Title: Introduction to Categorical Probability Mini-Course Lecture

Speakers: Tomas Gonda

Collection/Series: Introduction to Categorical Probability Mini-Course, Oct 1-7, 2025

Subject: Quantum Foundations

Date: October 07, 2025 - 3:30 PM

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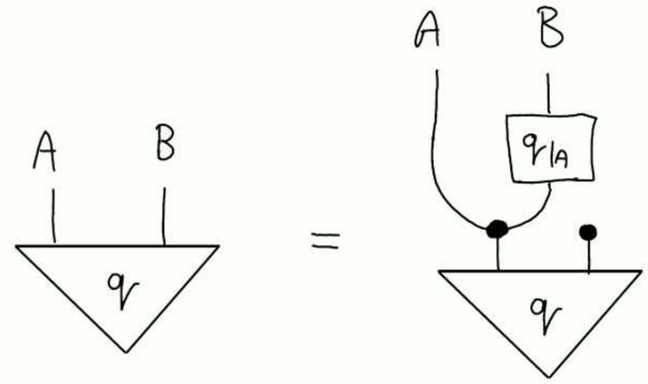
Part II.

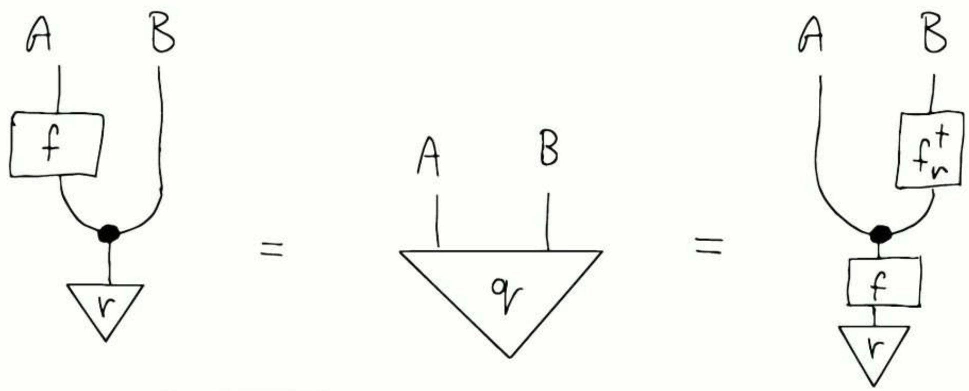
Additional concepts & axioms

Conditionals

(parametrized Bayes' theorem)

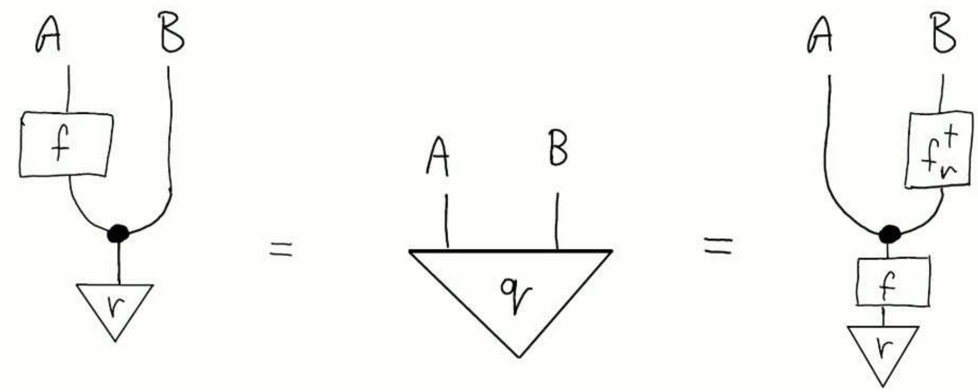


Bayesian Inverses

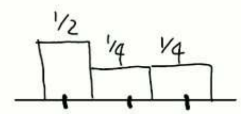


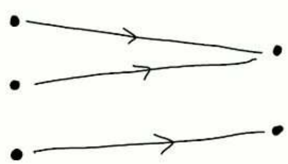
$f_r^\dagger$ is a Bayesian inverse of f w.r.t. r.

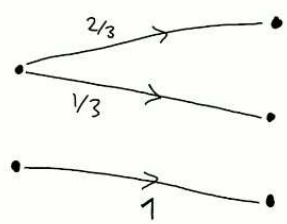
Bayesian Inverses



$f_r^\dagger$ is a Bayesian inverse of f w.r.t. r.

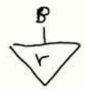
$B = \{0,1,2\}$ $A = \{0,1\}$ $r =$ 

f: 

$f_r^\dagger$: 

note $f \circ f_r^\dagger = id_A$

Bayesian Inverses

Consider two σ-algebras $\Omega \subseteq \Sigma$. $A = (X, \Omega)$, $B = (X, \Sigma)$
and a measure .

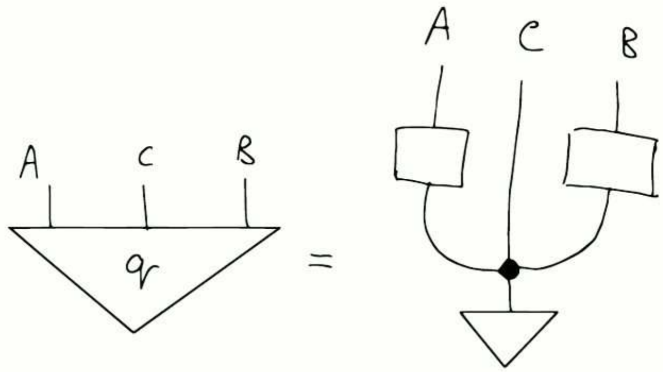
Then $f : B \to A$, $f(x) = x$ is measurable and

$$f_r^{+}(S \mid \_) := \mathbb{E}[1_S \mid \Omega] \qquad \text{for } S \in \Sigma$$

is its Bayesian inverse w.r.t. r.

Bayesian inverses generalize conditional expectations.

Conditional Independence ($A \perp\!\!\!\perp B \mid C$)

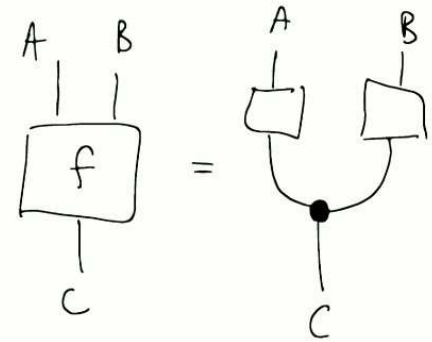


$$P(A, C, B) = P(A|C) \cdot P(C) \cdot P(B|C)$$

3 semigraphoid axioms follow by simple diagram manipulation

conditionals $\Rightarrow$ equivalence of def.

