C}^\times$.

But maybe this is the wrong way to think about the issue. The problem is that the primitive cocharacter $\alpha^\vee$ of $SL(2)$ "wants to be" the sum of two shorter things (possible when $G \ncong SL(2) \times Z(G)$).

The translation by $\alpha^\vee$ can't be written as the product of two shorter translations, but it can be written as the product of two shorter reflections in the affine Weyl group!

Natural generalization to higher rank: if A is a W-invariant subalgebra of $\mathbb{C}(T^*T^\vee)$, then we can let $A^\natural$ be the subalgebra generated by A and ∂_α for all simple roots α.

Theorem (W., Chan)

The ring $Z(A^\natural)$ is the functions on the open subset of $\mathrm{Hilb}^{\mathrm{free}}(\mathrm{Spec}(A))$ where $\prod \alpha \notin I$, intersected with the closure of the free orbits.

It's claimed in Bielawski–Foscolo that if $Sp(2n)$ is not a factor of G, then

$$A = \mathbb{C}[\mathfrak{M}_C(T, \mathbf{N})] \quad \Rightarrow \quad Z(A^\natural_\alpha) = \mathbb{C}[\mathfrak{M}_C(G, \mathbf{N})].$$

This is false in general (counter-example found by Chan), but there is a map $Z(A^\natural) \to \mathbb{C}[\mathfrak{M}_C(G, \mathbf{N})]$ which is surjective in many interesting cases.

Again, the problem is that sometimes you seem to need some factorizations of cocharacters that don't exist in the cocharacter lattice.

But they do exist in the affine Weyl group!

Theorem

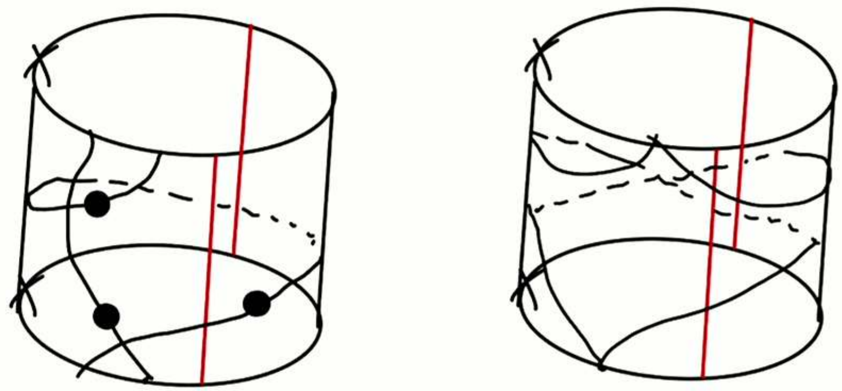
The ring $\mathbb{C}[\mathfrak{M}_C(G, \mathbf{N})]$ *is the endomorphism ring of an object in a larger category where objects are subspaces* $U \subset \mathcal{N}$ *and parahorics* $\mathcal{P} \subset \mathcal{G}$*, where we consider*

$$\mathrm{Hom}((U, \mathcal{P}), (U', \mathcal{P}')) = H_*^{BM}(U/\mathcal{P} \times_{\mathcal{Y}} U'/\mathcal{P}')$$

This category has a combinatorial presentation.

For quiver gauge theories, we can encode this presentation as cylindrical KLRW algebras.

In particular, we can see the issue with SL_2 or SL_3 factorization: if we send two points around the cylinder in the opposite directions, they cross in the back.



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