Title: Lecture - Statistical Physics (Core), PHYS 602

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Collection/Series: Statistical Physics (Core), PHYS 602, October 8 - November 7, 2025

Subject: Condensed Matter, Other

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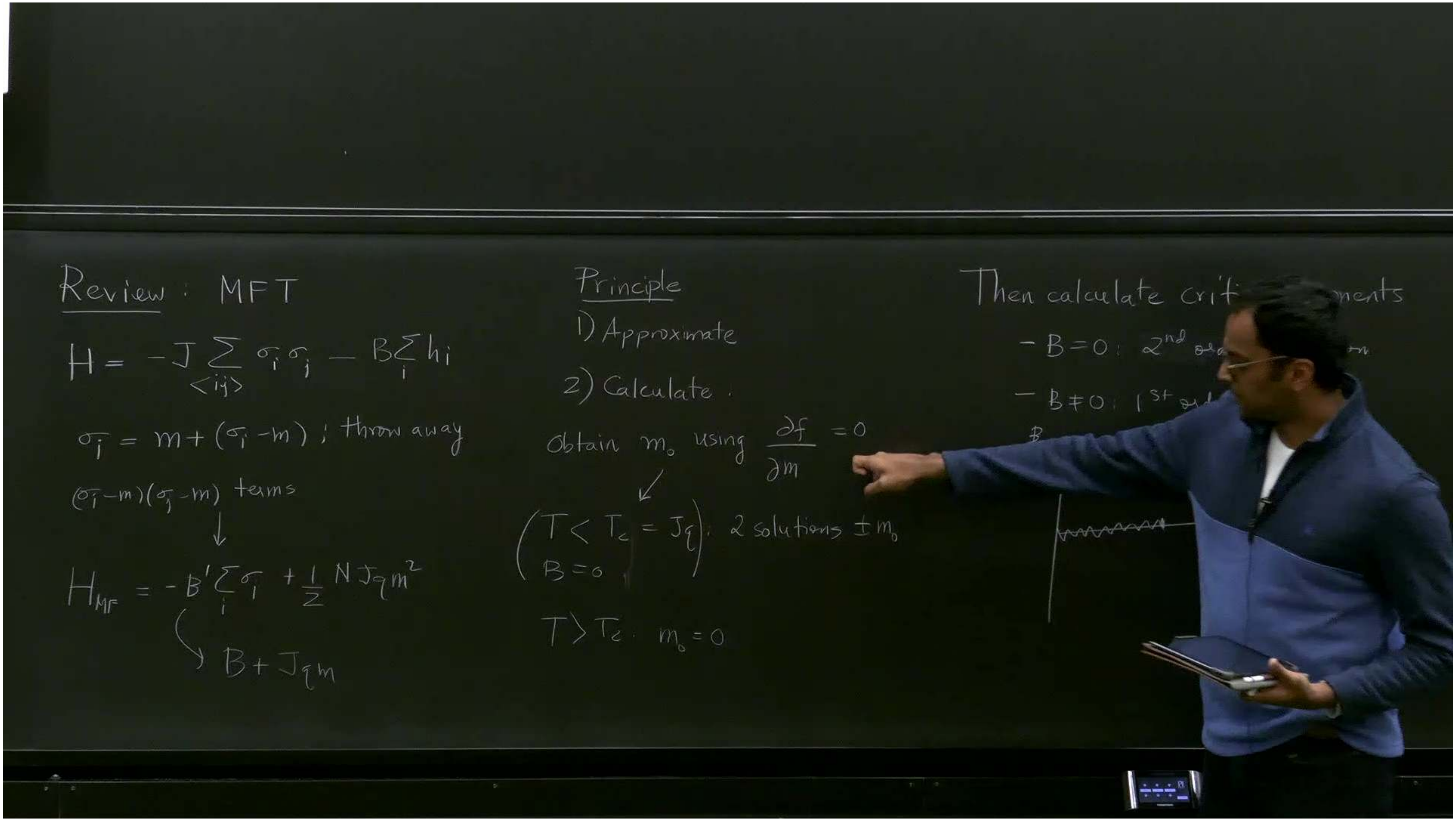

Review: MFT
$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i h_i$
$\sigma_i = m + (\sigma_i - m)$; throw away
$(\sigma_i - m)(\sigma_j - m)$ terms
$H_{MF} = -B' \sum_i \sigma_i + \frac{1}{2} N J q m^2$
$B + Jqm$
Principle
1) Approximate
2) Calculate.
Obtain m_0 using $\frac{\partial f}{\partial m} = 0$
$T < T_c = Jq$; 2 solutions $\pm m_0$
$B = 0$
$T > T_c$. $m_0 = 0$
Then calculate
$- B = 0$: 2^{nd}
$- B \neq 0$: 1^{st}

Review : MFT

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i h_i$$

$\sigma_i = m + (\sigma_i - m)$; throw away

$(\sigma_i - m)(\sigma_j - m)$ terms

$\downarrow$

$$H_{MF} = -B' \sum_i \sigma_i + \frac{1}{2} N J q m^2$$

$\hookrightarrow B + Jqm$

Principle

1) Approximate

2) Calculate :

Obtain m_0 using $\frac{\partial f}{\partial m} = 0$

$\left(\begin{array}{l} T < T_c = Jq \\ B = 0 \end{array} \right)$: 2 solutions $\pm m_0$

$\hookrightarrow \sim \sqrt{T - T_c}$

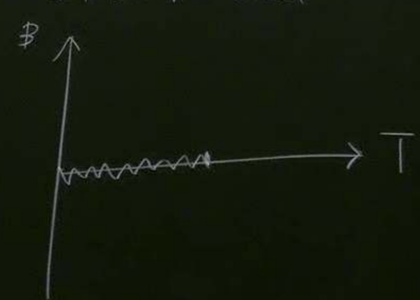
for $T \lesssim T_c$

$T > T_c$: $m_0 = 0$

Then calculate critical exponents

– $B = 0$: 2^{nd} order transition

– $B \neq 0$: 1^{st} order



...tical exponents

...er transition

...n

$\longrightarrow T$

approximate $1-\frac{T_c}{T}$ in powers of $(T-T_c)$

$$1-\frac{T_c}{T-T_c+T_c}$$

$$= 1-\frac{T_c}{T_c\left(1+\frac{T-T_c}{T_c}\right)}$$

$$= 1-\left(1-\frac{T-T_c}{T_c}+\left(\frac{T-T_c}{T_c}\right)^2-\cdots\right)$$

$$\frac{T-T_c}{T_c}-\left(\frac{T-T_c}{T_c}\right)^2+\cdots$$

* Critical exponents should only be computed near 2^{nd} order phase transition

* MFT critical exponents have no dimensional dependence!

* Critical exponents should only be computed near 2^{nd} order phase transition $C_V \sim |T-T_c|^{-\alpha}$.

