Title: Lecture - Statistical Physics (Core), PHYS 602

Speakers: Naren Manjunath

Collection/Series: Statistical Physics (Core), PHYS 602, October 8 - November 7, 2025

Subject: Condensed Matter, Other

Date: October 20, 2025 - 2:45 PM

URL: https://pirsa.org/25100018

gauge symmetry $M^2 A_\mu A^\mu$

sweet

Recap (MFT, last week)

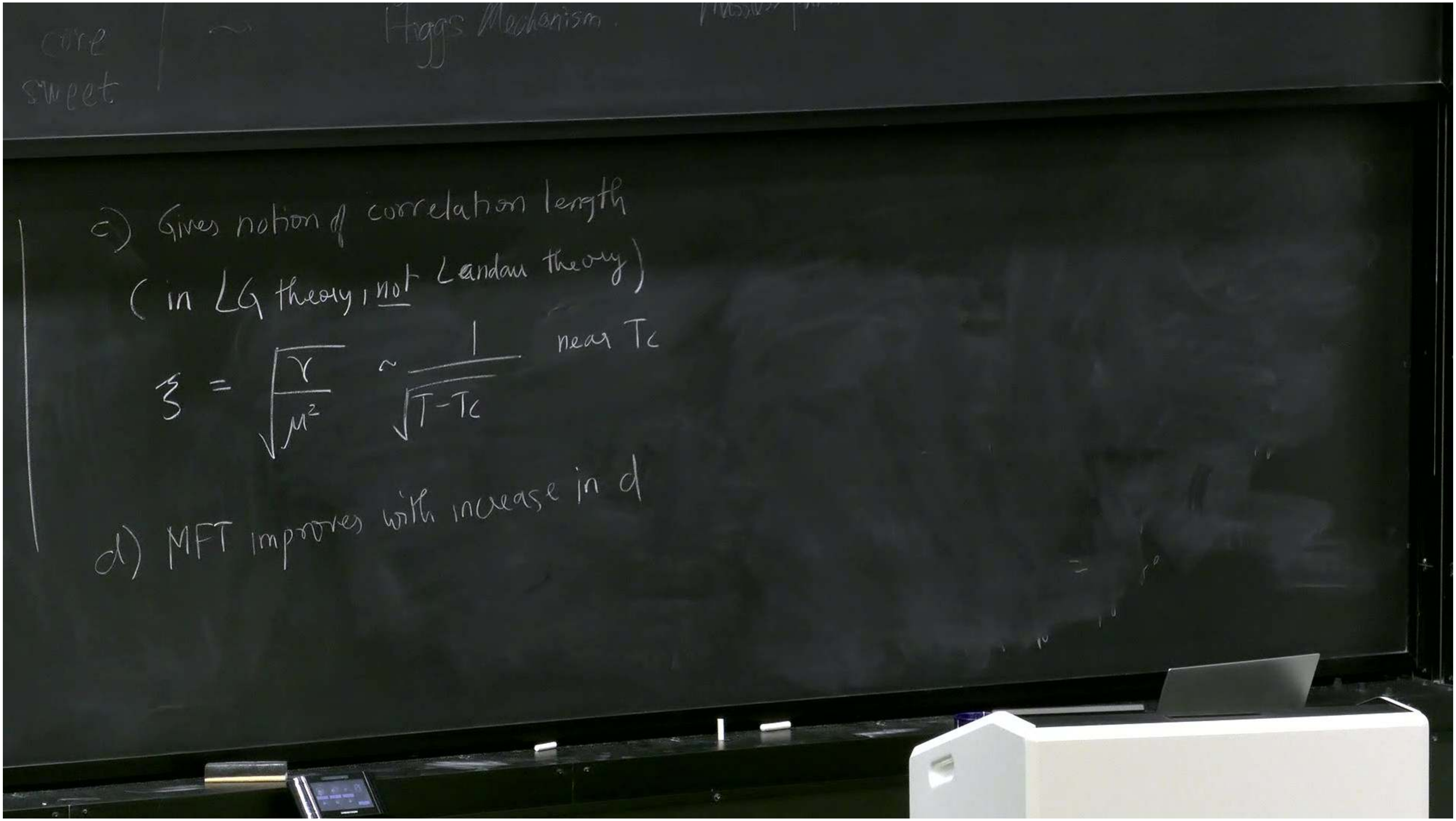
Mean field theory for Ising model

a) fails in $d=1$ (no ordering - domain wall argument) $(d_\ell = 1)$

b) Exact critical exponents in $d \geq 4$ $\quad (d_u = 4)$

$$\frac{\int_0^\xi \langle \phi(x)\phi(0)\rangle_c}{\int_0^\xi m_0^2} \ll 1$$

c) Gives n


Higgs Mechanism.
core
sweet
c) Gives notion of correlation length
(in LG theory, not Landau theory)
$\xi = \sqrt{\frac{\gamma}{\mu^2}} \sim \frac{1}{\sqrt{T-T_c}}$ near T_c
d) MFT improves with increase in d

