Title: Lecture - Statistical Physics (Core), PHYS 602

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Subject: Condensed Matter, Other

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Recap: RG (Gaussian model)

$$S_0[\phi] = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 \right) \quad (u=0)$$

$$= \int d\bar{k}\ \phi_{-k}(k^2+r)\phi_k$$

a) Coarse-grain: $S[\phi] = S_0[\phi_<] + S_0[\phi_>] + \underbrace{S_{int}[\phi_<, \phi_>]}_{0}$

(no explicit integration needed)

b) Rescal

for any op

(this resu

b) Rescale. $x \to \frac{x}{b}$ (or $k \to kb$)

for any operator $\mathcal{O}$, $\mathcal{O} \to \mathcal{O} b^{\Delta_{\mathcal{O}}}$ s.t. $\mathcal{L}$ is invariant

(this results in a rescaling of coupling constants)

– Consider the term $\int d^d x \, \frac{1}{2} (\nabla\phi)^2 \xrightarrow{\text{rescaling}} \int \frac{d^d x}{b^d} \, b^2 (\nabla\phi)^2 \, b^{2\Delta_\phi}$; if action is invariant

then $b^{2-d+2\Delta_\phi} = 1$

$$\Rightarrow \Delta_\phi = \frac{d-2}{2}$$

- Consider $\int d^dx \left(\frac{1}{2} r \phi^2\right) \longrightarrow \int \frac{d^dx}{b^d}\left(\phi^2 b^{d-2} r\right)$

$$= \int r b^{-2} d^dx \, \phi^2.$$

For action to be invariant, we must also rescale $r \longrightarrow r b^2$ to absorb the rescaling of ϕ.

$$\Longrightarrow \tilde{r} = r b^2 \,\Big]$$

r grows under RG!

Rescaling interaction term:

$$\int d^d x \, \frac{u}{4!} \phi^4 \longrightarrow \int \frac{d^d x}{b^d} (u) \phi^4 b^{2(d-2)}$$

$$= u \, b^{d-4} \int d^d x \, \phi^4.$$

We need $\tilde{u} = u \, b^{4-d}$

$$\begin{cases} d > 4: & u \text{ \underline{decreases} under rescaling} \quad \text{irrelevant} \\ d < 4: & \underline{u \text{ increases}} \\ d = 4: & u \text{ does not change} \end{cases}$$

A coupling constant g is

$$\begin{cases} \underline{\text{relevant}}, \text{ if } g \longrightarrow g\, b^{\Delta_g}, & \Delta_g > 0 \\ \underline{\text{irrelevant}}, \text{ if} & \Delta_g < 0 \\ \underline{\text{marginal}}, \text{ if} & \Delta_g = 0 . \end{cases}$$

RG fixed points (Gaussian model)

We found

$$\tilde{r} = b^2 r$$

write $b = e^{\ell}$

$$r(\ell) = e^{2\ell} r(0)$$

$$\frac{dr(\ell)}{d\ell} = 2r(\ell)$$

we get $\boxed{\frac{dr}{d\ell} = 2r}$

fixed point corresponds to $\frac{dr}{d\ell} = 0$

$\Longrightarrow \boxed{r_* = 0 \;(\text{and } u_* = 0)}$ "Gaussian fixed point"

* Correlation length $\xi \to \xi/b$ under RG $\left(\xi = \sqrt{\frac{1}{r}} \right.$

$\Rightarrow$ at fixed point $\xi = 0$ or ∞.

$\xi = \infty$: critical point $(T = T_c)$

$\xi = 0$: corresponds to $r = \infty$. (one phase of the model)

* RG is not really a 'group'

Perturbative RG (Wilsonian RG)

$$Z = \int \mathcal{D}\phi_< \, e^{-S_0[\phi_<]} \int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]} \, e^{-S_{int}[\phi_<, \phi_>]}$$

$S_0[\phi_>]$ → non-interacting action for fast modes

$$= \int \mathcal{D}\phi_< \, e^{-S_0[\phi_<]} \underbrace{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]}}_{Z_0} \quad \frac{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>] - S_{int}[\phi_<, \phi_>]}}{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]}}$$

$$\Rightarrow \boxed{g_* = 0 \ (\text{and } \ldots_*)}$$

$2r(\lambda)$

$$u\phi^4 = u(\phi_< + \phi_>)^4 = u(\underbrace{\phi_>^4} + \phi_< \phi^3 + \cdots)$$

fast modes

$$\ldots, \phi_>\} \Big\} \left\langle e^{-S_{int}} \right\rangle_{fast}$$

$$S_0 \approx \int d^dx \, \tfrac{1}{2}\left((\nabla\phi)^2 + r\phi^2\right) + B(x)\phi(x)$$