$\Rightarrow$ weak union



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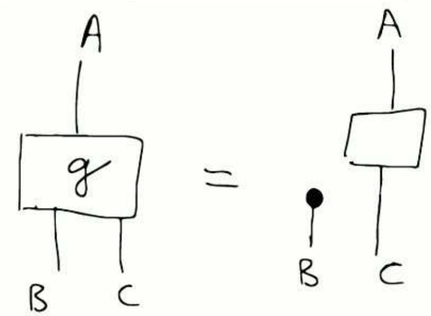
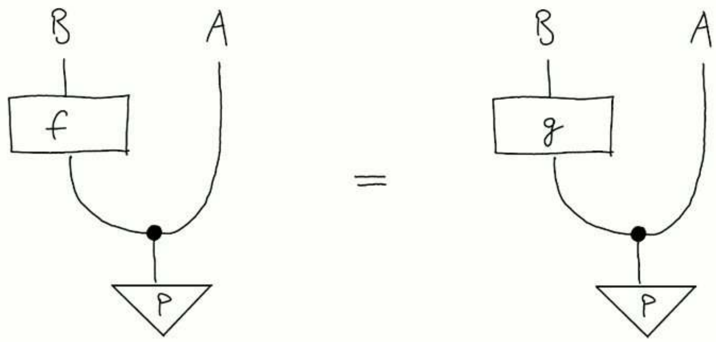


$$P(A \mid B, C) = P(A \mid C)$$

Almost surely



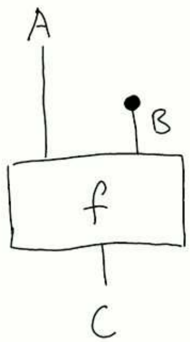
in **FinStoch** :

$$f(b \mid a) = g(b \mid a)$$

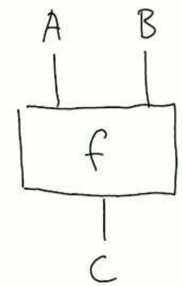
$$\forall\ b \in B \qquad \forall\ a \in \mathrm{supp}(p) = \{ a \mid p(a) \neq 0 \}$$

Positivity

(no destructive interference)



deterministic $\Rightarrow$



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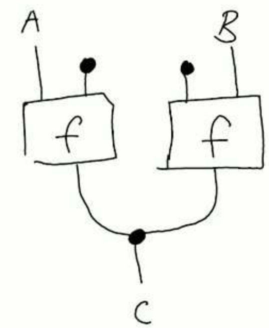


$0 = P(a) = \sum_{b \in B} P(a,b) \quad \Rightarrow \quad P(a,b) = P(a) \cdot P(b) = 0.$

conditionals $\Rightarrow$ positivity

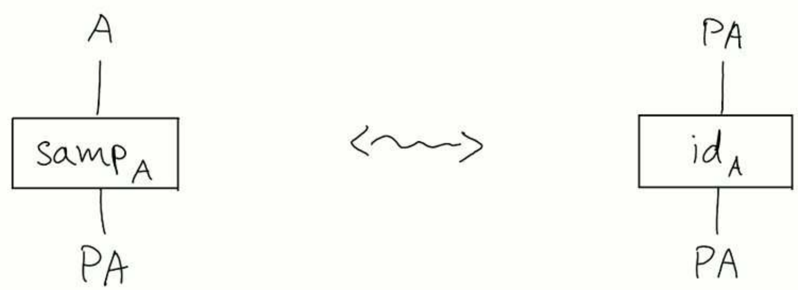
✓

CHausStoch
cC^*Alg

✗

FinStoch$\pm$

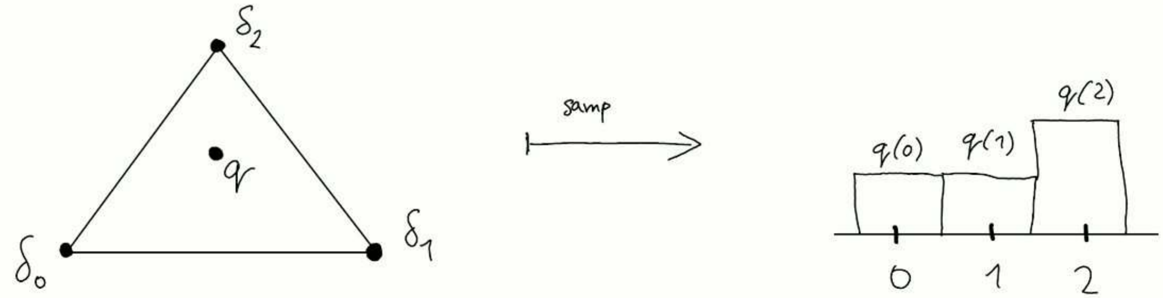
Distribution object



deterministic

sampling map in **BorelStoch**

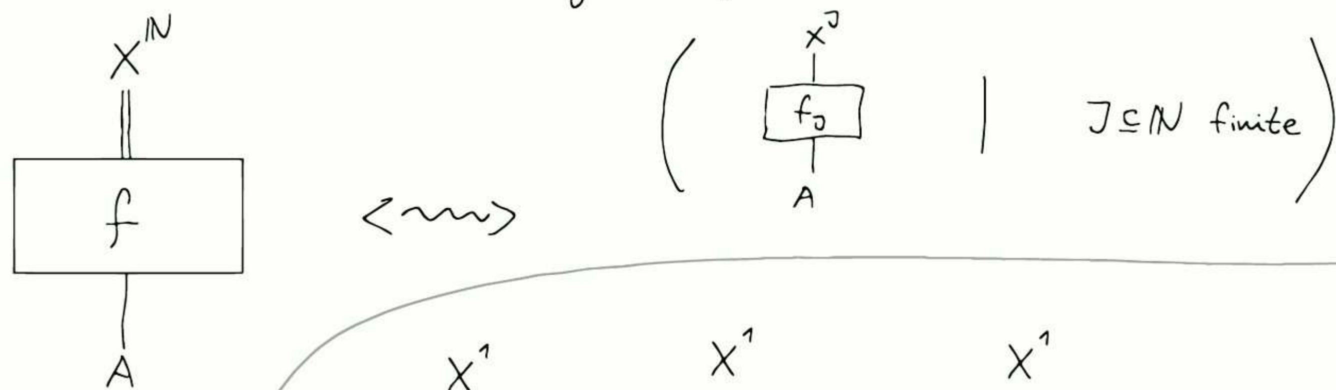
$$\mathrm{samp}_A(S \mid \mu) = \mu(S) \qquad \text{for } \mu \in PA, \; S \in \Sigma_A$$