Connecting Ising model to continuum (LG) field theory

$Z =$

* we want a continuum description so we ~~can~~ use tools of analysis

(1) $\sigma_i \longrightarrow \phi_i$; ϕ_i is continuous - valued

(11) "continuum limit" ϕ is defined at each point in space.

tricky!

$$Z = \sum_{\{\sigma_i\}} e^{-\frac{\beta}{2}\left(\vec{\sigma}^T \hat{J} \vec{\sigma}\right)} \quad \Big\} \sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j$$

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_N \end{pmatrix} \begin{pmatrix} J_{11} & J_{12} & & \\ J_{21} & J_{22} & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{pmatrix} \equiv \vec{\sigma}$$

<u>Claim</u>: $$e^{\frac{\beta}{2}\vec{\sigma}^T J \vec{\sigma}} = \frac{1}{(2\pi)^N \det(J\beta)} \times \int d\vec{\Phi}\, e^{-\frac{\beta}{2}\vec{\Phi}^T J^{-1}\vec{\Phi} + \beta\vec{\sigma}^T\vec{\Phi}}$$

$$\vec{\Phi} = (\phi_1, \phi_2, \ldots, \phi_N)$$

$$\int_{-\infty}^{\infty} d\vec{\phi}\; e^{-\frac{1}{2}\vec{\phi}^T J^{-1}\vec{\phi} + \vec{\sigma}^T\vec{\phi}}$$

$$= \int_{-\infty}^{\infty} d\vec{\phi}\; e^{-\frac{1}{2}\left(\vec{\phi} - J\vec{\sigma}\right)^T J^{-1}\left(\vec{\phi} - J\vec{\sigma}\right) + \frac{1}{2}\vec{\sigma}^T J\vec{\sigma}}$$

redefine $\vec{\phi}' = \vec{\phi} - J\vec{\sigma}$

$$= e^{\frac{1}{2}\vec{\sigma}^T J\vec{\sigma}} \times \int d\vec{\phi}'\, e^{-\frac{1}{2}\vec{\phi}'^T J^{-1}\vec{\phi}'} = e^{\frac{1}{2}\vec{\sigma}^T J\vec{\sigma}} \times \sqrt{\frac{(2\pi)^N}{\det J^{-1}}}$$

redefine σ & J to recover the claim

$$\rightarrow Z = \sum_{\{\sigma_i\}} \int d\Phi \, e^{-\frac{\beta}{2}\Phi^T J^{-1}\Phi + \beta \sigma^T \Phi} \times \frac{1}{\underbrace{\sqrt{(2\pi)^N \det J\beta}}_{\text{Smooth function of } T!\ \text{ignore.}}}$$

$$= \prod_i \int d\phi \, e^{-\frac{\beta}{2}\Phi^T J^{-1}\Phi} \, (2\cosh\beta\phi_i)$$

$$\equiv \int \mathcal{D}\phi_i \, e^{-S[\phi_i]}, \quad S[\phi_i] = \frac{\beta}{2}\Phi^T J^{-1}\Phi - \sum_i \log(2\cosh\beta\phi_i)$$

Part 2 of continuum limit

a) Fourier transform $\phi_i \to \phi(\bar{q})$

b) Expand about small $\bar{q}$ and let $\sum_{\bar{q}} \longrightarrow \int \frac{d^d q}{(2\pi)^d}$

c) Fourier transform back, now write $\phi(\bar{x})$ instead of ϕ_i

tion $\left(\sum_{i,j} \phi_i (\sigma^{-1})_{ij} \phi_j\right)$

Wilsonian RG

Comments

Big goal: a) how do microscopic details affect macroscopic properties

↳ what is the role of interactions in LG theory?

b) When is some microscopic parameter important?

c) Compute critical exponents in $d=2,3$ more accurately.

d) Relations between critical exponents? (Tutorial) gauge symmetry $M^2 A_\mu A^\mu$

sweet

The RG procedure (in momentum space)

(1) Coarse-g

Given $Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$,

DK

$$S[\phi] = \int d\bar{k} \; \phi_{-k}\left(k^2 + r\right)\phi_k + u\, \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}$$

$$d\bar{k} = \frac{d^d k}{(2\pi)^d}$$

$$r \to \frac{\mu^2}{\text{Tong}}$$

Higgs Mechanism. massless particle

① Coarse-grain

a) Separate out high-k and low-k modes.

