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Proof of a-theorem: On Renormalization Group Flows in Four Dimensions, Komargodski and Schwimmer, 2011

Luohan Wang

Oct 2025

Irreversibility of RG flow

- Zamolodchikov (1986) proved that, in 2D QFT, there is a function $c(g^i)$ of the coupling g^i, decreases monotonically along RG flow:

$$\frac{\mathrm{d}c}{\mathrm{d}\ln\mu} = \beta^i(g)\frac{\partial c(g)}{\partial g^i} \leq 0. \tag{1}$$

- This is **c-theorem** $\rightarrow$ RG flow is irreversible
- c is defined by

$$c = 2z^4\langle T(x)T(0)\rangle, \;\; \langle T^\mu_\mu\rangle = -\frac{c}{12}R \text{ is the trace anomaly in 2D} \tag{2}$$

- Is there a c-theorem in 4D? Cardy's (1988) conjecture of **a-theorem**

Proof of a-theorem

The idea of this paper:

1. Conformal symmetry of a CFT is spontaneously broken by the trace anomaly $\langle T^\mu_\mu \rangle \rightarrow$ there is a Goldstone boson: **the dilaton** τ.
2. The effective action of the dilaton encodes the anomaly difference between the UV and IR:
$$S[\tau] = (a_{\mathrm{UV}} - a_{\mathrm{IR}}) \int_x (\text{4pt vertex of } \tau) + \cdots \tag{3}$$
3. The positive definiteness of 2→2 dilaton scattering cross section gives $a_{\mathrm{UV}} - a_{\mathrm{IR}} > 0$

Trace anomaly of CFT

1. Classically, a theory $S[\phi, g]$ with traceless energy-momentum tensor is invariant under Weyl transformation $g_{\mu\nu} \to g_{\mu\nu} e^{-2\sigma}$:

$$\delta S \simeq \int_x \sqrt{-g}\; T_{\mu\nu} \delta g^{\mu\nu} \simeq \int_x \sqrt{-g}\; T^\mu_\mu \sigma = 0 \tag{4}$$

2. However, the corresponding effective action $W[g] = -i \ln Z$ may obtain a nonzero $\langle T^\mu_\mu \rangle$. This is the trace anomaly as a quantum effect.

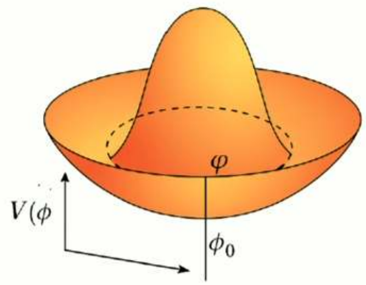
3. In 4D curved space,

$$\langle T^\mu_\mu \rangle \sim (c W_{abcd} W^{abcd} - a E_4 + b \Box R) \tag{5}$$

4. SSB of conformal symmetry

SSB and Goldstone bosons: Mexican hat

Spontaneous Symmetry Breaking



$V(\phi) = -\mu^2\phi^2 + \lambda\phi^4$

Mexican hat potential

$$\Phi(x) = \frac{1}{\sqrt{2}}(\rho(x) + v)e^{i\pi(x)/v}, \quad (6)$$

with the potential

$$V(x) = \lambda(|\Phi|^2 - v^2/2)^2. \quad (7)$$

The full theory is invariant under Global U(1), but the vacuum $\langle\Phi\rangle = v/2$ chooses a specific phase when $v \neq 0$.
So the symmetry is spontaneously broken: the Lagrangian is invariant, but the vacuum isn't.
Under global U(1), $\Phi \to e^{i\alpha}\Phi$, $\pi \to \pi + \alpha/v$ to compensate for the broken U(1).