* MFT critical exponents have no dimensional dependence!

$M_0 \sim |T_c - T|^{\beta}$

$|T-T_c|^{-\gamma} \propto \frac{\partial M}{\partial B}\Big|_{+}$

See §§ 1.2, 1.3 of David Tong

$M \propto B^{1/\delta}$ at $T = T_c$

	MFT	$d=1$	$d=2$
α	0 (jump)	no phase transition!	0 (no jump)
β	$1/2$		$1/8$
γ	1	→ Completely wrong!	$7/4$
δ	3		15 Not great

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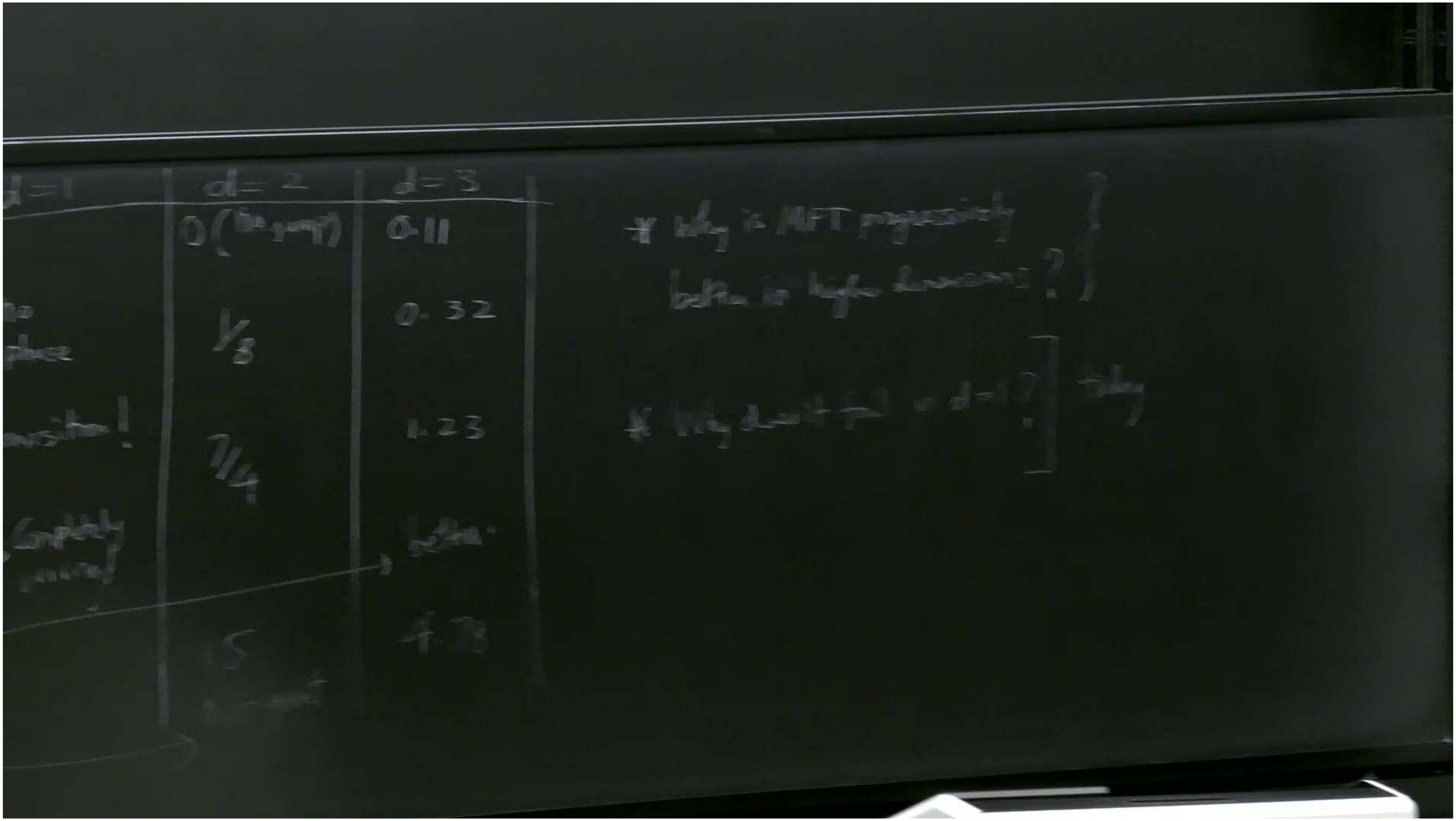
Sees: 1.2, 1.3 of David Tong

$M \propto B^{1/\delta}$ at $T = T_c$

	MFT	d=1	d=2	d=3
α	0 (jump)	no phase transition!	0 (no jump)	0.11
β	1/2		1/8	0.32
γ	1	Completely wrong!	7/4	1.23
				better
δ	3		15 Not great	4.78


d = 2
d = 3
0.11
0.32
1/8
1.23
7/4
5

$=1$	$d=2$	$d=3$
	0 (no jump)	0.11
ase	$1/8$	0.32
nsition!	$7/4$	1.23
ompletely wrong!		better.
	15 Not great	4.78

* Why is MFT progressively better in higher dimensions?

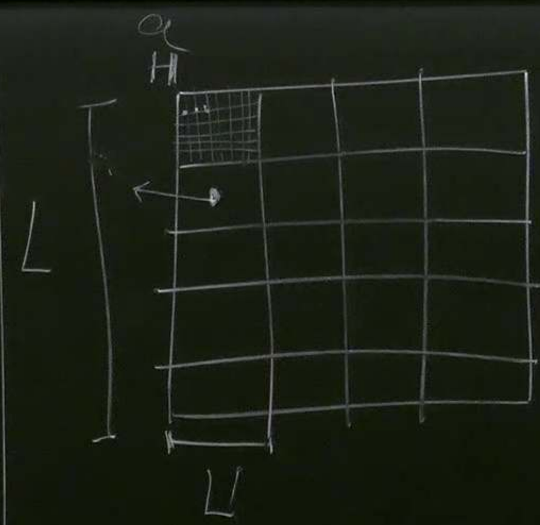
* Why does it fail in $d=1$? today

Landau - Ginzburg theory

$$\boxed{m \longrightarrow m(\bar{x})}$$

* Can "derive" as a continuum limit of Ising model

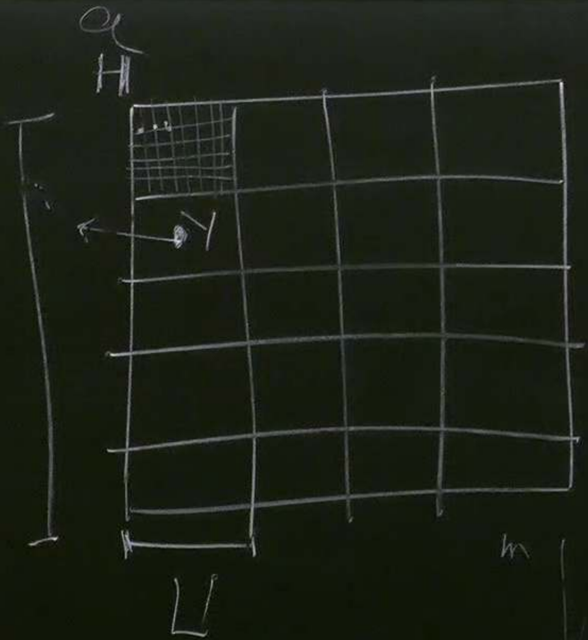
* Also explain based on symmetries.