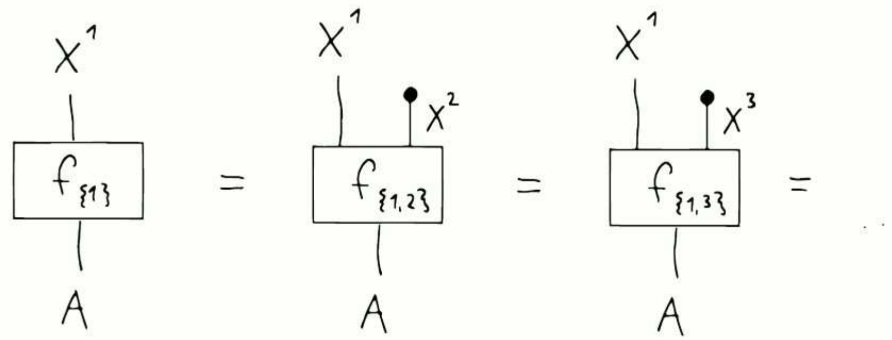


Infinite parallel composites

infinite tensor product $\approx$ compatible family of finite products

$X^{\mathbb{N}} \in \mathcal{C}$ is a Kolmogorov product if





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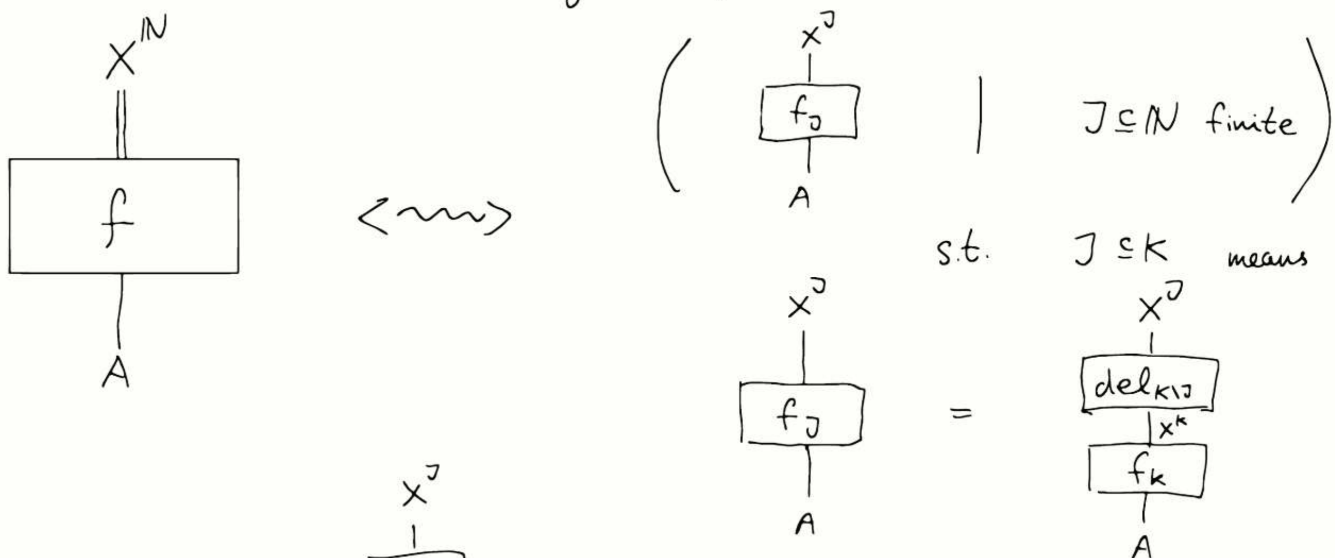

and all  are deterministic.

Infinite parallel composites

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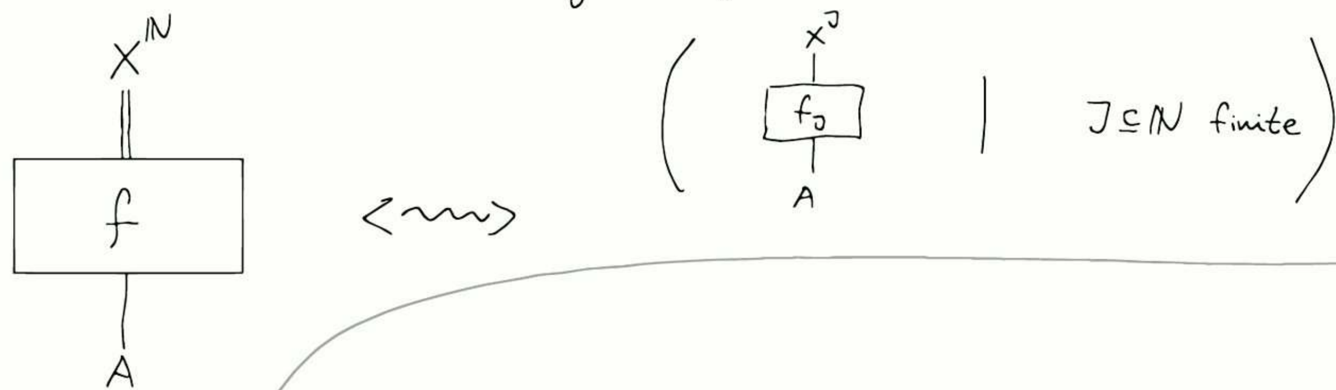


f deterministic $\iff$ $\forall J$: f_J deterministic

Infinite parallel composites

infinite tensor product ≈ compatible family of finite products

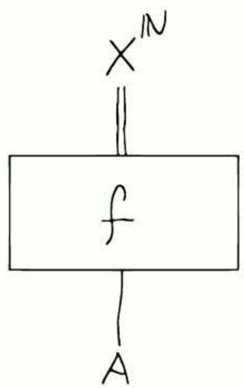
countable Kolmogorov products

✓	X
BorelStoch	FinStoch
qMeasStoch	Gauss
cC*Alg	FinSetMulti

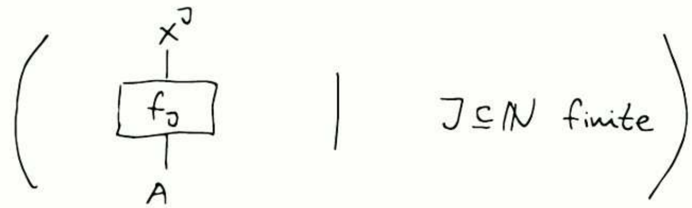
Infinite parallel composites

infinite tensor product ≈ compatible family of finite products

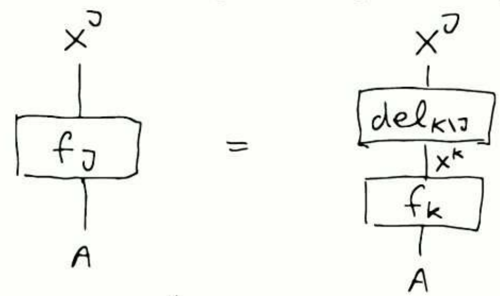
$X^{\mathbb{N}} \in \mathcal{C}$ is a Kolmogorov product if