Wess-Zumino's trick in a-theorem

1. If we start from an original CFT that is invariant under Weyl transform, but its trace anomaly in the VEV $\langle T^\mu_\mu \rangle$ breaks the conformal symmetry spontaneously
2. There is a Goldstone boson of this SSB, namely the dilaton τ, whose variation of effective action under $g_{\mu\nu} \to g_{\mu\nu} e^{-2\sigma}$ reproduces the variation of action due to trace anomaly

$$\delta_\sigma S_{\text{anomaly}} = \int \mathrm{d}^4 x \sqrt{-g}\sigma(cW_{abcd}W^{abcd} - aE_4 + b'\Box R) = \delta_\sigma S_{eff}[\tau] \tag{8}$$

3. Along the RG flow: $CFT_{UV} \to CFT_{IR} + dilatons$

Theory of dilatons

Under Weyl transformation: $g_{\mu\nu} \to g_{\mu\nu} e^{-2\sigma}$, $\tau \to \tau + \sigma$, so that $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$ is Weyl invariant. The kinetic term of dilaton is obtained by constructing a Weyl invariant theory using $\hat{g}_{\mu\nu}$ up to two derivatives:

$$S_{kin} = f^2 \int d^4x \sqrt{-\hat{g}} \hat{R} \overset{g_{\mu\nu}=\eta_{\mu\nu}}{=} f^2 \int_x e^{-2\tau} (\partial\tau)^2. \tag{9}$$

There are also conformal invariant interaction terms constructed from $\hat{R}^2, \hat{E}_4, \hat{W}_{abcd}\hat{W}^{abcd}$. But none of them contributes to the leading order 2 to 2 dilaton scattering.

The anomaly term of τ, by using Eq. (8), is

$$\begin{aligned} S_{anomaly} = & -a \int_x \sqrt{-g} (\tau E_4 + 4 G^{\mu\nu} \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box \tau + 2(\partial\tau)^4) \\ & + c \int_x \sqrt{-g} \tau W_{abcd} W^{abcd} \end{aligned} \tag{10}$$

Matching anomaly

The anomaly is constructed by a local functional of curvature invariants.
$\rightarrow$ total anomaly is determined only by UV physics and does not change from UV to IR. (t'Hooft, 1980).
This does not mean the coefficients a, c in the trace anomaly do not change from UV to IR. It means that their changes must be compensated for by the dilaton:

$$
\begin{aligned}
S_{IR}[g] &= \mathrm{CFT}_{IR}[g] + \frac{1}{6} f^2 \int_x \sqrt{-\hat{g}}\hat{R} \\
&- (a_{UV} - a'_{IR}) \int_x \sqrt{-g}(\tau E_4 + 4G^{\mu\nu}\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\tau + 2(\partial\tau)^4) \\
&+ (\text{Other interaction terms}), \qquad (11)
\end{aligned}
$$

$a'_{IR} = a_{IR} + a_{scalar}$, where a_{scalar} is the anomaly contributed by the dilaton.

$2 \to 2$ dilaton scattering

In flat space, the IR action becomes

$$S_{IR} = \mathrm{CFT}_{IR} + \int_x \left(f^2 e^{-2\tau}(\partial\tau)^2 + (a_{UV} - a'_{IR})(4(\partial\tau)^2\Box\tau - 2(\partial\tau)^4)\right) + (\text{other terms}), \tag{12}$$

so the leading order of $2 \to 2$ dilaton scattering only depends on the $a_{UV} - a'_{IR}$ term:

$$\mathcal{A}(s,t) = \frac{a_{UV} - a'_{IR}}{f^4}(s^2 + t^2 + u^2) + \cdots \overset{\text{on shell}}{=} \frac{2(a_{UV} - a'_{IR})}{f^4} s^2 + O(s^4), \tag{13}$$

so

$$a_{UV} - a'_{IR} = \frac{f^4}{\pi}\int_{s'>0} \mathrm{d}s' \frac{Im\mathcal{A}}{s'^3} \xrightarrow{\text{optical theorem}}$$
$$a'(\mu) = a_{UV} - \frac{f^4}{\pi}\int_{s'>\mu} \mathrm{d}s' \frac{\sigma(s')}{s^2} < a_{UV} \tag{14}$$