$a \ll L' \ll L$

box centers :

odel



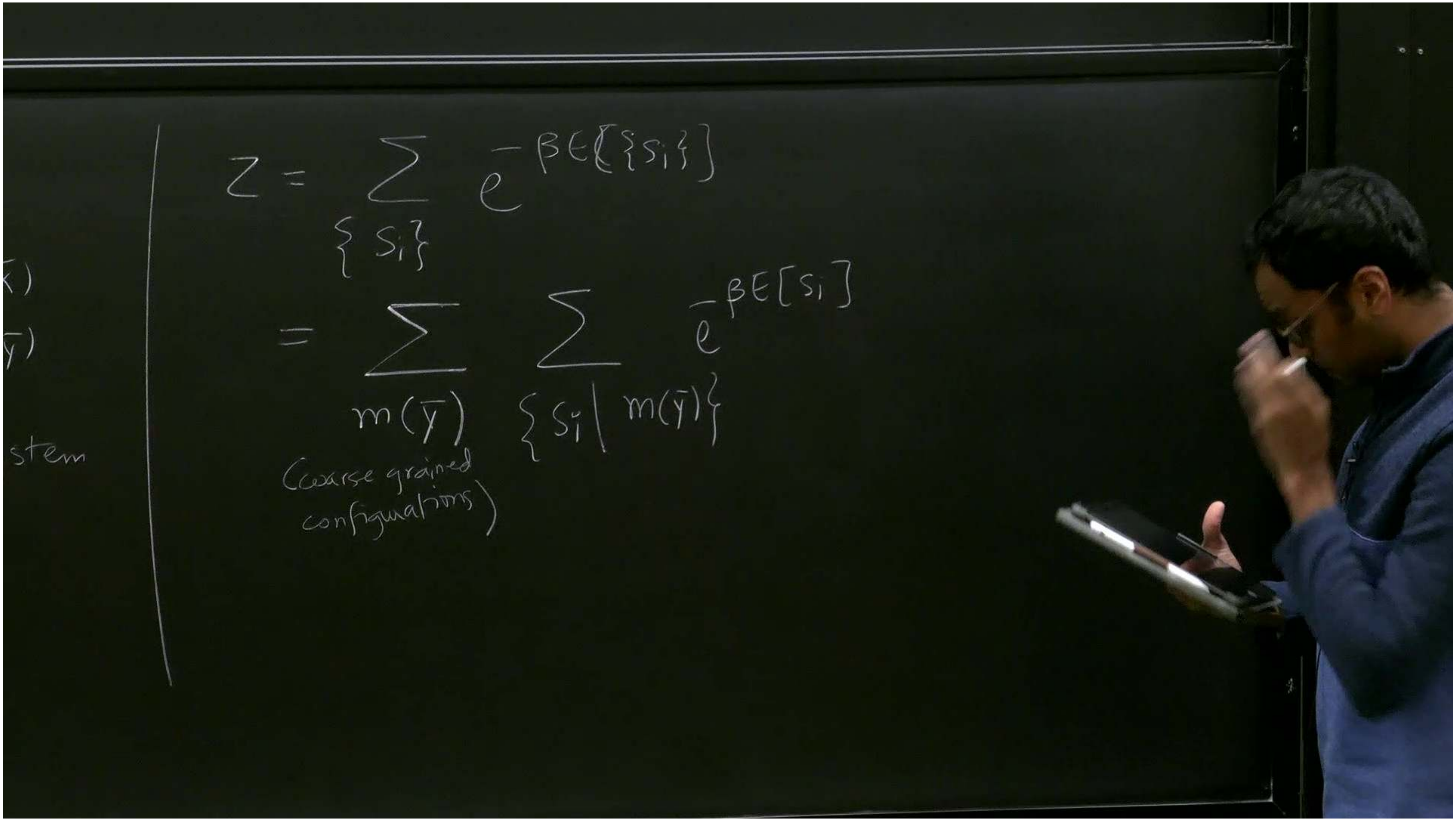
$a \ll L' \ll L$

box centers : $\bar{y}$

$$m(\bar{y}) = \frac{1}{(L')^d} \sum_{\bar{x} \in \mathrm{box}(\bar{y})} m(\bar{x})$$

Goal: describe the system using $m(\bar{y})$

$Z =$



$Z = \sum_{\{s_i\}} e^{-\beta E[\{s_i\}]}$
$= \sum_{m(\bar{y})} \sum_{\{s_i | m(\bar{y})\}} e^{-\beta E[s_i]}$
(coarse grained configurations)

