Title: Lecture - Combinatorial QFT, CO 739-002

Speakers: Michael Borinsky

Collection/Series: Combinatorial QFT, CO 739-002, September 4 - December 2, 2025

Subject: Mathematical physics, Quantum Fields and Strings

Date: October 07, 2025 - 2:00 PM

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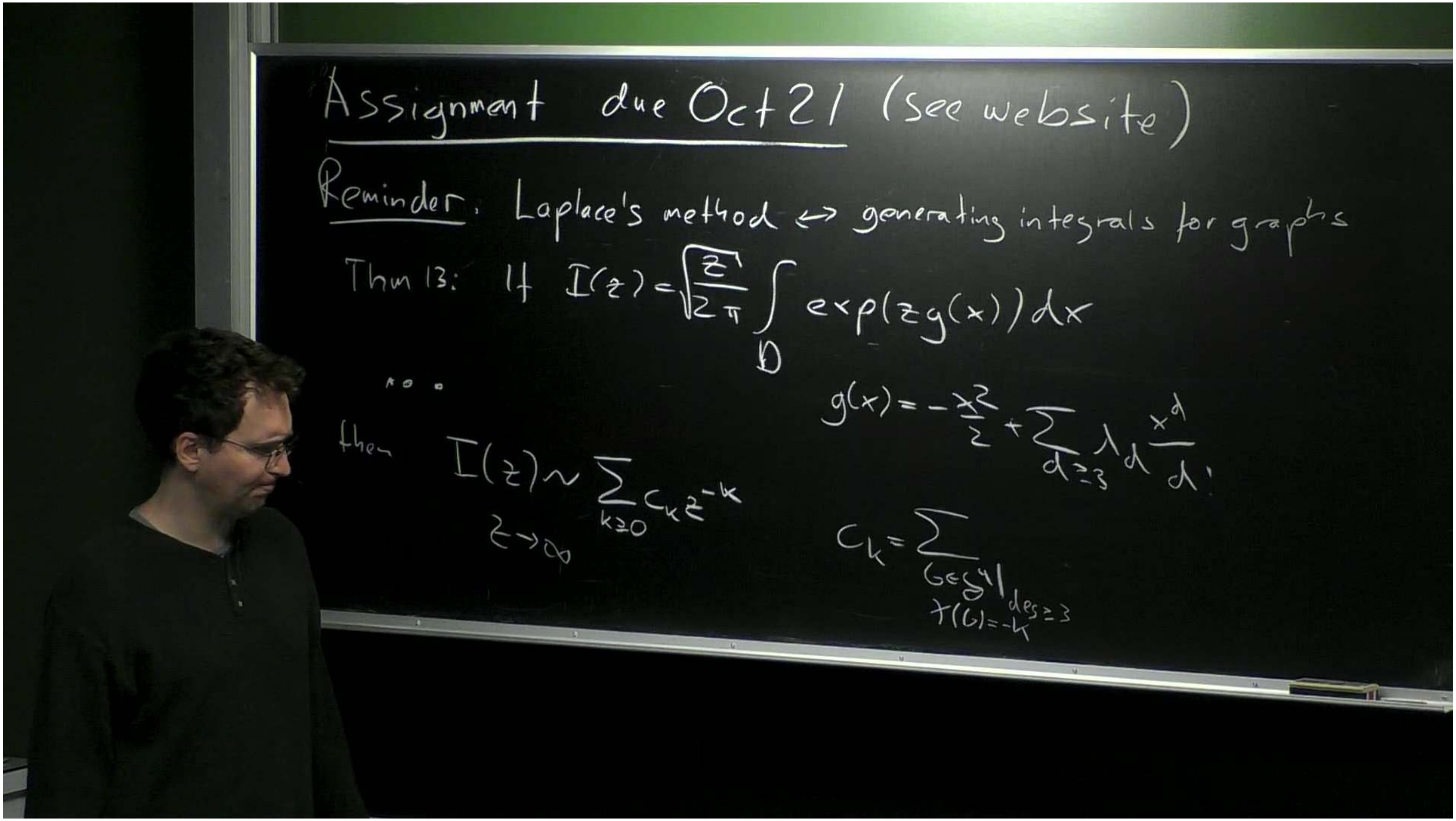
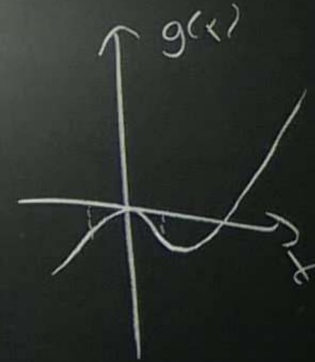