Deformed CFT$_{UV}$

If the CFT$_{UV}$ breaks conformal symmetry explicitly by adding relevant operators $\lambda_i O_i$ to the UV CFT, where $[O_i] = \Delta_i < 4$. (Operatorial anomaly)

$$S_{matter} = S_{matter}[\Phi_i, M_i] \tag{15}$$

Prescription: Remove operatorial anomaly by coupling dilatons to the matter field: $\Omega_i = e^{-\tau}$, $M_i \to M_i\Omega_i$, and add a dilaton kinetic term, so that the theory becomes Weyl invariant

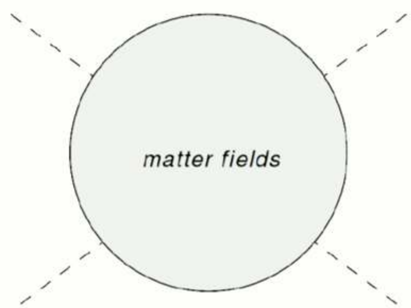
$$S' = S_{matter}[\Phi_i, M_i\Omega_i] + f^2 \int_x (\partial\Omega)^2. \tag{16}$$

Now S' is free of operatorial anomaly and has $T^\mu_\mu = 0$.

Matching anomaly in deformed CFT

Set $M_i << f$, so that the coupling of dilatons to the matter field is **arbitrarily weak**.
$\rightarrow$ Thus, we recover the original theory that is only perturbed by an infinitesimal coupling to the dilaton field. $(M_i/f \rightarrow 0)$
Then everything becomes the same. We obtain the same $S_{IR}[g]$ as before.
Note that

1. a'_{IR} is replaced by a_{IR}, because now we also have dilatons in UV theory
2. $(a_{UV} - a_{IR})$ is the coefficient after integrating out the matter field:



Refs.

① Rychkov : 1505.00967

② Melo : 1905.11099

$$S = \int d^d x \; \frac{1}{2}(\partial\varphi)^2 + \frac{g}{4!}\mu^{\epsilon}\varphi^4$$

$d = 4-\epsilon$

$[g] = 0$

$[\varphi] = \frac{d-2}{2}$

$\epsilon = 0, \quad \varphi^n$

$\epsilon \neq 0, \quad V_n$

$[V_n] = n\delta + \gamma_n$

classical exp.

Anomal. dim.

$S^{\text{eff}}_{\Lambda}(\phi) \quad \in \leq$

momentum space RG.

① Coarse graining: $\phi = \phi^{+} + \phi^{-}$

$$Z_{\Lambda_0} = \int D\phi^{+} D\phi^{-}\, e^{-S_{\Lambda_0}(\phi^{+}+\phi^{-})} \overset{!}{=} \int D\phi^{-}\, e^{-S^{\text{eff}}_{\Lambda}(\phi^{-})}$$

$$e^{-S^{\text{eff}}_{\Lambda}(\phi^{-})} = \int D\phi^{+}\, e^{-S_{\Lambda_0}(\phi^{+}+\phi^{-})}$$

Output:

$\Rightarrow \; S^{\text{eff}}_{\Lambda}(\phi^{-})$

e.g.: $S_{\Lambda_0}(\phi) = \int d^dx \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda_4}{4!}\varphi^4 \right)$

$S_{\Lambda}(\phi^{-}) = \Big(Z'_{\phi} \qquad m \qquad \lambda(\Lambda)$

Output:

$\Longrightarrow S^{eff}_{\Lambda}(\phi^-)$

e.g: $S_{\Lambda_0}(\phi) = \int d^dx \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda_4}{4!}\varphi^4 \right)$