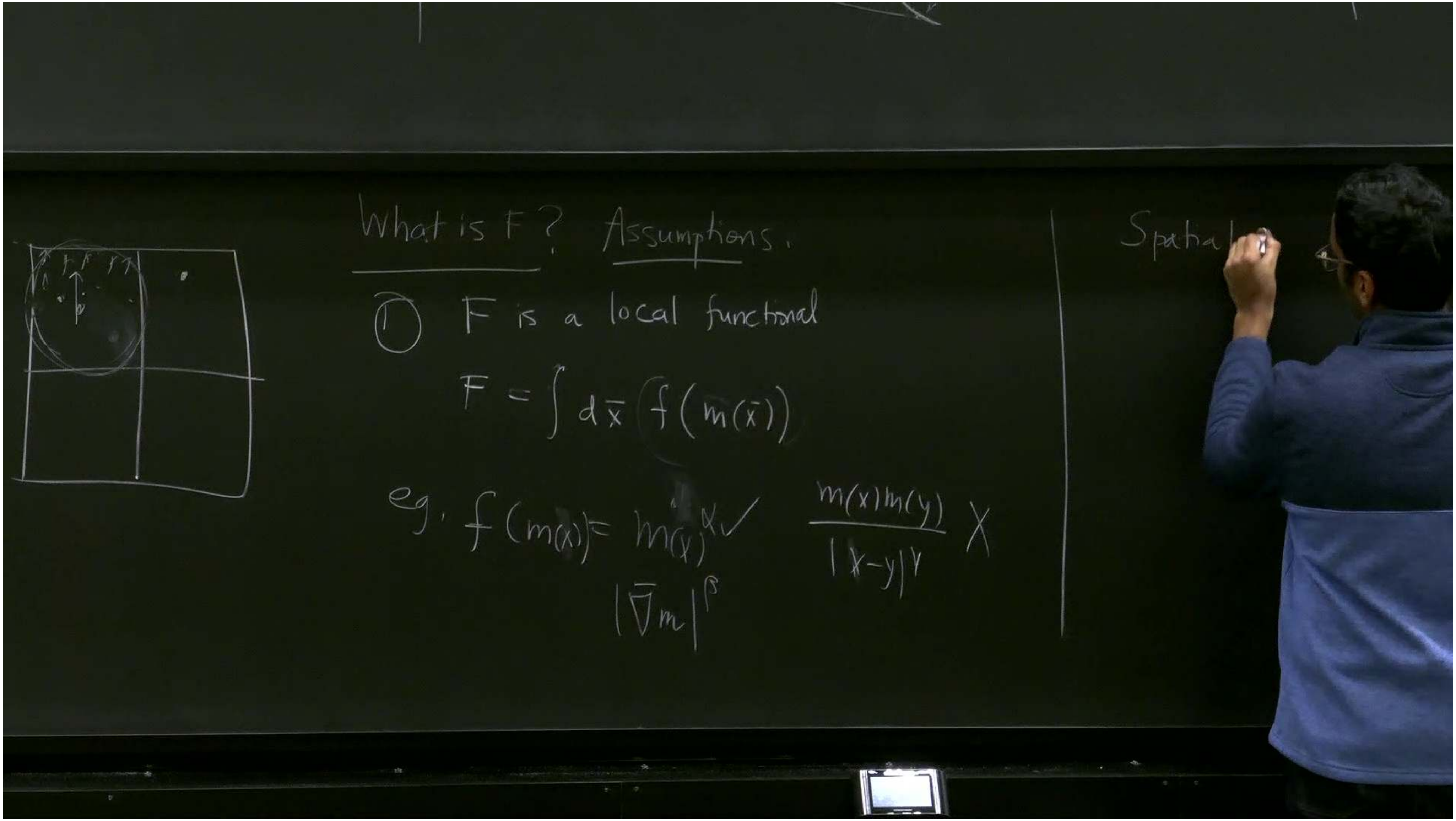
$$Z = \sum_{\{s_i\}} e^{-\beta E[\{s_i\}]}$$

$$= \sum_{m(\bar{y})} \underbrace{\sum_{\{s_i \mid m(\bar{y})\}} e^{-\beta E[s_i]}}_{e^{-\beta F[\bar{m}]}}$$

(coarse grained configurations)

$$= \sum_{m(\bar{y})} e^{-\beta F[\bar{m}(y)]}$$

$F[\bar{m}(y)] \longrightarrow$ Landau - Ginzburg functional


What is F? Assumptions.
① F is a local functional
$F = \int d\bar{x}\, f(m(\bar{x}))$
eg. $f(m(x)) = m(x)^{\alpha}$ ✓
$\frac{m(x)m(y)}{|x-y|^{\gamma}}$ X
$|\bar{\nabla} m|^{\beta}$
Spatial

② Symmetries.

- from spatial symmetry terms like $(x^2+y^2)\,f(x,y)$ X

(no preferred origin)

$$f = \sum_j c_j\, m^j$$

when $B=0$ we know there is $\mathbb{Z}_2$ symmetry

taking $m \to -m$.

$\lambda_j (\nabla m)^j \Rightarrow$ When $B=0$, $c_j = 0$ when j is odd.

$\lambda_j = 0$

(3) Analyticity : f has a Taylor expansion

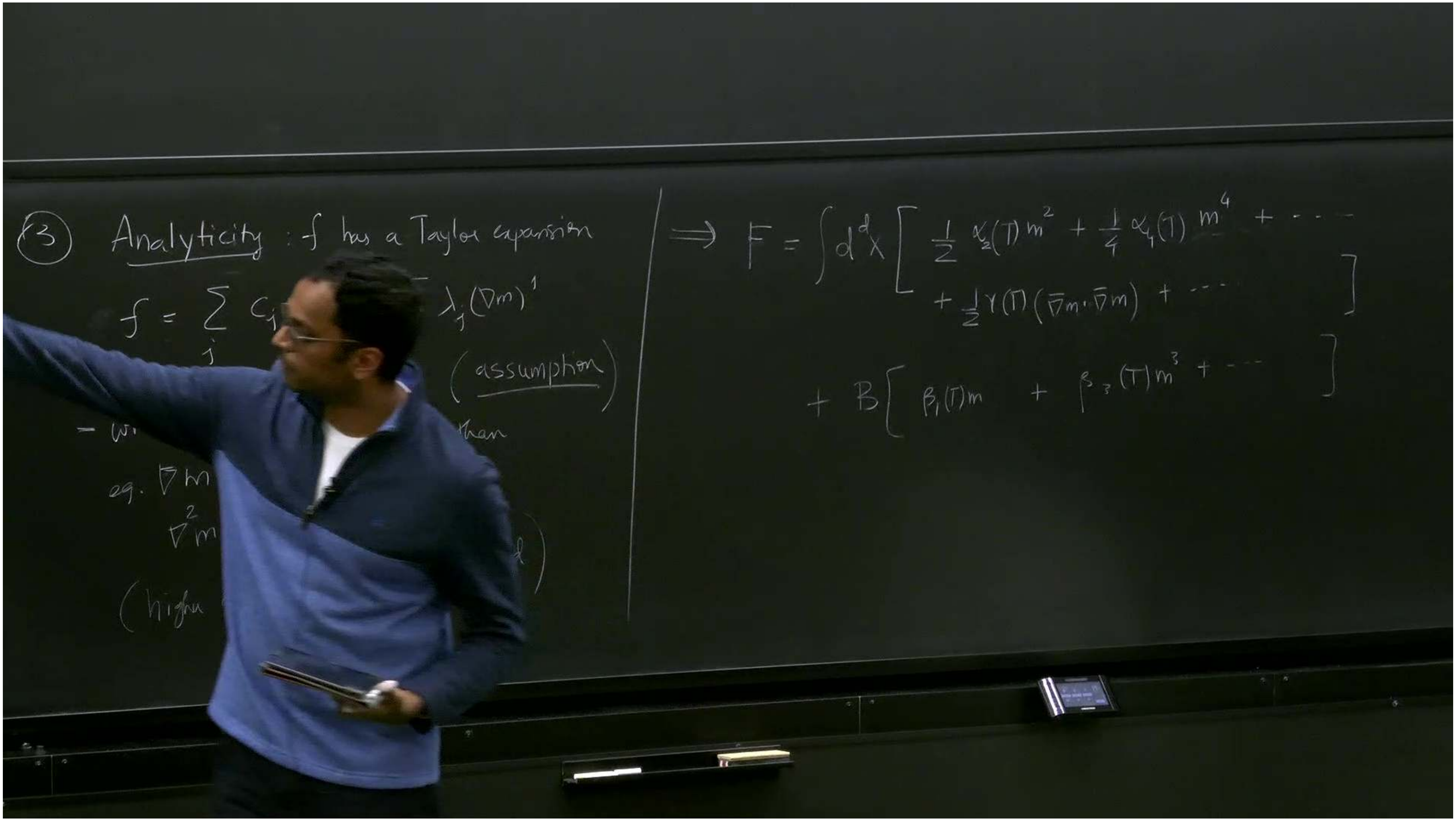
$$f = \sum_j c_j m^j + \sum_j \lambda_j (\nabla m)^j$$