Assignment due Oct 21 (see website)
Reminder: Laplace's method ↔ generating integrals for graphs
Thm 13: If $I(z) = \sqrt{\frac{z}{2\pi}} \int_D \exp(z g(x))\, dx$
$g(x) = -\frac{x^2}{2} + \sum_{d \geq 3} \lambda_d \frac{x^d}{d!}$
then $I(z) \underset{z \to \infty}{\sim} \sum_{k \geq 0} c_k z^{-k}$
$c_k = \sum_{G \in \mathcal{G}'|_{\deg s \geq 3},\ \chi(G) = -k}$

Example: (See Assignment 1.1)

$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\varepsilon}^{\varepsilon} \exp\left(z\left(-\frac{x^2}{2} + \frac{x^3}{3!}\right)\right) dx$$



E.g. $\varepsilon = \frac{1}{10}$

$$\Rightarrow I(z) = \sum_{k=0}^{R-1} c_k z^{-k} + O(z^{-R}) \quad \text{for all } R \geq 0$$

where $$c_k = \sum_{G \in \mathcal{G}^4|_{\deg=3},\ \chi(G)=-k} \frac{1}{|\mathrm{Aut}\, G|}$$

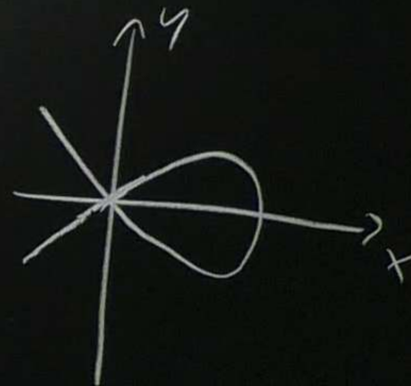
$\int_\gamma D\varphi\, DA_t\, DA_{\bar z}\, e^{\dots}$

$\int D\varphi_{\dots}\, D\bar\varphi_{\dots}\, DA_t\, D\bar A_t\, DA_{\bar z}\, D\bar A \dots$

Example Combinatorial statement derived from analytics ⟷ Combinatorics correspondence

Intuition: It would be nice to have a Gaussian ...

$$-\frac{y^2}{2} = -\frac{x^2}{2} + \frac{x^3}{3!}$$



Think: $-\frac{y^2}{2} = -\frac{x(y)^2}{2} + \frac{x(y)^3}{3!}$ $\quad x(y) = \sum_{n\geq 0} b_n^{\pm} y^n$

Puiseux Theorem: Solve locally as power series (locally convergent!)

Puiseux Theorem: Solve locally as power series) ((locally convergent!)