$S_\Lambda(\phi^-) = \quad Z'_\phi \qquad m(\Lambda) \qquad \lambda(\Lambda)$

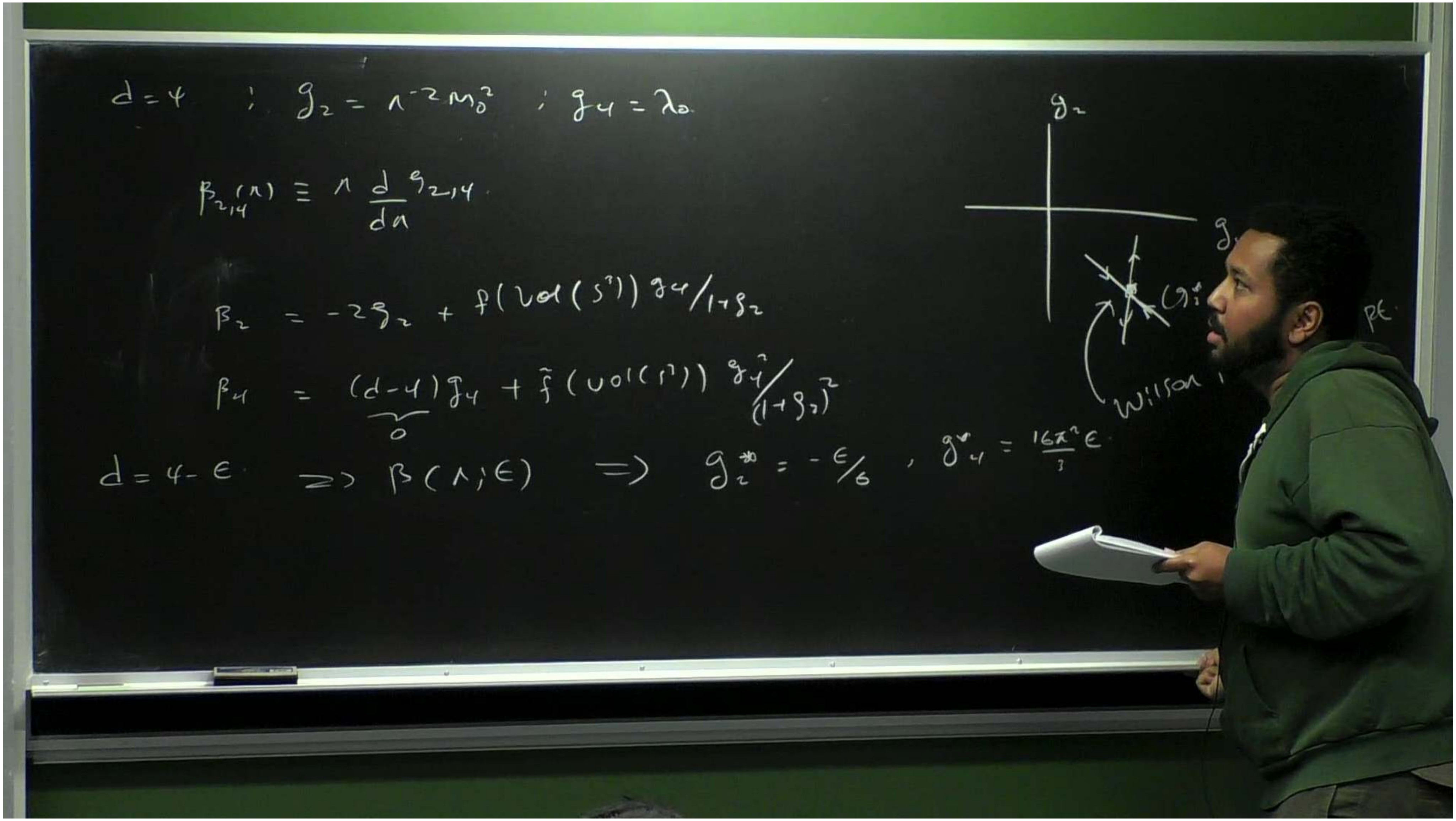
③ fix scaling units of φ^-

$\phi' = \sqrt{Z_\phi}\,\phi^-$ $\qquad \Longrightarrow$ k. term is $\frac{1}{2}(\partial\phi')^2$

$\left(\frac{\Lambda_0}{\Lambda}\right)^{d-2}$ $\qquad \gamma_\phi$ an. dim of ϕ.