- when fluctuations are slow, (assumption)

eg. ∇m is more important than $\nabla^2 m$

(higher derivatives can be ignored)

$$\Rightarrow F = \int d^d x \Big[$$


③ Analyticity : f has a Taylor expansion
f = ∑ c
(assumption)
than
eg. ∇m
(higher
⟹ F = ∫ d^d x [1/2 α_2(T) m^2 + 1/4 α_4(T) m^4 + ...
+ 1/2 γ(T) (∇m · ∇m) + ...]
+ B [β_1(T) m + β_3(T) m^3 + ...]

(higher derivatives can be ignored)

Relation to Landau MFT

$\to$ m is constant in space

let's do a saddle point approximation for f

$$\delta F = \int d^dx \left[\alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m + \cdots \right] \delta m$$

$$\delta \int \gamma (\vec{\nabla} m \cdot \vec{\nabla} m) = \delta \int m \cdot \nabla(\gamma \nabla m)$$

$$= -\int \delta m \, \gamma \nabla^2 m$$

In SP approx we require $\frac{\delta F}{\delta m} = 0$

$(B=0)$

$$\Longrightarrow \alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m = 0$$

If m is constant we recover the Landau MFT equation

f_{Landau}

in SP approx we require $\frac{\delta F}{\delta m} = 0$
$(B=0)$

$\Rightarrow \alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m = 0$

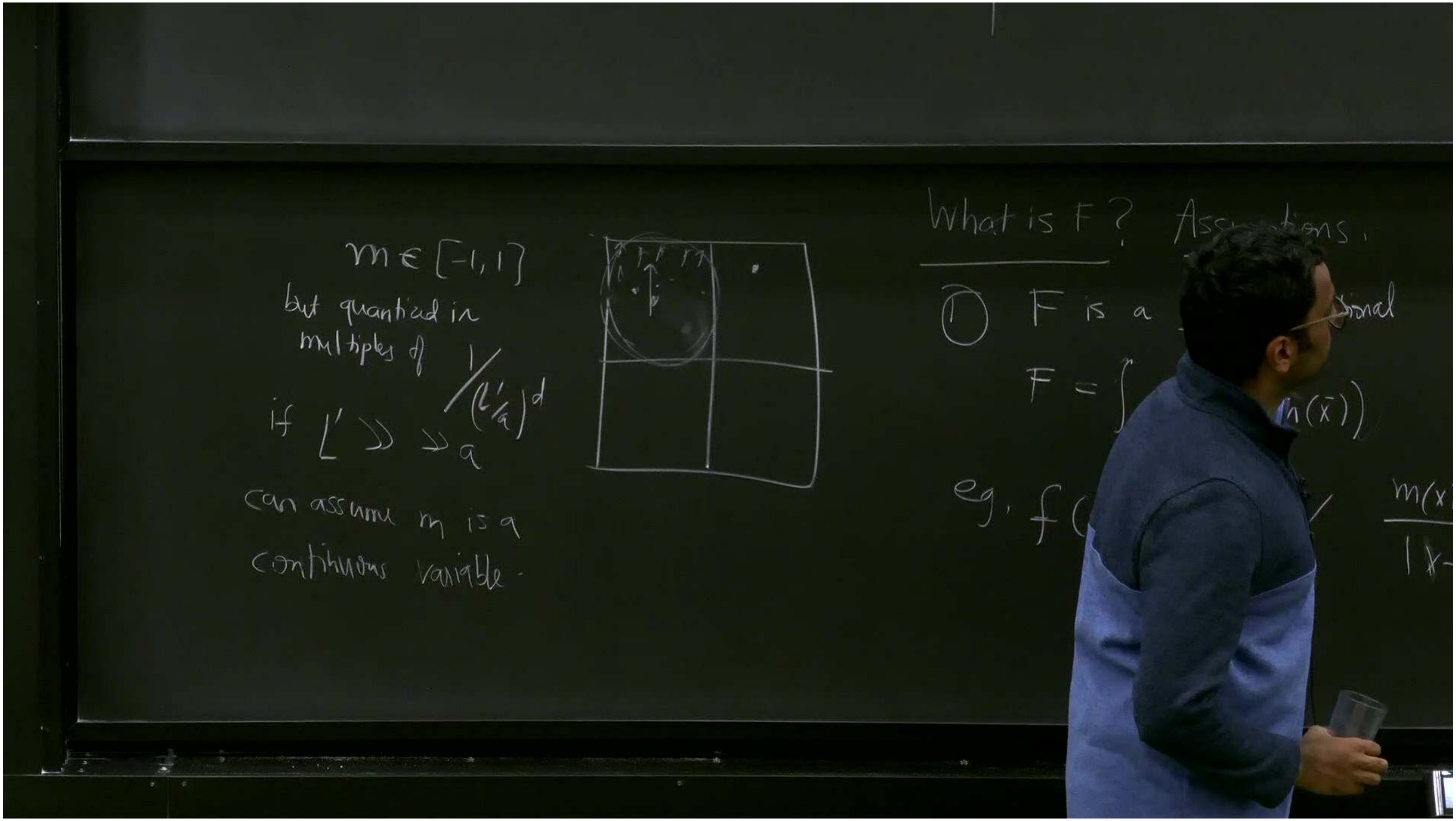
If m is constant we recover the Landau MFT equation

We got a similar eqn from

$$\left[f_{Landau} \sim \frac{1}{3}(T-T_c)\, m^2 + \frac{T}{12}\, m^4 + \ldots \right.$$

$$\left. \text{by taking } \frac{\partial f}{\partial m} = 0 \right]$$

* Landau MFT is the saddle point approx of LG theory



m ∈ [-1, 1]
but quantized in
multiples of 1/(L'/a)^d
if L' >> >> a
can assume m is a
continuous variable.
What is F? Assumptions.
F is a
F =

Motivated by coarse graining we take

2 separate "continuum limits". (both are realistic assumptions)

- $m(\bar{y})$ takes continuous values

- $\bar{y}$ takes continuous values

What is F? Assumptions.