Defining $Z = \int \mathcal{D}\phi_< \, e^{-\tilde{S}[\phi_<]}$, we see that

$\langle e^{-\Omega} \rangle \equiv \int dx$

$$\tilde{S}[\phi_<] = S_0[\phi_<] - \underbrace{\log Z_0}_{\text{ignore}} - \log\left(\langle e^{-S_{int}} \rangle_{fast} \right)$$

Last term is simplified by cumulant expansion :

Ω = prob. distribution

$$\langle e^{-\Omega} \rangle \equiv \int \underset{\downarrow}{dx}\ \underset{\downarrow}{e^{-\Omega(x)}}\ \underbrace{p(x)}_{e^{-S_0[\phi_>]}}$$

$$\mathcal{D}\phi_> \qquad S_{int}$$

$$= e^{-\langle \Omega \rangle + \frac{1}{2}\left(\langle \Omega^2 \rangle - \langle \Omega \rangle^2 \right) + \cdots}$$

$$\Rightarrow \tilde{S}[\phi_<] = S_0[\phi_<] - \log Z_0 + \langle S_{int} \rangle_{fast} - \frac{1}{2}\left(\langle S_{int}^2 \rangle_{fast} - \langle S_{int} \rangle_{fast}^2 \right) + \cdots$$

Result

$$\tilde{S}[\phi_<] = S_0[\phi_<] + \langle S_{int} \rangle_f - \frac{1}{2}\left(\underbrace{\langle S_{int}^2 \rangle_f - \langle S_{int} \rangle_f^2}_{} \right) + \cdots$$

$u\phi^4$

$\parallel$

$u\phi_<\phi_<\phi_>\phi_> + \cdots$

$$u^2 \underbrace{\phi_<\phi_<\phi_<\phi_<}_{} \underbrace{\phi_>\phi_>\phi_>\phi_>}_{}$$

b) Rescal

for any op

(this resu

– Consider th

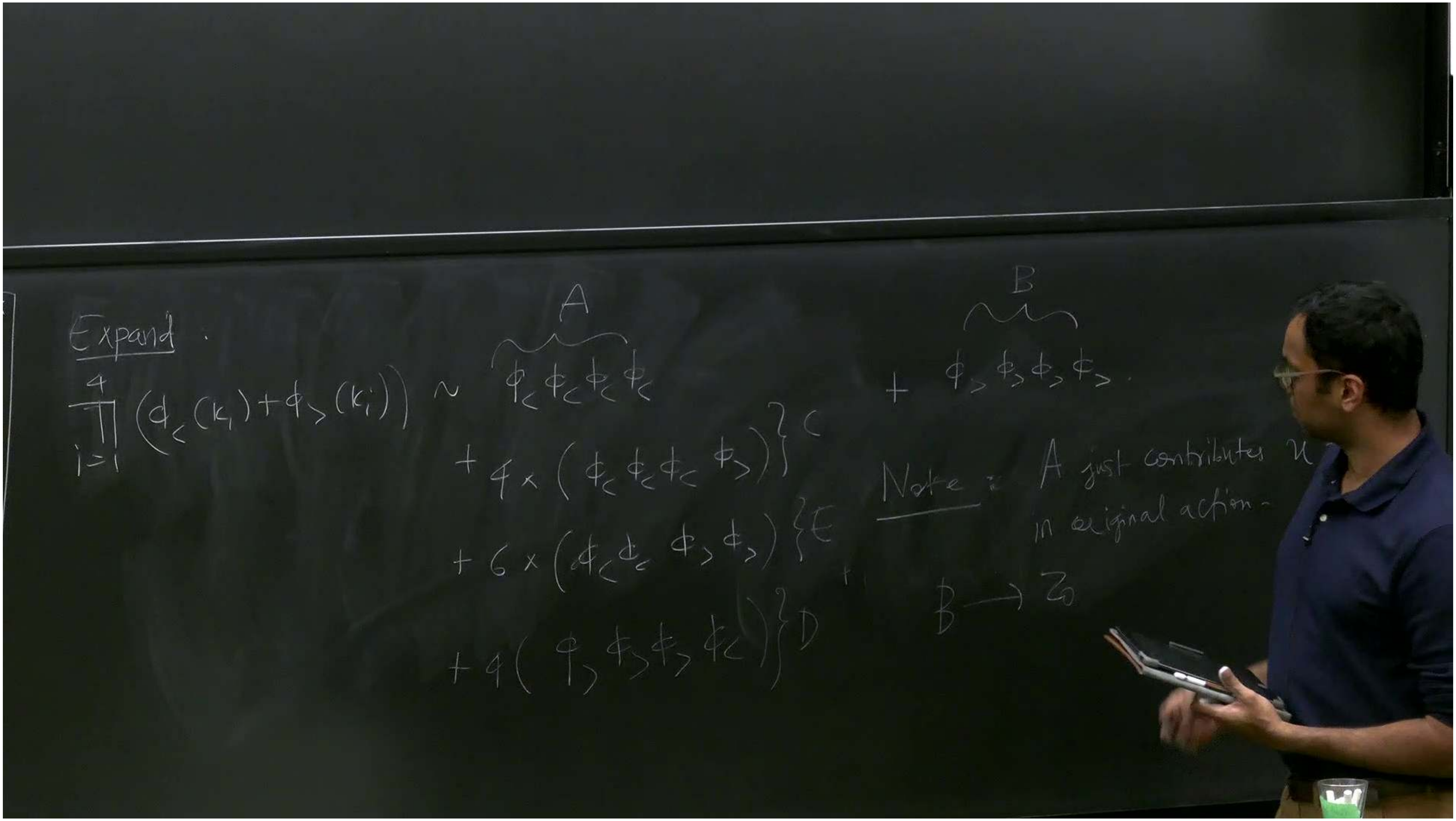
Note :