expand $\Lambda \simeq \Lambda_0$

$$\phi' = \left(1 + \left(\frac{d-2}{2} \overbrace{- \frac{1}{2}\Lambda \frac{d \log Z'_\phi(\Lambda)}{d\Lambda}\Big|_{\Lambda=\Lambda_0}}^{|||}\right)\left(1 - \Lambda/\Lambda_0\right) + \cdots\right)\phi$$


d = 4 ; g_2 = Λ^-2 m_0^2 ; g_4 = λ_0
β_2,4(Λ) ≡ Λ d/dΛ g_2,4
β_2 = -2g_2 + f(vol(S^3)) g_4/1+g_2
β_4 = (d-4) g_4 + f̃(vol(S^3)) g_4^2/(1+g_2)^2
d = 4 - ε => β(Λ;ε) => g_2^* = -ε/6 , g_4^* = 16π^2 ε/3
g_2
Wilson

$\xrightarrow{\quad} 2-\frac{\epsilon}{2}=\delta$

$\epsilon \neq 0$; V_n Primaries $[V] = n\delta + \gamma_n$.

$\gamma_n(\epsilon) = \underline{\underline{\gamma_{1,n}}}\,\epsilon + \underline{\underline{\gamma_{2,n}}}\,\epsilon^2 + O(\epsilon^2)$ ① ?

WF f.pt is a CFT ② ✓ (PT)

Multiplet Recomb. ③ ? Crucial

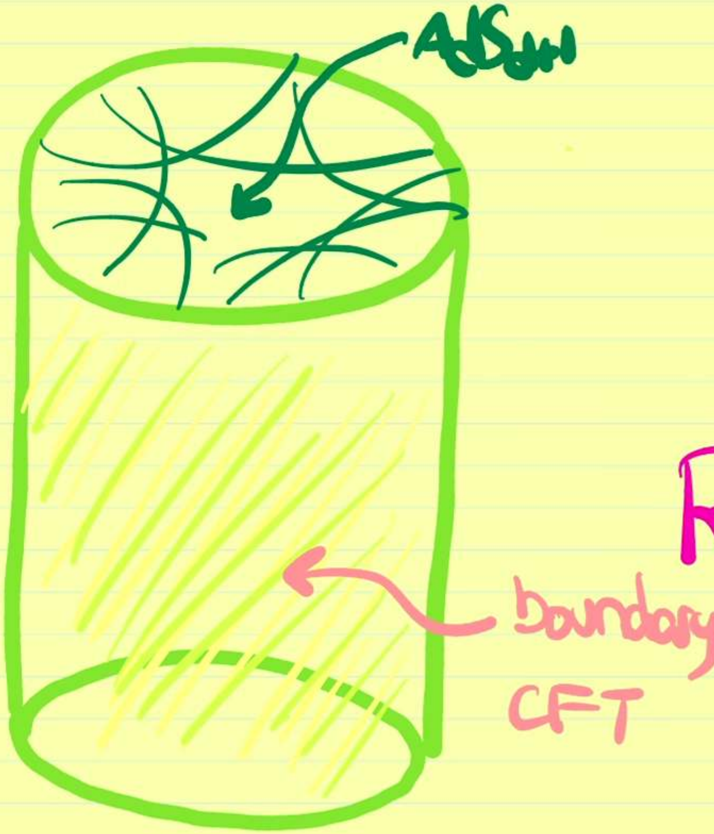
Assumptions.

EoM: $\Box\varphi = \frac{1}{3!} g_* \mu^{\epsilon} \varphi^3$

$\Longrightarrow \Box V_1 = \alpha V_3$.

Holographic & Wilsonian RG flows

I. Heemskerk, J. Polchinski 6 oct 2010



what is the holographic dual of Wilsonian Renormalization?