(1) F is a local functional

$$F = \int d\bar{x}\, f(m(\bar{x}))$$

e.g. $f(m(x)) = m(x)^{\alpha}$ ✓

$|\bar{\nabla} m|^{\beta}$

$\frac{m(x)}{|x-}$

Friday 1pm – 2pm (Oct 17), Workroom

in space

for f

$\ldots] \, \delta m$

in SP approx we require $\dfrac{\delta F}{\delta m} = 0$

$(B=0)$

$\Rightarrow \alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m = 0$

If m is constant we recover the Landau MFT equation

We got a similar eqn from

$\left[f_{Landau} \sim \frac{1}{3}(T-T_c)\, m^2 + \frac{T}{12} m^4 + \ldots \right.$

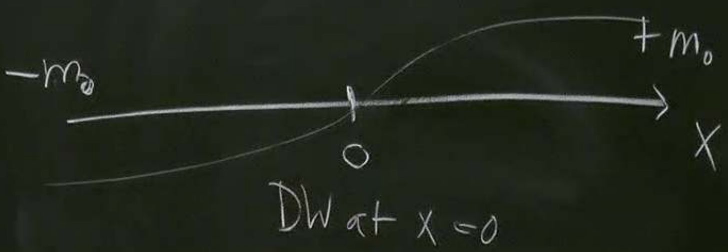
by taking $\left. \dfrac{\partial f}{\partial m} = 0 \right]$

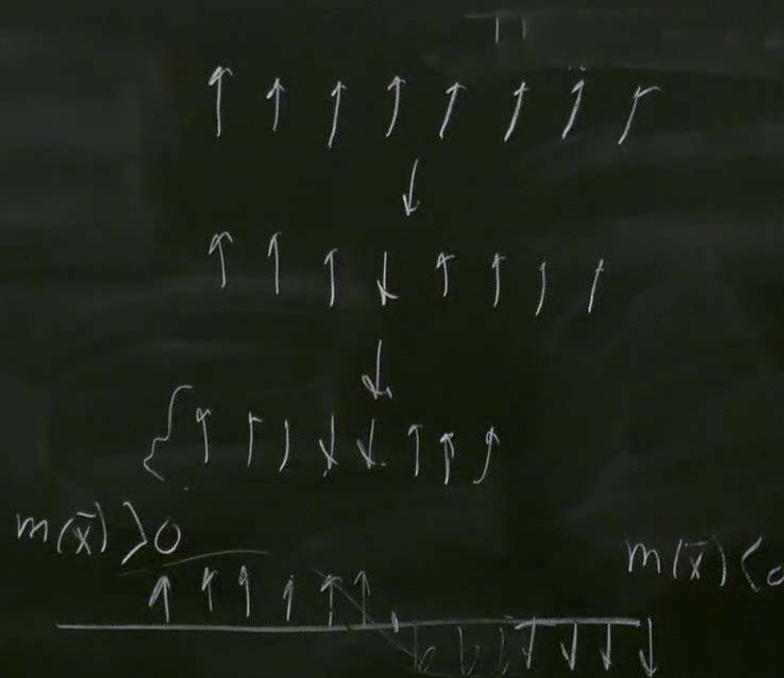
* Landau MFT is the <u>saddle point</u> approx of homogeneous LG theory

$|\nabla m|$

<u>No ordering in $d=1$</u> (too easy to create domain walls)

* Consider a single DW on a 1d slice





$\chi_j(\nabla m) \Longrightarrow$ when $B=0$, $C_j = 0$

$\chi_j = 0$

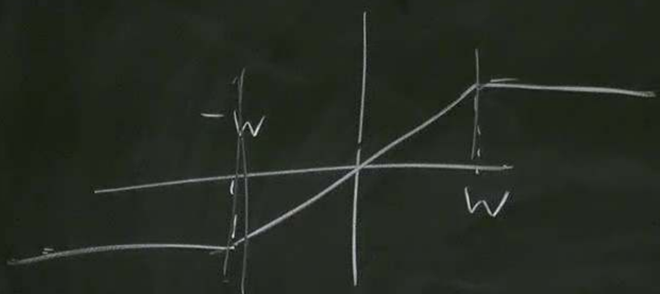
From EOM,

$$\gamma \frac{d^2 m}{dx^2} = \alpha_2 m + \alpha_4 m^3$$

Can be solved as

$$m = m_0 \tanh\left(\frac{x}{W}\right), \quad \underbrace{W = \sqrt{\frac{-2\gamma}{\alpha_2}}}_{\text{width of DW}}$$

ΔF_{PW} is contributed by the $\int \gamma |\nabla m|^2 d^dx$ term



$$\Delta F = \gamma \; W \underbrace{\left(\frac{\Delta m}{\Delta x}\right)^2}_{\frac{m_0}{W}} \underbrace{\left(L^{d-1}\right)}_{\text{ignore when } d=1}$$

$$= \left(L^{d-1}\, \gamma \, \frac{m_0^2}{W}\right)$$

$$x$$

$$(\ldots x_d)\left(\int dx\,|\nabla m|^2\right) = L^{d-1}\gamma\left(\sqrt{\frac{\alpha_2}{\alpha_4}}\right)^2\sqrt{\frac{\alpha_2}{\gamma}} = \sqrt{\frac{\gamma\alpha_2^3}{\alpha_4^2}}\;L^{d-1}$$