$$S_{int} = \frac{u}{4!} \int d^d x \, \phi(x)^4$$

(1) $\left\{ \phi_k = \int d\bar{k} \, \phi(x) e^{i\vec{k}\cdot\vec{x}} \right.$

$$= \frac{u}{4!} \int_0^\Lambda \left[\prod_{i=1}^{4} d\vec{k}_i \left(\phi_< (k_i) + \phi_> (k_i) \right) \right] \underbrace{(2\pi)^d \delta(k_1+k_2+k_3+k_4)}_{x\text{-integration}}$$

(2)


Expand :
$\prod_{i=1}^{4} \left(\phi_<(k_i) + \phi_>(k_i) \right) \sim \underbrace{\phi_< \phi_< \phi_< \phi_<}_{A} + \underbrace{\phi_> \phi_> \phi_> \phi_>}_{B}$
$+ 4 \times (\phi_< \phi_< \phi_< \phi_>) \} C$
$+ 6 \times (\phi_< \phi_< \phi_> \phi_>) \} E$
$+ 4 (\phi_> \phi_> \phi_> \phi_<) \} D$
Note : A just contributes u
in original action.
$B \longrightarrow Z_0$

- Consider the term $\int d^d x \, \frac{1}{2} (\nabla\phi)^2 \xrightarrow{\text{rescaling}} \int \frac{d^d x}{b^d}$ ... invariant

then $b^{2-d+2\Delta_\phi} = 1$

$\Rightarrow \Delta_\phi = \frac{d-2}{2}$

We're left with

$$\langle S_{int} \rangle_f = \int 6 \frac{u}{4!} \, \phi_<(k_1)\phi_<(k_2) \, \delta^d(k_1+k_2+k_3+k_4)(2\pi)^d \int \langle \phi_>(k_3)\phi_>(k_4) \rangle_f \, d\bar{k}_3 \, d\bar{k}_4 \quad (3)$$
$d\bar{k}_1 \, d\bar{k}_2$