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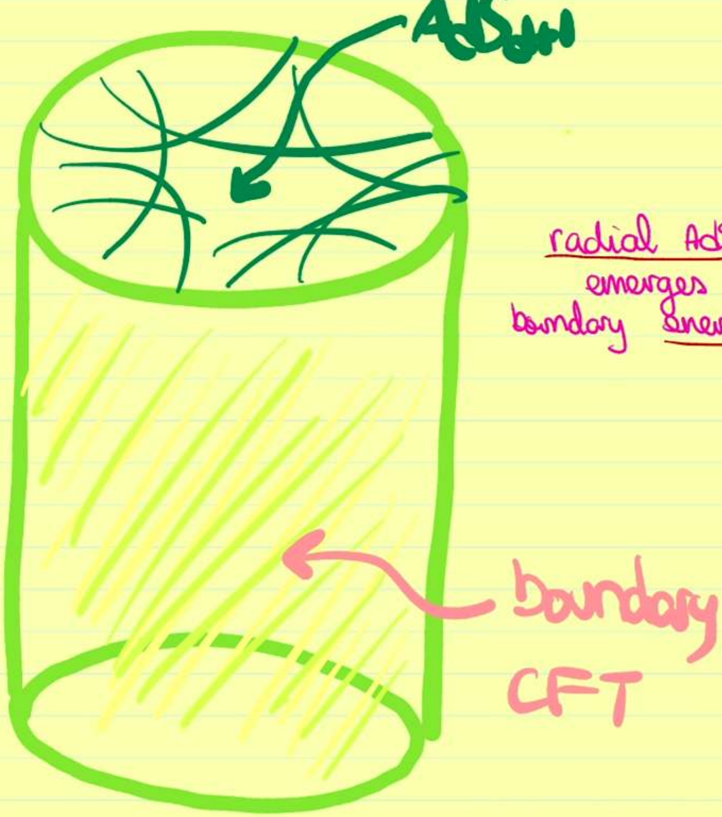
Where/what is Renormalization in the AdS/CFT dictionary?



earlier work:

radial AdS coord emerges from boundary energy scale

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[6] L. Susskind and E. Witten, "The holographic bound in anti-de Sitter space," arXiv:hep-th/9805114.

Holographic Renormalization Group

[28] J. de Boer, E. P. Verlinde and H. L. Verlinde, "On the holographic renormalization group," JHEP **0008**, 003 (2000) [arXiv:hep-th/9912012];

J. de Boer, "The holographic renormalization group," Fortsch. Phys. **49**, 339 (2001) [arXiv:hep-th/0101026];

E. P. Verlinde and H. L. Verlinde, "RG-flow, gravity and the cosmological constant," JHEP **0005**, 034 (2000) [arXiv:hep-th/9912018].

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Usual Holographic Renormalization story:

correlation functions

$$Z[\phi_{(0)}] = e^{-W[\phi_{(0)}]} = \Big\langle \exp \int d^dx \, \underbrace{\phi_{(0)}}_{\text{source}} \, \mathcal{O}(x) \Big\rangle_{CFT_d}$$

Boundary cond. = sources

$$\text{AdS/CFT}: \quad W[\phi_0] = S^{\text{on-shell}}_{\text{SUGRA}}[\phi] \Big|_{\boxed{\phi(z,x)\cdot z^{\Delta-d} \xrightarrow[z\to 0]{} \phi_{(0)}}}$$

IR divergence

Problem: $S^{\text{on-shell}}_{\text{SUGRA}}$ is infinite! $\text{vol}(\text{AdS}) = \infty$

Cancel the divergencies by cutting AdS at $z=\varepsilon$

$$ds^2_{AdS_{d+1}} = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right)$$ and adding

Usual Holographic Renormalization story:

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Boundary cond. = sources

AdS/CFT : $\quad W[\phi_0] = S^{\text{on-shell}}_{SUGRA}[\phi] \Big|_{\boxed{\phi(z,x)\cdot z^{\Delta-d} \xrightarrow[z\to 0]{} \phi_{(0)}}}$

Problem: $S^{\text{on-shell}}_{SUGRA}$ is infinite! $\quad \underbrace{\mathrm{vol}(AdS) = \infty}_{\text{IR divergence}}$

Cancel the divergencies by cutting AdS at $z = \varepsilon$

$ds^2_{AdS_{d+1}} = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}\, dx^\mu dx^\nu\right)$ and adding counterterms.

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Previous ideas

- IR cut-off in bulk ~ UV cut-off in bdry.
- Study $S^{\text{on-shell}}_{\text{gravity}}[\varepsilon]$ relate it to renormalization in the boundary
- Holographic dual (geometries) of boundary flows

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(Note : not answered)

Today

Why? How can we derive this?