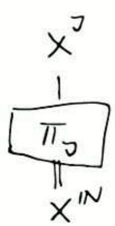
$\langle\!\rightsquigarrow\!\rangle$



s.t. $J \subseteq K$ means



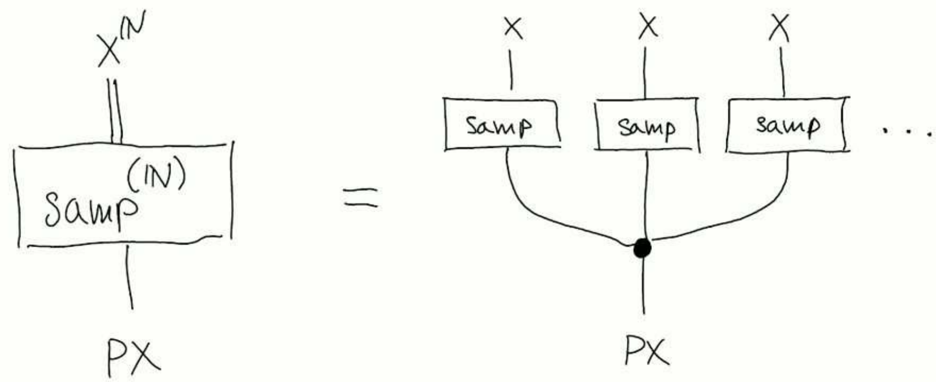
and all



are deterministic.

Infinite parallel composites

e.g. the "IID morphism":



produces a sequence of independent samples.

Infinite parallel composites

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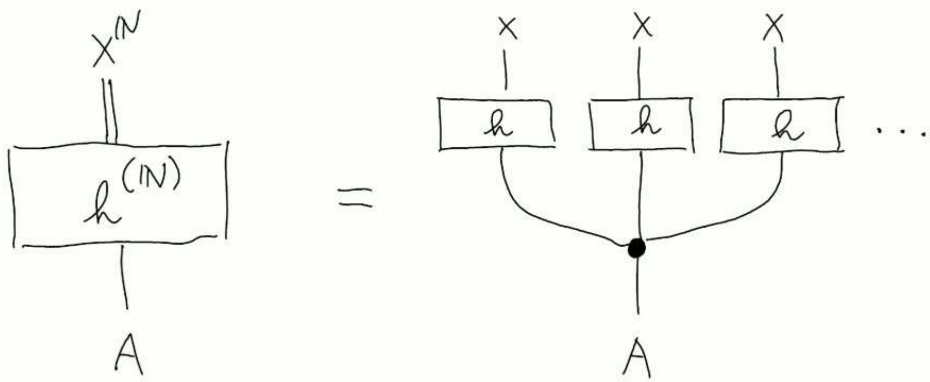

Infinite parallel composites

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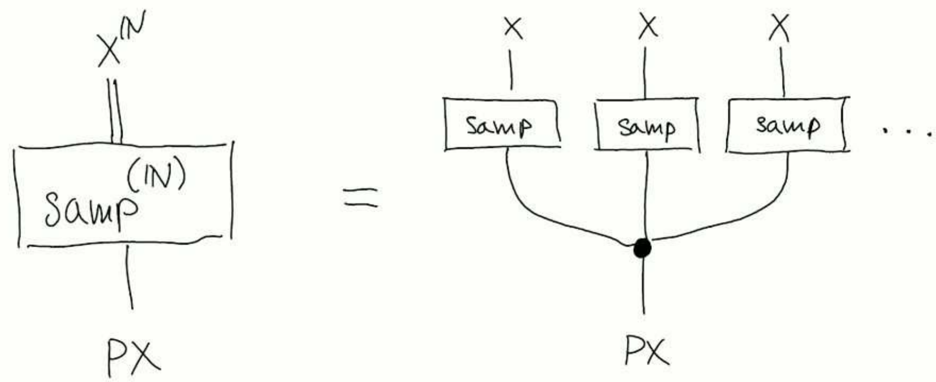
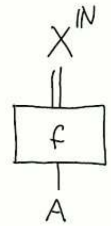
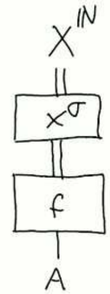


produces a sequence of independent samples.

de Finetti's Theorem



is exchangeable if



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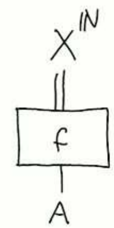


for all $\sigma \in S_{\mathbb{N}}$.

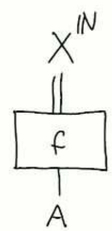
Theorem: Assume $\mathcal{C}$ is a Markov category with

i) conditionals

ii) countable Kolmogorov products

iii) distribution objects

If f is exchangeable, then



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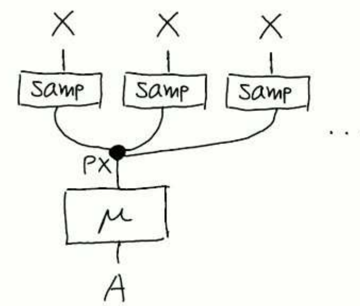


Strong law of large numbers

IID sampling from a distribution p. $p(1) = p(-1) = \frac{1}{2}$

trial	1	2	3	4	5	6	7	8	9	10	…	n
outcome	1	1	−1	1	−1	−1	1	−1	1	−1		a_n
relative fr. of $(-\infty, 0]$	0	0	1/3	1/4	2/5	1/2	3/7	1/2	4/9	1/2	…	$\frac{\lvert\{i \leq n \mid a_i \leq 0\}\rvert}{n}$
average	1	1	1/3	1/2	1/5	0	1/7	0	1/9	0	…	$\frac{1}{n}\sum_{i=1}^{n} a_i$

strong LLN: sample average converges to $E[p]$ (p^N-almost surely)

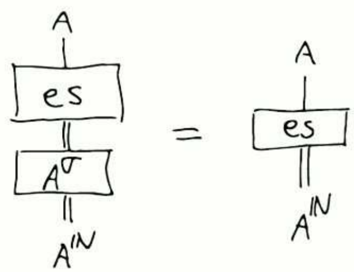
Glivenko-Cantelli Theorem: empirical distribution converges to the CDF of p (p^N-almost surely).