Recall from Lec. 6 (or look up App B of DK's notes):

$$\langle \phi_>(k_3)\phi_>(k_4) \rangle_f = \frac{(2\pi)^d \delta(k_3+k_4)}{k_3^2 + r}$$

$$= \int d\bar{k}_1 \frac{6u}{4!} \phi_<(k_1)\phi_<(-k_1) \times I_1 \quad (4), \qquad \boxed{I_1 = \int_{\Lambda/b}^{\Lambda} d\bar{k}_3 \frac{1}{k_3^2 + r}}$$

after 1st order PT in S_{int},

$$\tilde{S}[\phi_<] = \int d\bar{k} \left\{ \left(\frac{k^2+r}{2}\right) |\phi_<(k)|^2 + \frac{u}{4} I_1 |\phi_<(k)|^2 + \frac{u}{4!} \phi_<^4 \right\}$$

$\langle S_{int}$

I_1

Step (a) took

$$r \longrightarrow r + \frac{u}{2} I_1$$

$$u \longrightarrow u$$

Step (b): Rescale.

we know

$$r \rightarrow b^2 r$$

$$u \rightarrow b^{4-d} u$$

$$I_1 \rightarrow I_1 b^{d-2}$$

Step (b) takes

$$u \rightarrow b^{4-d} u$$

$$r \rightarrow b^2 r + \frac{1}{2} u b^{4-d} I_1 b^{d-2}$$

$$= b^2 \left(r + \frac{u}{2} I_1 \right)$$

$$\langle S_{int} \rangle =$$

$$I_1 = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+r} = \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} dq \frac{q^{d-1}}{q^2+r}$$

$$= \frac{S_d}{(2\pi)^d} \frac{\Lambda^{d-1}}{\Lambda^2+r} \left(\Lambda - \frac{\Lambda}{b}\right) = \frac{S_d}{(2\pi)^d} \frac{\Lambda^d (1-1/b)}{\Lambda^2+r}$$

$4-d$ u

RG (1st order) gives

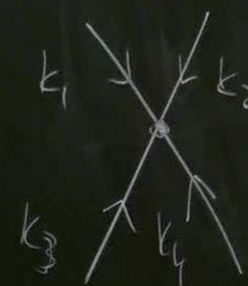
$$\boxed{\begin{aligned} r &\longrightarrow b^2\left(r + \frac{u}{2} I_1\right) \\ u &\longrightarrow b^{4-d}\, u \end{aligned}}$$

this is not so interesting as it forces $u \to 0$ or ∞ (unphysical)

$\Longrightarrow$ We need to do 2nd order PT!

Need to evaluate $-\frac{1}{2}\left(\langle S_{int}^2 \rangle_f - \langle S_{int} \rangle_f^2 \right)$

To make things easier, use diagrammatic notation



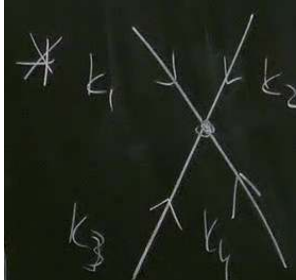
$$\equiv \frac{u}{4!} (2\pi)^d\, \delta(k_1 + k_2 + k_3 + k_4)$$

$4!$ $\quad I_1 = \int_{\Lambda/b} \ldots \frac{}{k_3^2 +}$

$\Rightarrow$ We need to do 2^{nd} order PT!

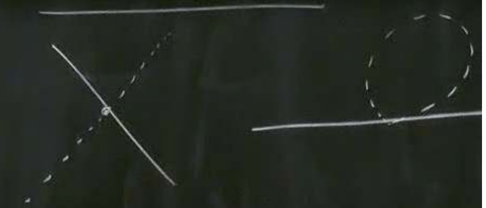
Need to evaluate $-\frac{1}{2}\left(\langle S_{int}^2 \rangle_f - \langle S_{int} \rangle_f^2 \right)$

To make things easier, use diagrammatic notation :



$* \quad \equiv \frac{u}{4!}(2\pi)^d \, \delta(k_1+k_2+k_3+k_4)$; ——— : slow ; - - - - - - : fast

Internal lines



$1/b$

e PT!