- IR cut-off in bulk ~ UV cut-off in bdry.

- Study $S^{\text{on-shell}}_{\text{gravity}}[\varepsilon]$ relate it to renormalization in the boundary.

Imitating Wilson's idea : Integrate out dof near the boundary and define a QFT with UV cut-off.

Integrating bulk fields in the region $z < \ell$ ⟷ Integrating CFT modes up to energy ~ $1/\ell$

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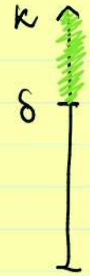
Hamilton-Jacobi ⟷ Polchinski's RG eqn



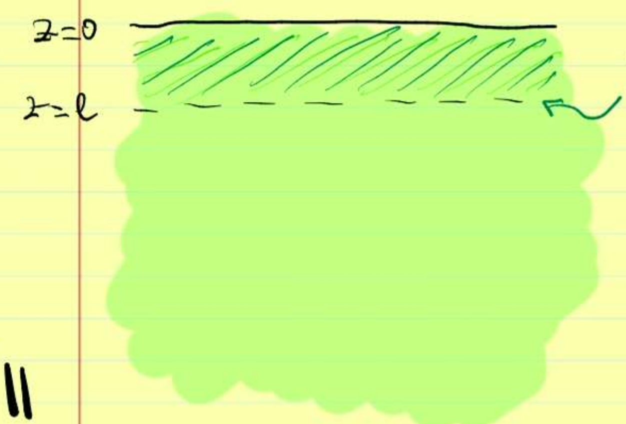
Wilson says

$$Z = \int DM \, e^{-S} = \int DM_{k\delta<1} \, DM_{k\delta>1} \, e^{-S} = \int DM_{k\delta<1} \, e^{-S(\delta)}$$

Apply it to gravity : (1) Fixed background

Asymptotically AdS_{d+1} : $ds^2 = n^2(z,x)\,dz^2 + h_{\mu\nu}(z,x)\,dx^\mu dx^\nu$

$$n \xrightarrow[z\to 0]{} \frac{L}{z} \qquad h_{\mu\nu} \xrightarrow[z\to 0]{} \frac{L}{z}\,\delta_{\mu\nu}$$



$\tilde{\phi}$

$$Z = \int D\phi \, e^{-k^2 S} = \int D\phi|_{z>\ell} \, D\tilde{\phi} \, D\phi|_{z<\ell} \, e^{-k^2 S|_{z>\ell} - k^2 S|_{z<\ell}} =$$

$$= \int D\tilde{\phi} \; \Psi_{IR}(\ell,\tilde{\phi}) \; \Psi_{UV}(\ell,\tilde{\phi})$$

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- what is S_0? } Not answered
- what is δ exactly? ⇝ there are estimates

Starting point:

$$\Psi_{IR}(\ell, \vec{\phi}) = \int DM\Big|_{\kappa \delta < 1} \exp\left\{ -S_0 + \frac{1}{\kappa^2}\int d^d x\, \tilde{\phi}^i\, \mathcal{O}_i \right\} \tag{2.3}$$

Integrating bulk fields in the region $z < \ell$ $\longleftrightarrow$ Integrating CFT modes up to energy $\sim 1/\ell$

However, Ψ_{UV} is not a Wilsonian eff. action

multi-trace even if Ψ_{UV} only has single trace

$$e^{-\kappa^2 S(\delta)} = \int D\tilde{\phi}\, \exp\left\{ \frac{1}{\kappa^2}\int d^d x\, \tilde{\phi}^i \mathcal{O}_i \right\} \Psi_{UV}(\delta, \tilde{\phi})$$

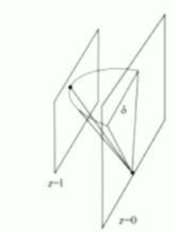

Figure 1: The scale δ on which the Wilsonian action is localized.