Strong law of large numbers

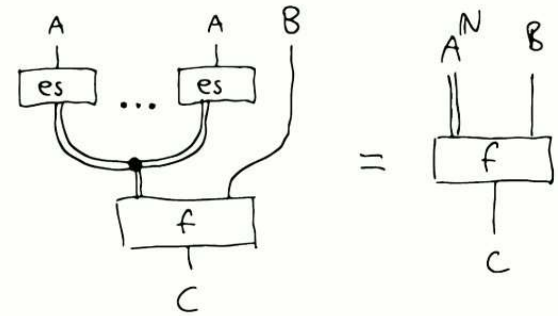
Empirical distribution is $es\left((-\infty, r] \mid (a_i)_{i\in\mathbb{N}}\right) = \lim_{n\to\infty} \dfrac{|\{i \leq n \mid a_i \leq r\}|}{n}$

An empirical sampling morphism $es : A^{\mathbb{N}} \to A$ satisfies

(i)



(ii)



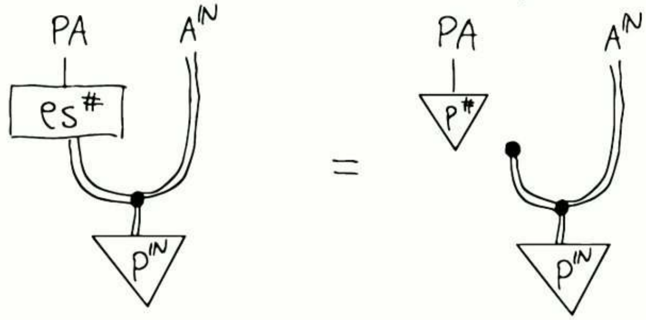
for any exchangeable f.

✓

BorelStoch

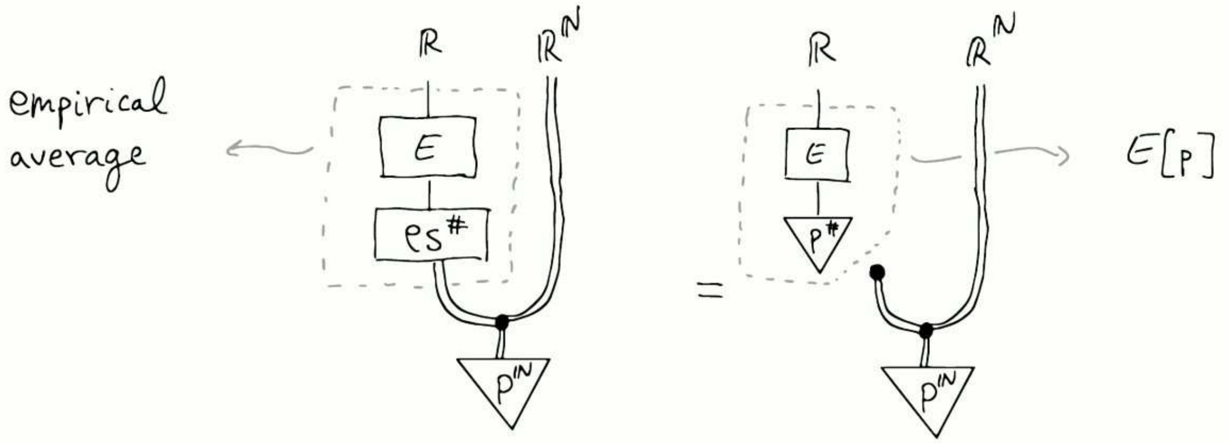
Strong law of large numbers

Theorem (synthetic Glivenko-Cantelli Theorem): For a state 



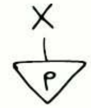
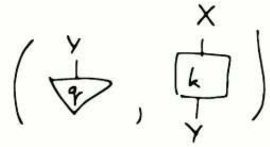
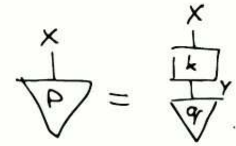
For $A = \mathbb{R}$ in **BorelStoch** we get the GC Theorem.

Strong law of large numbers



Using $\boxed{E}$ ($P\mathbb{R} \to \mathbb{R}$) $: \mu \longmapsto \int_{x\in\mathbb{R}} x\,\mu(dx)$, we get strong LLN.

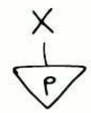
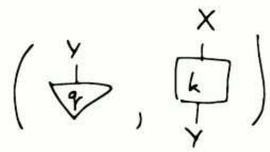
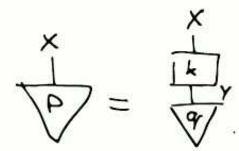
Ergodic Decomposition Theorem

A decomposition of  is  s.t. .

- in **FinStoch** : $p(x) = \sum_y k(x|y)\, q(y)$ eg $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{3} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{2}{3}$

$$\begin{pmatrix} p(x) \\ p(x') \end{pmatrix} = \begin{pmatrix} k(x|y) \\ k(x'|y) \end{pmatrix} q(y) + \begin{pmatrix} k(x|y') \\ k(x'|y') \end{pmatrix} q(y')$$

Ergodic Decomposition Theorem

A decomposition of  is  s.t. .

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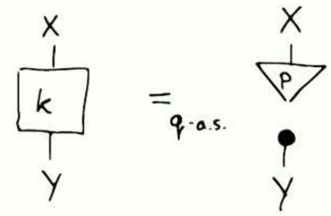
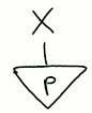
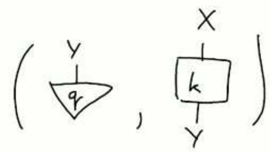
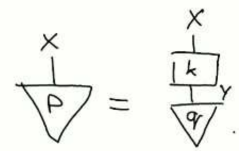
A trivial decomposition (q, k) is s.t. 

Ergodic Decomposition Theorem

A decomposition of  is  s.t. 

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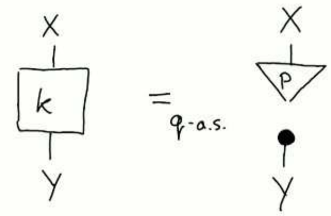
A trivial decomposition (q, k) is s.t. 

p is indecomposable if all its decompositions are trivial.

- think convexly extremal

- In a positive Markov category, indecomposable $\Leftrightarrow$ deterministic

Ergodic Decomposition Theorem

A dynamical system is a monoid M, an object $X \in \mathcal{C}$ and a homomorphism $m \in M \longmapsto$ $\begin{array}{c} X \\ | \\ \boxed{t_m} \\ | \\ X \end{array}$

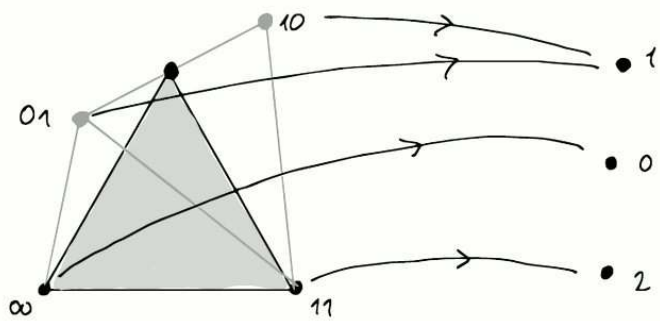
e.g. $M = \mathbb{N}$; $X \in$ **BorelStoch** $\rightsquigarrow$ a discrete-time Markov chain where t_1 is the transition probability matrix for a time step.