Define $$\phi_<(k) = \begin{cases} \phi(k), & |k| < \Lambda/b \\ 0, & \text{o.w.} \end{cases}$$

$$\phi_>(k) = \begin{cases} 0, & |k| < \Lambda/b \\ \phi, & \text{o.w.} \end{cases}$$

$$\phi(k) = \phi_<(k) + \phi_>(k)$$

$*$ the inte

Now, $S[\phi] = S_<[\phi_<] + S_>[\phi_>] + S_{int}[\phi_<, \phi_>]$

eg. $u\, \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4}$

$= u\,(\phi^<_{k_1} + \phi^>_{k_1})(\phi^<_{k_2} + \phi^>_{k_2})(\phi^<_{k_3} + \phi^>_{k_3})(\phi^<_{k_4} + \phi^>_{k_4})$

$$= u\left(\begin{array}{l} \phi^< \phi^< \phi^< \phi^< \longrightarrow S_<[\phi_<] \\ + \phi^> \phi^> \phi^> \phi^> \longrightarrow S_>[\phi_>] \\ + \phi^> \phi^> \phi^< \phi^< \xrightarrow[\phi^>]{\text{integrate}} \# \phi^< \phi^< \end{array} \right.$$

modes $\quad \phi(k) = \phi_<(k) + \phi_>(k)$

* the interaction term changes the value of r in the theory after integrating out $\phi_>$!

(2) Rescale variables:

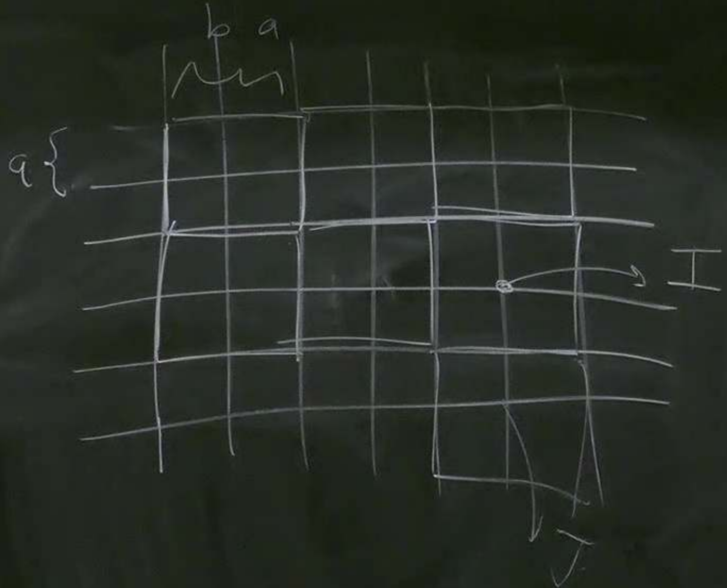
$$\tilde{k} = bk$$

$$\tilde{\phi}(\tilde{k}) = \frac{\phi_<(k)}{b^{\#}}$$

$$\equiv \int \prod_i d\phi_i\, e \qquad , \; S(\{\phi_{ij}\}) = \frac{1}{2}$$

...d - Stratonovich transformation

$$\Big(\sum_{ij} \phi_i \sigma$$



1) Coarse grain:

a) divide into boxes of size $b \times b$

"integrate out" degrees of freedom inside the $b \times b$ box

define $\left\{ \sigma_I = \frac{1}{b^d} \sum_{i \in box(I)} \sigma_i \right\}$

rewrite H in

σ_I

$\Phi^T J^{-1} \Phi$ $= e^{\frac{1}{2}\sigma^T J \sigma} \times \sqrt{\frac{(2\pi)^N}{\det J^{-1}}}$

-d - Stratonovich transformation

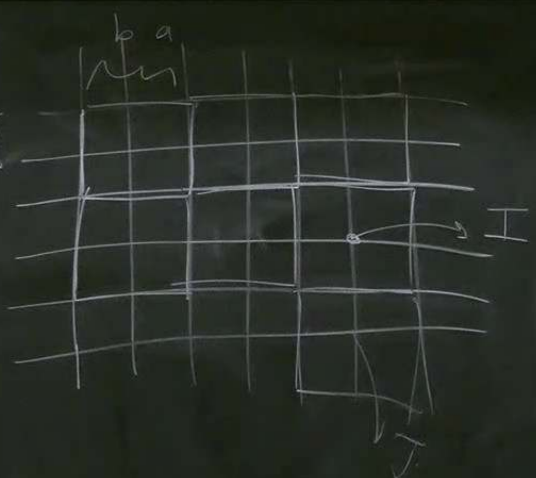
$$= \prod_i \int d\Phi \, e^{-\frac{\beta}{2}\Phi^T J^{-1}\Phi} \, (2\cosh\beta\phi_i)$$

Smooth function of T!