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The partition function does not depend on the choice of ℓ!

$$0 = \frac{d}{d\ell} Z = \frac{d}{d\ell} \langle e^{-\kappa^2 S(\delta)} \rangle_\delta$$

Ψ_{UV}, Ψ_{IR} obey

$$\begin{cases} \kappa^2 \partial_\ell \Psi_{IR}(\ell, \tilde{\phi}) = H(\tilde{\phi}, \tilde{\pi}) \Psi_{IR}(\ell, \tilde{\phi}) \\ \kappa^2 \partial_\ell \Psi_{UV}(\ell, \tilde{\phi}) = -H(\tilde{\phi}, \tilde{\pi}) \Psi_{UV}(\ell, \tilde{\phi}) \end{cases}$$

(Hamiltonian formalism thinking of z as time)

MAIN RESULT OF THE PAPER : $\kappa^2 \partial_\ell e^{-\kappa^{-2} S(\delta)} = \kappa^2 \partial_\ell \delta \, \partial_\delta e^{-\kappa^{-2} S(\delta)} = -H\left(\kappa^2 \frac{\delta}{\delta\theta}, i\theta\right) e^{-\kappa^{-2} S(\delta)}$ (2.15)

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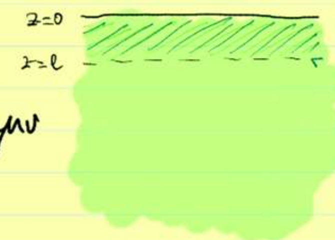
$$\kappa^2 \partial_\ell e^{-\kappa^{-2}S(\delta)} = \kappa^2 \partial_\ell \delta\, \partial_\delta e^{-\kappa^{-2}S(\delta)} = -H\left(\kappa^2 \frac{\delta}{\delta\theta}, i\theta\right) e^{-\kappa^{-2}S(\delta)}$$

$$\Big\downarrow \quad \kappa^2 \to 0$$

Wilsonian eff. action $\rightsquigarrow$ $\delta_\ell S(\delta) = H\left(-\frac{\delta S(\delta)}{\delta\theta}, i\theta\right)$ Hamilton-Jacobi eqn

radially evolves wrt H

(2) Dynamical background



Use Fefferman-Graham and integrate out $h_{\mu\nu}$

- This approach is natural in renorm.
  but isn't gauge invariant

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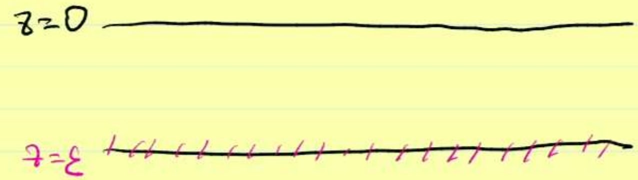
Not today 😢 but in the paper 😊

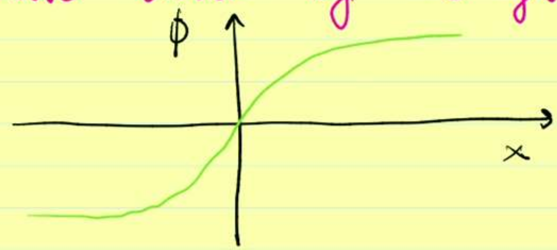
- **Examples :**
  - Scalar Field in AdS: YOU CAN'T IGNORE DOUBLE TRACE OPS !
    (no backreaction)
  - Gauge fields in AdS
  - Domain wall flow
  - Backreaction
- Comparison with the literature
  - Projections : (do we include all states ?)
    stringy states ?
  - Holographic RG flows
- 10/11 Wilsonian renormalization in CFT with matrices ~ more on multi-traces ...

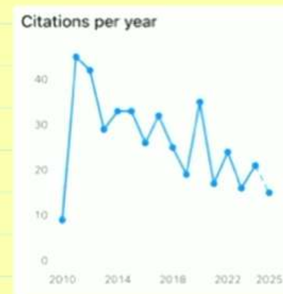
In 2025.

- Braneworld holography



- Holographic dual of RG flows





- How to choose a physical cut-off?
  ex: when insert nonlocal ops

- Symmetries in AdS/CFT

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