Ergodic Decomposition Theorem

An invariant state p (with output X) is ergodic if for every invariant deterministic observable c (from X to R), the composite of p followed by c is deterministic.

Example: $X = \mathbb{Z}_2 \times \mathbb{Z}_2$, $M = \mathbb{Z}_2$, $t_0 = |\ |$ $t_1 =$

Invariant states



universal invariant observable

$X_{inv} = \{0, 1, 2\}$

$r(a, b) = a + b$

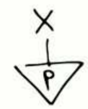
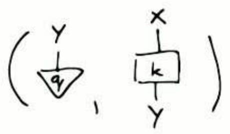
ergodic states:

δ_{00} δ_{11} $\frac{1}{2}\delta_{01} + \frac{1}{2}\delta_{10}$

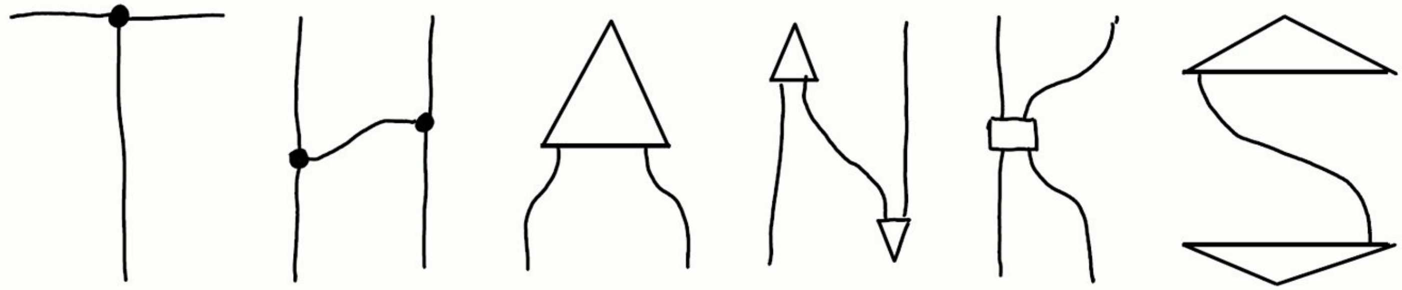
Ergodic Decomposition Theorem

Theorem: Assume that $(t_m)_{m \in M}$ is a deterministic dynamical system such that

- X_{inv} exists
- each deterministic $f: X \to R$ has a Bayesian inverse.

Then every invariant  has a decomposition  where k is q-a.s. ergodic.

In **BorelStoch**, every invariant measure is a mixture of ergodic ones.


THANKS

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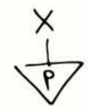
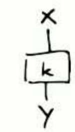
Then every invariant  has a decomposition ( , )
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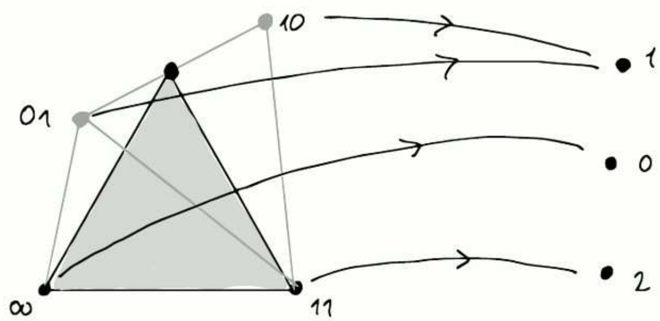
In **BorelStoch**, every invariant measure is a mixture of ergodic ones.

Ergodic Decomposition Theorem

An qas invariant $k: A \to X$ is qas ergodic if for every invariant deterministic observable $\begin{array}{c} R \\ | \\ \boxed{c} \\ | \\ X \end{array}$, the composite $\begin{array}{c} R \\ | \\ \boxed{c} \\ | \\ \boxed{k} \\ | \\ A \end{array}$ is qas deterministic.

Example: $X = \mathbb{Z}_2 \times \mathbb{Z}_2$, $M = \mathbb{Z}_2$, $t_0 = \mid \; \mid$ $t_1 =$