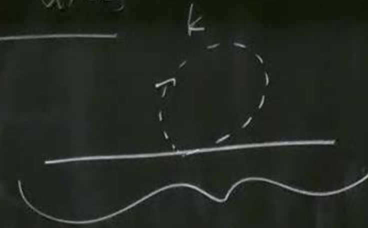
$\langle S_{int}^2 \rangle_f - \langle S_{int} \rangle_f^2$)

grammatic notation :

$(k_1+k_2+k_3+k_4)$; ———— : slow

- - - - - - - : fast

* Internal lines



$\equiv \langle \phi_k \phi_{-k} \rangle_f$

* loop gives extra $\int \frac{d^d q}{(2\pi)^d}$

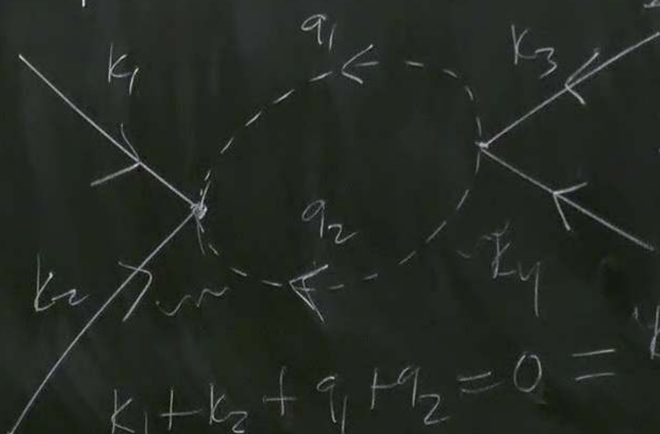
Correction to η from $\langle S_{int}^2 \rangle$:

8 total lines, need to integrate out fast modes

& get $4 \times$ —— $\Rightarrow$ start with $4 \times$ ——

$4 \times$ - - - -

$\times\, 6^2$



$$k_1 + k_2 + q_1 + q_2 = 0 = k_3 + k_4 - q_1 - q_2.$$

$+ 4(\phi, \phi, \phi, \phi_c)\}^D$

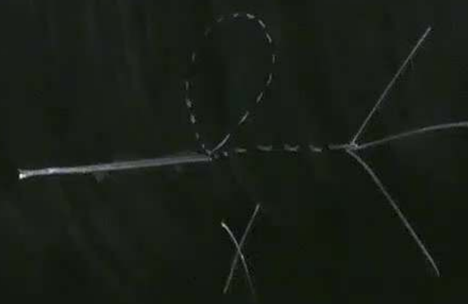
(B) $\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2$:

disconnected and connected diagrams

→ disconnected diagrams eg

$\Longrightarrow$ only look for connected diagrams.

(C) $\bar{k}$ conservation at each vertex $\Longrightarrow$



(2 × —— per vertex)

Check: only o

diagram sum

$+ \; B \; + \; C \; + \; D \; + \; E \; \equiv$

$\langle S_{int} \rangle^2$ → disconnected diagrams eg

...grams

...nnected diagrams.

...tion at each vertex ⇒

— per vertex) or 4 —

or 4 ----

Check: only one diagram survives at $\mathcal{O}(\lambda^2)$

$\lambda\,\phi_4 \qquad \lambda\phi^4$

$6\;\phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}\big)$

Result

$$\tilde{S}[\phi_<] = S_0[\phi_<] + \langle S_{int} \rangle_f - \frac{1}{2}\left(\underbrace{\langle S_{int}^2 \rangle_f - \langle S_{int} \rangle_f^2} \right) + \cdots$$

$$-\frac{1}{2}\left(\frac{u}{4!}\right)^2 \times 36 \int \left(\delta(k_1+k_2+k_3+k_4) \times (2\pi)^d \phi_<(k_1)\phi_<(k_2)\phi_<(k_3)\phi_<(k_4) \right.$$
$$\left. d\bar{k}_1 \, d\bar{k}_2 \, d\bar{k}_3 \, d\bar{k}_4 \right)$$

$$\times \int \frac{d^d q_1}{(2\pi)^d} G_0(q_1) \, G_0(k_1+k_2-q_1)$$

Λ/b

ϕ^4 term of original theory

$$G_0(q) = \frac{1}{q^2+r}$$

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2+r)^2} \approx \frac{S_d}{(2\pi)^d} \frac{\Lambda^d(1-1/b)}{(\Lambda^2+r)^2}$$

$\rightarrow$ After 2^{nd} order PT, and Fourier transform

$$\tilde{S}[\phi_<] = \int d^dx \, \frac{1}{2}\left((\nabla\phi)^2 + \left(r + \frac{u}{2} I_1\right)\phi^2 \right) + \frac{1}{4!}\left(u - \frac{u^2}{24^2} \times 36 \, I_2 \right)\phi^4 + \cdots$$