$$m_0 = \boxed{\sqrt{\left|\frac{\alpha_2}{\alpha_4}\right|}}$$

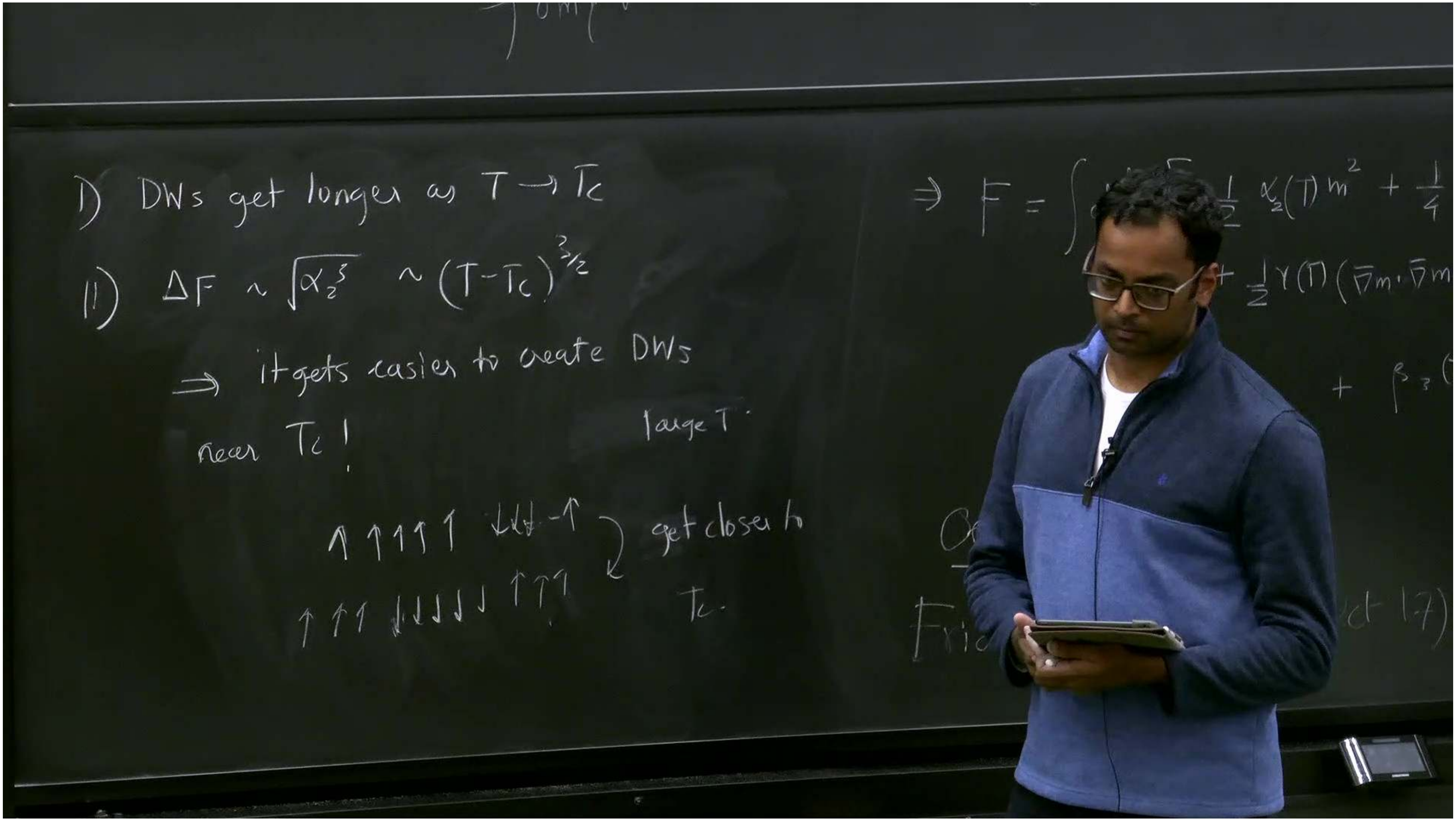
$$W \propto \sqrt{\left|\frac{\gamma}{\alpha_2}\right|}$$