Invariant states



universal invariant observable

$X_{inv} = \{0, 1, 2\}$

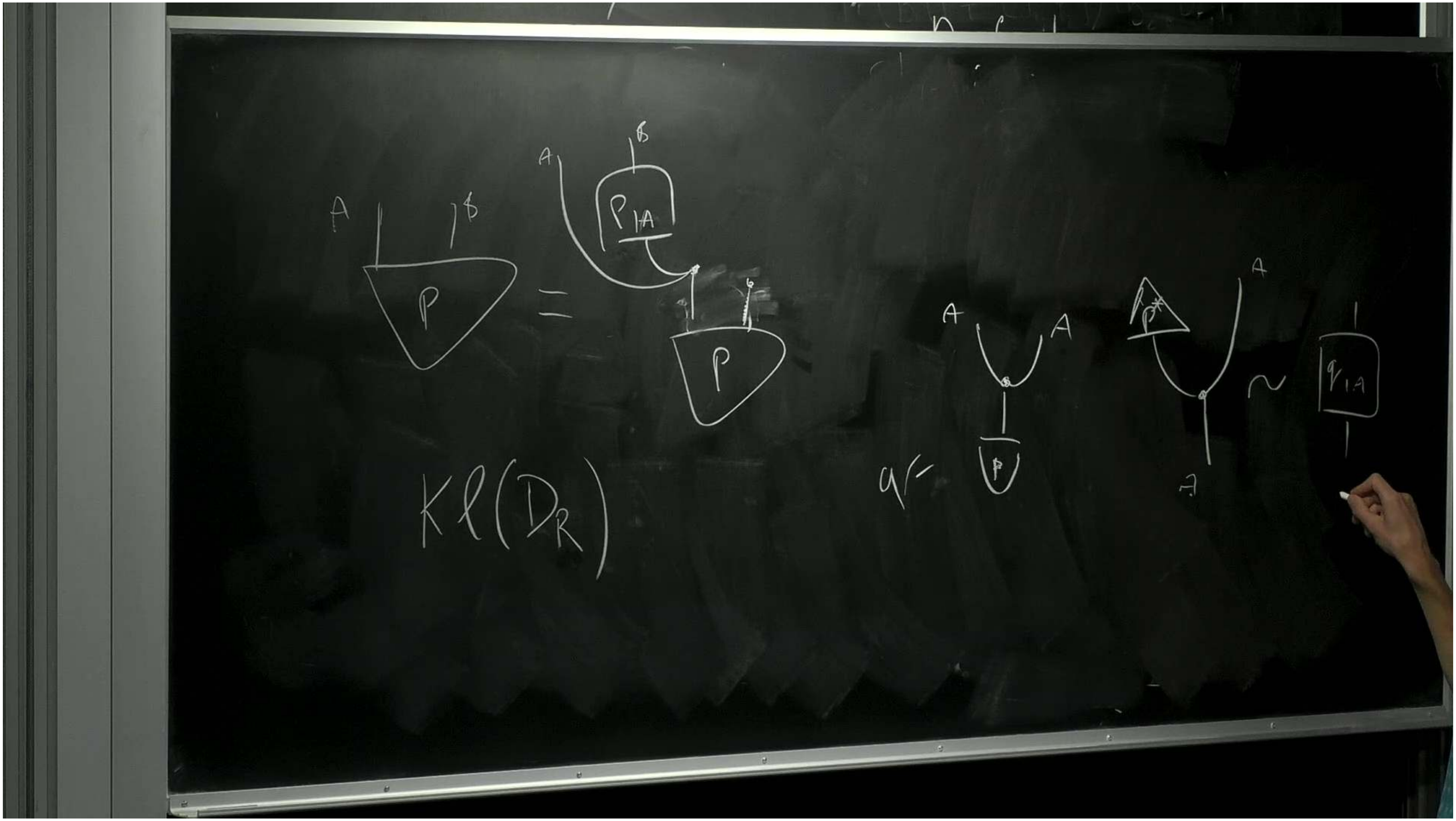
$r(a, b) = a + b$

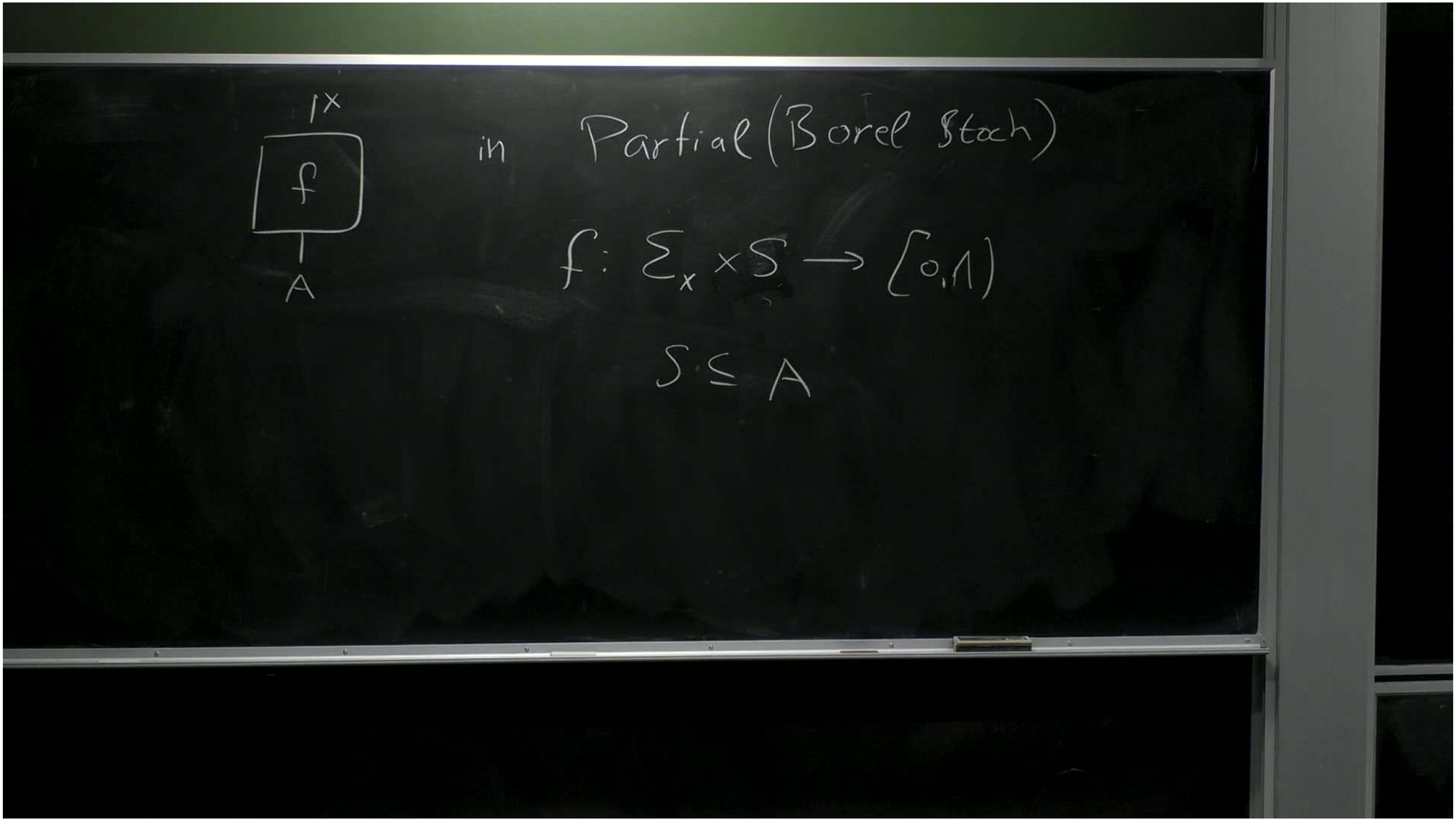
ergodic states:

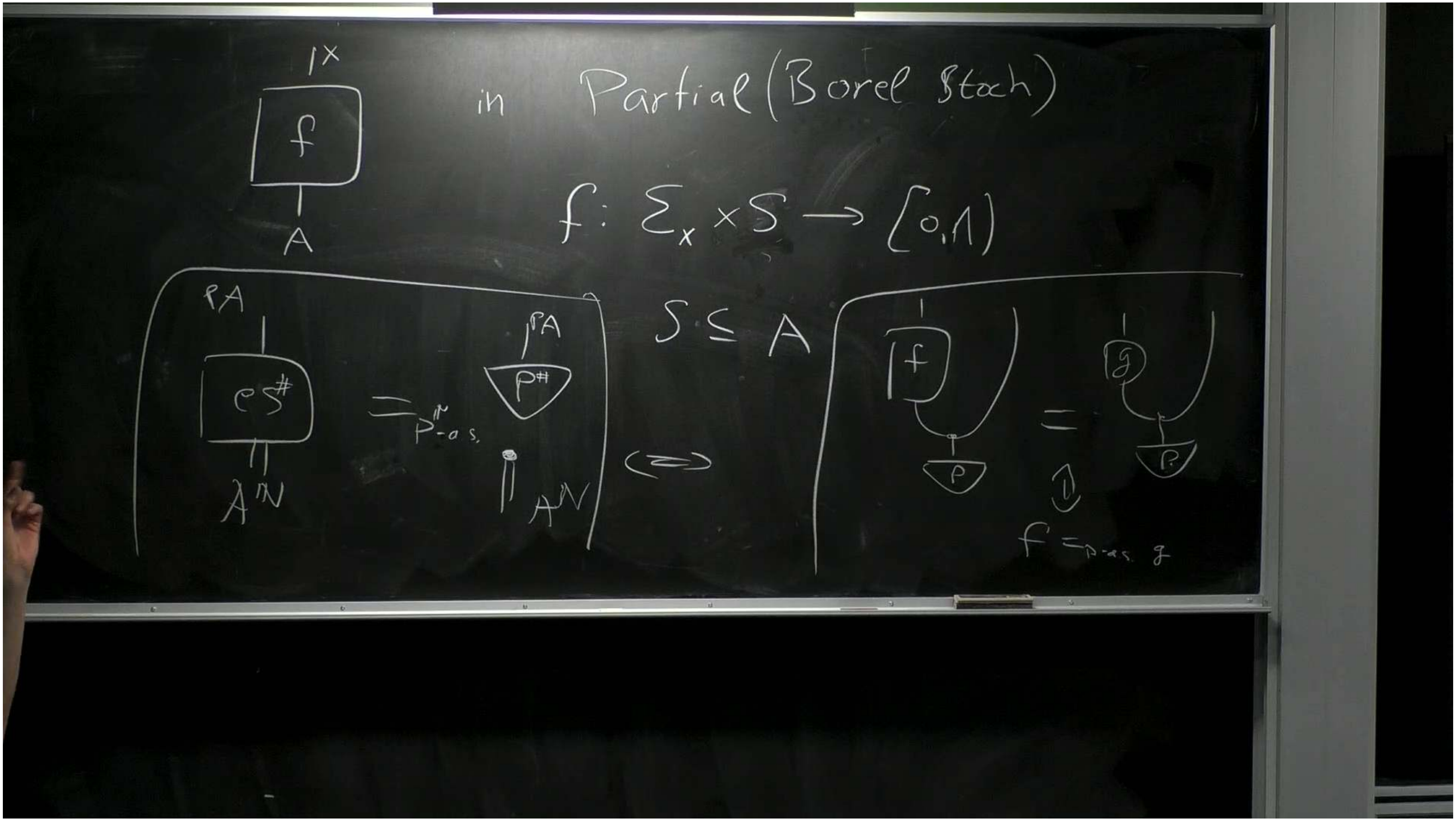
δ_{00} δ_{11} $\frac{1}{2}\delta_{01} + \frac{1}{2}\delta_{10}$

Other Theorems

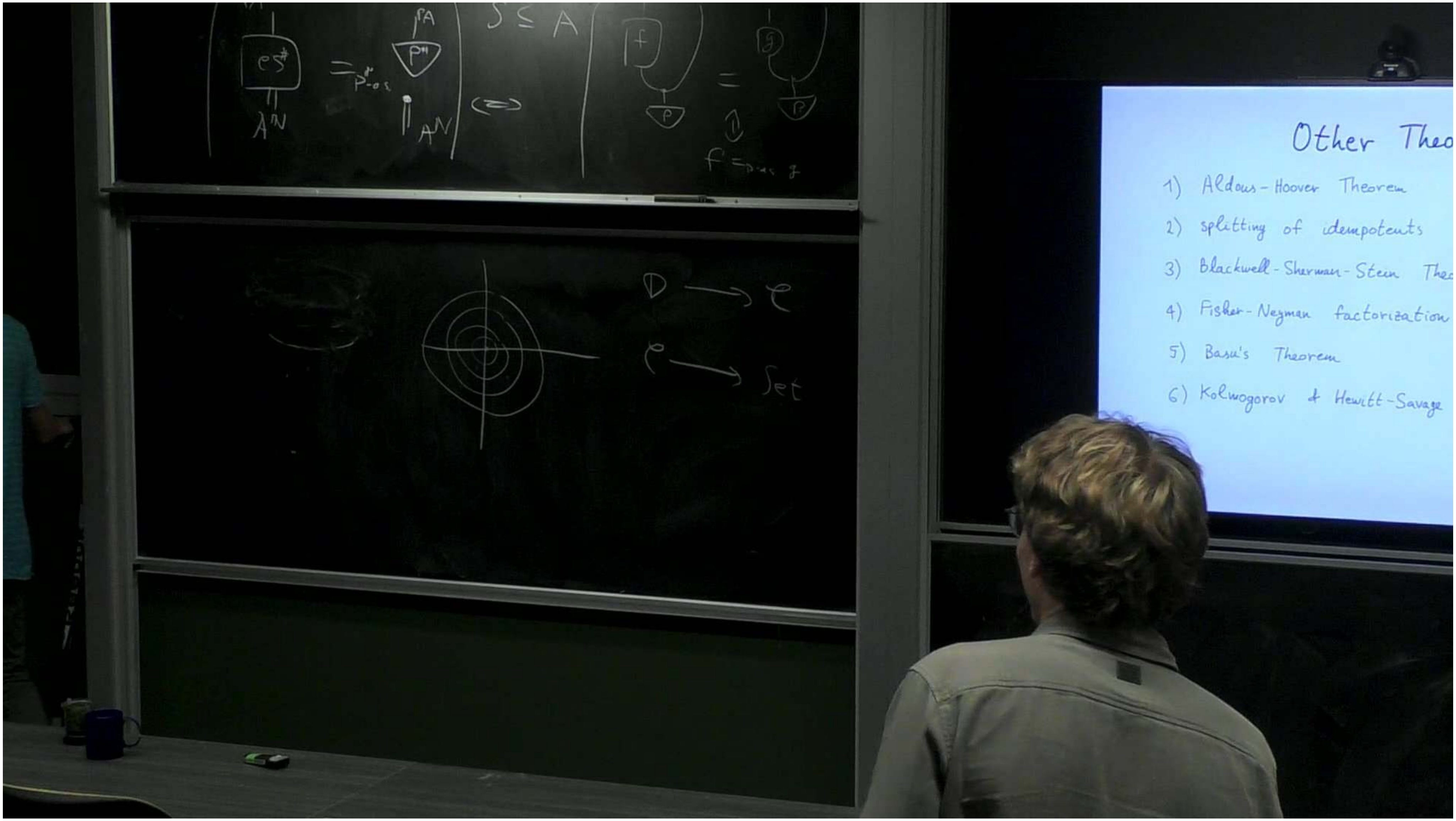
1) Aldous – Hoover Theorem
2) splitting of idempotents
3) Blackwell - Sherman - Stein Theorem
4) Fisher - Neyman factorization
5) Basu's Theorem
6) Kolmogorov & Hewitt-Savage zero/one laws


Kl(D_R)


X
f
A
in Partial(Borel Stoch)
f : Σ_X × S → [0,1]
S ⊆ A



in Partial(Borel Stoch)
f: Σx × S ⟶ [0,1)
S ≤ A
f
A
ℙA
es#
A^ℕ
p#
⟺
f =p-a.s. g


Other Theo
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