Ignore.

$$\equiv \int \prod_i d\phi_i \, e^{-S[\phi_i]}, \quad S[\{\phi_i\}] = \frac{\beta}{2}\Phi^T J^{-1}\Phi - \sum_i \log(2\cosh\beta\phi_i)$$

$$\left(\sum_{ij}\phi_i (J^{-1})_{ij}\phi_j\right)$$



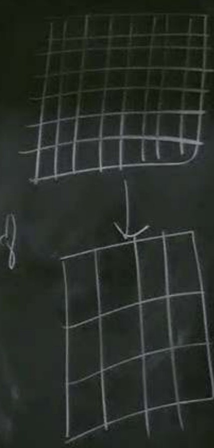
1) Coarse grain.

a) divide into boxes of size $b \times b$

"integrate out" degrees of freedom inside the $b\times b$ box

define $\left\{ \sigma_I = \frac{1}{b^d} \sum_{i \in \mathrm{box}(I)} \sigma_i \right\}$ rewrite H in terms of σ_I

b) Rescale: $X \to X/b$



$(*) \longrightarrow$ Hubbard - Stratonovich transformation

Example : Gaussian model $(u=0)$

$$S[\phi] = \int d^dx \left(\frac{1}{2} |\nabla\phi|^2 + \frac{r}{2}\phi^2 \right)$$

$$= \int d\bar{k} \; \phi_{-\bar{k}} (k^2 + r) \phi_{\bar{k}}$$

$$\phi_k = \phi_k^{<} + \phi_k^{>}$$

$S[\phi] = \int d\bar{k}$

when we integrate out ϕ_J the ... vanish by symmetry.

redefine σ & J to recover the claim

$$\to Z = \sum_{\{\sigma_I\}} \int d\Phi \, e^{-\frac{\beta}{2}\Phi^T J^{-1}\Phi + \beta\sigma^T\Phi} \times \frac{1}{\underbrace{\sqrt{(2\pi)^N \det J\beta}}_{\text{smooth function of } T!\ \text{ignore.}}}$$

$$= \prod_i \int d\Phi \, e^{-\frac{\beta}{2}\Phi^T J^{-1}\Phi} \left(2\cosh\beta\phi_i\right)$$

$$\equiv \int \prod_i d\phi_i \, e^{-S[\phi_i]}, \quad S[\{\phi_i\}] = \frac{\beta}{2}\underbrace{\Phi^T J^{-1}\Phi}_{\left(\sum_{ij}\phi_i (J^{-1})_{ij}\phi_j\right)} - \sum_i \log\left(2\cosh\beta\phi_i\right)$$

Note in this case $S[\phi] = S[\phi^<] + S[\phi^>]$; $S_{int}[\phi^<, \phi^>] = 0.$

$\Rightarrow$ integrating out $\phi^>$ doesn't affect the couplings in $S[\phi^<]$

$$\Rightarrow \text{after step 1, } Z = \underbrace{e^{-F^>}}_{\text{after integrating } \phi^>} \times \int \mathcal{D}\phi^< \, e^{-\beta S[\phi^<]}$$

2) Rescale.

$\tilde{k} = kb \quad \left(\tilde{x} = x/b \right)$

want to ensure that the $\frac{\gamma}{2}(\nabla\phi)^2$ term always has coupling $\gamma = 1$.

rescale ϕ:

$$S[\tilde{\phi}_<] = \int d^d x \left(\frac{1}{2}(\nabla\phi_<)^2 + \frac{\gamma}{2}\phi_<^2 \right) \qquad \text{define } \tilde{\phi} = \frac{\phi_<}{z(b)}$$

$$\downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$b^{-d} d^d x \left(\frac{1}{2} b^2 (\nabla\phi_<)^2 + \frac{\gamma}{2}\phi_<^2 \right) \qquad \phi_< = \tilde{\phi}\, z(b).$$

we want $b^{2-d} \times Z^2 = 1 \Rightarrow Z = b^{\frac{d-2}{2}} = b^{\Delta_\phi}$

$\frac{d-2}{2}$ is called the scaling dimension of ϕ

Scaling dimension of r:

the r term becomes $r \cdot b^{-d} \times \left(\frac{\phi^2}{2} b^{d-2}\right)$.

the action is invariant if we redefine $(r' = b^2 r)$

d)

ϕ^{ζ}

$b^{\#}$

The fact that r grows with RG means it becomes more important

For any coupling g s.t. $g \to b^{\Delta_g} g$ under RG.

$\Delta_g > 0 \iff g$ is a relevant coupling.

$\Delta_g < 0 \iff g$ is an irrelevant coupling $\to$ $\left(u \phi^4 d^d x\right)$

$\Delta_g = 0 \iff g$ is a marginal coupling: u

$$u \to b^{2(d-2)} b^{-d} = b^{d-4}$$

$$\tilde{u} = b^{4-d} u$$