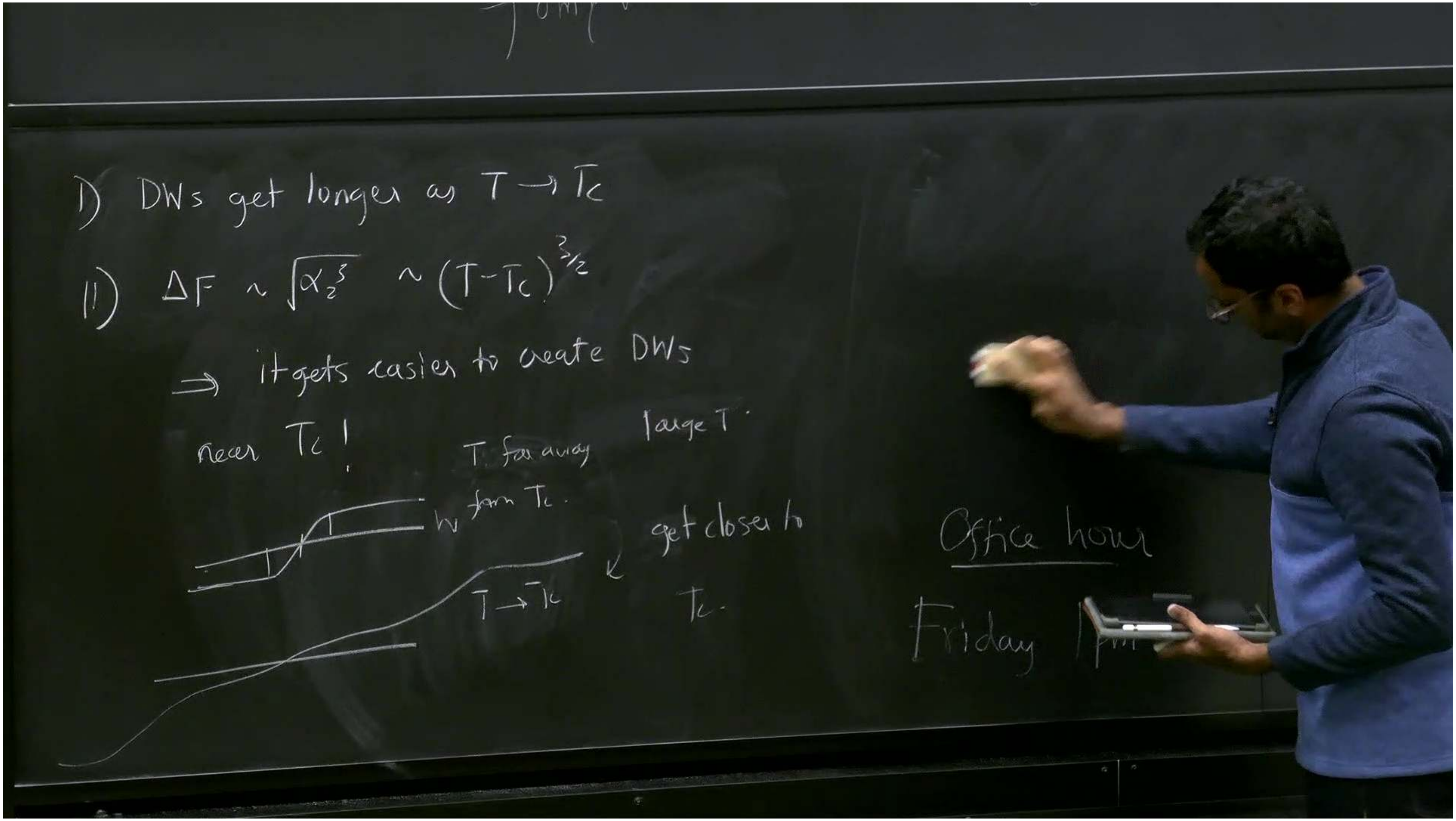
from Landau theory

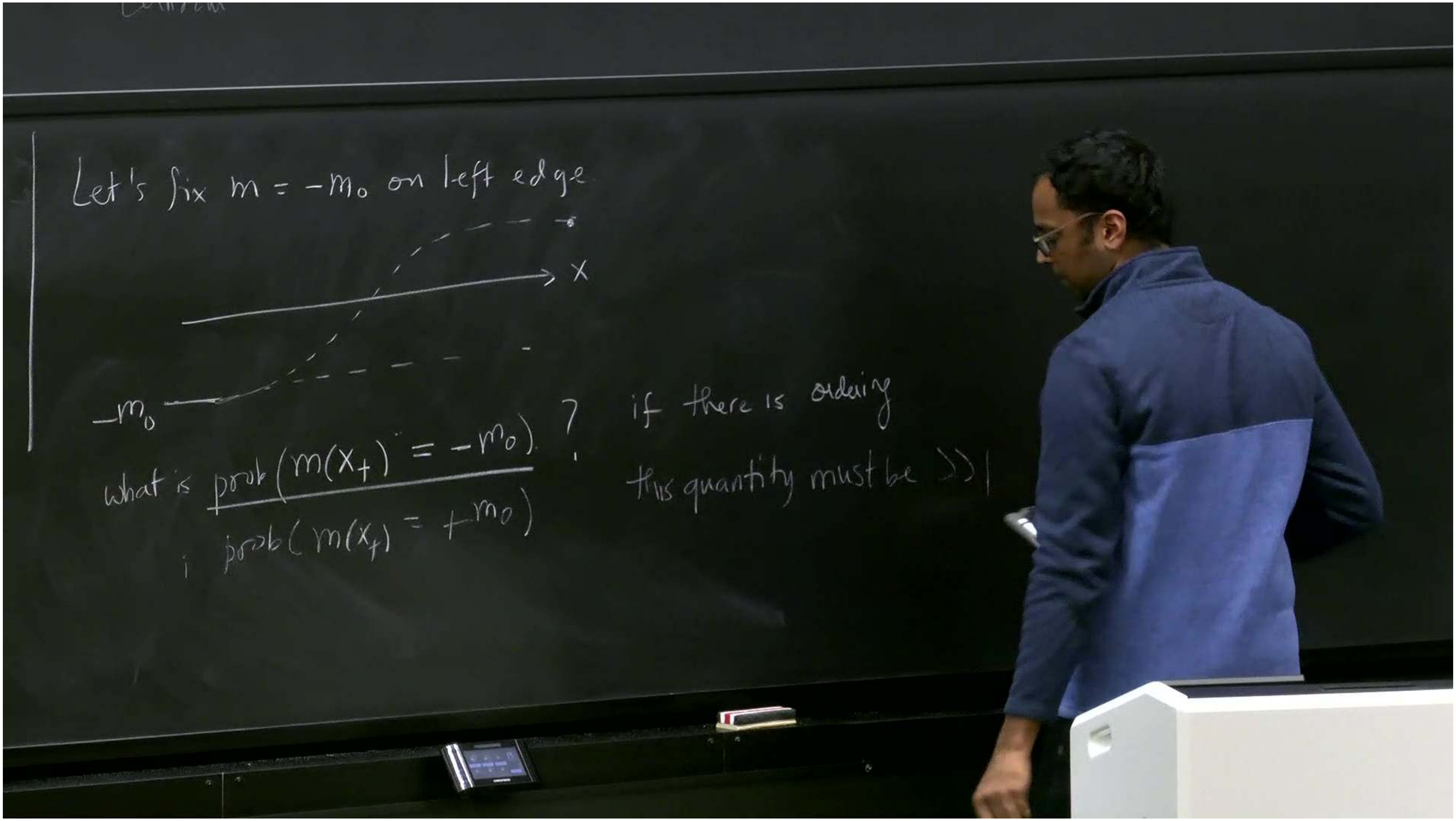
we expect $\alpha_2 \sim (T - T_c)$

$\alpha_4 \sim T$

$\Rightarrow T \to T_c\ ,\ W \to \infty$


I) DWs get longer as T → Tc
II) ΔF ~ √α₂³ ~ (T − Tc)^3/2
⇒ it gets easier to create DWs
near Tc !
large T
get closer to
Tc.


I) DWs get longer as $T \to T_c$
II) $\Delta F \sim \sqrt{\alpha_2^3} \sim (T-T_c)^{3/2}$
$\Rightarrow$ it gets easier to create DWs
near T_c !
large T.
T far away
from T_c.
w
get closer to
T_c.
$T \to T_c$
Office hour
Friday 1pm


Let's fix $m = -m_0$ on left edge
x
$-m_0$
what is $\frac{\text{prob}(m(x_+) = -m_0)}{\text{prob}(m(x_+) = +m_0)}$?
if there is ordering
this quantity must be $\gg 1$

$T \to T_c$ $\quad T_c$.

$$\frac{P(m(x_f) = -m_0)}{P(m(x_+) = +m_0)} = \frac{P(\text{no DW}) + P(2\,\text{DW}) + P(4\,\text{DW}) + \cdots}{P(1\,\text{DW}) + P(3\,\text{DW}) + P(5\,\text{DW}) + \cdots} = \frac{S_{even}}{S_{odd}}$$

note

$$p(1\,\text{DW}) = \left((\#\text{locations}) \times e^{-\beta \Delta F_{DW}} \right) \frac{1}{Z}$$

$$\#\text{locations} = L/W$$

$$p(n\,\text{DWs}) = \frac{1}{Z} \left(\left(\frac{L}{W}\right)^n e^{-\beta n \Delta F_{DW}} \right) \frac{1}{n!}$$

$$X = \frac{L}{W} e^{-\beta \Delta F_{DW}}$$

$$S_{even} = \frac{1}{2}\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{\cosh x}{2}$$

$$S_{odd} = \frac{\sinh x}{2}$$

$$\frac{S_{even}}{S_{odd}} = \frac{1}{\tanh x} = \frac{1}{\tanh\left(\frac{L}{W} e^{-\beta\Delta F}\right)}$$

$e^{-\beta\Delta F}$ is independent of L.

$\Rightarrow$ as $L \to \infty$ the ratio $\to \frac{1}{\tanh(\infty)}$

$= 1$

$\sqrt{|\alpha_2|}$ as $T \to T_c$, $W \to \infty \longrightarrow$ (1).

From EOM,

$$\gamma \frac{d^2 m}{dx^2} = \alpha_2 m + \alpha_4 m^3$$

Can be solved as

$$m = m_0 \tanh\left(\frac{x}{W}\right), \quad \underbrace{W = \sqrt{\frac{-2\gamma}{\alpha_2}}}_{\text{width of DW}}$$

No ordering in $d=1$
(LG theory corrects the failing of Landau theory)

* there is ordering in $d>1$

$\Rightarrow$ $d_l = 1$ is lower critical dimension for Ising model