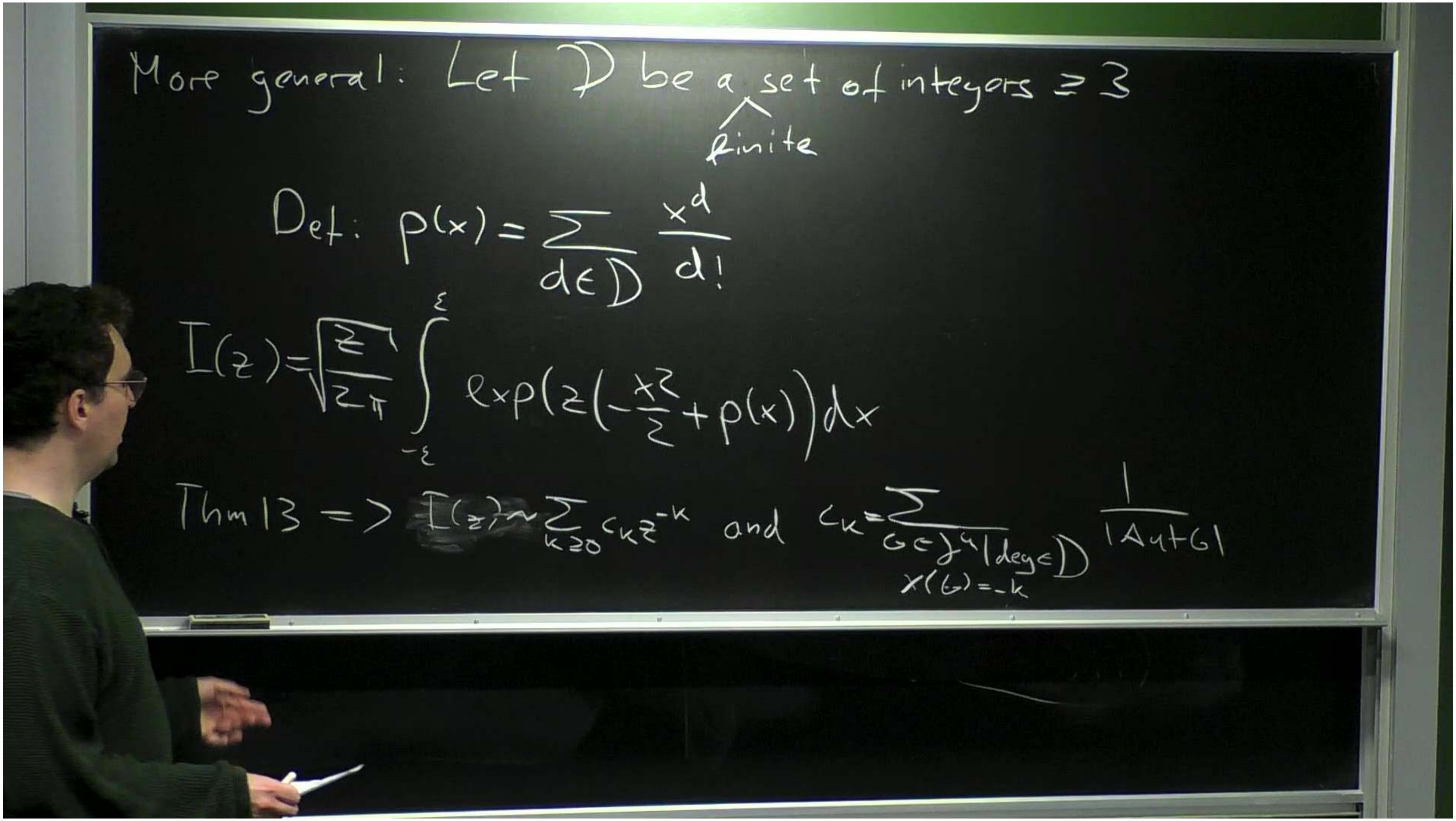
$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\varepsilon'}^{\varepsilon'} \exp\left(-z\frac{y^2}{2}\right) \frac{\partial x(y)}{\partial y} \, dy$$

Jacobian

$$\varepsilon' = \sqrt{\varepsilon^2 - \frac{\varepsilon^3}{3}} \approx \varepsilon$$

use Taylor exp of $x(y)$ as in proof of Thm 13.

$$I(z) \sim \sum_{k \geq 0} c_k z^{-k} \qquad c_k = (2k-1)!! \, [y^{2k}] \frac{\partial x}{\partial y}$$


More general: Let $\mathcal{D}$ be a finite set of integers ≥ 3
Def: $p(x) = \sum_{d\in\mathcal{D}} \frac{x^d}{d!}$
$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\varepsilon}^{\varepsilon} \exp\left(z\left(-\frac{x^2}{2}+p(x)\right)\right)dx$
Thm 13 $\Rightarrow$ $I(z) \sim \sum_{k\geq 0} c_k z^{-k}$ and $c_k = \sum_{\substack{G\in\mathcal{G}^{u} \mid \deg\in\mathcal{D} \\ \chi(G)=-k}} \frac{1}{|\mathrm{Aut}\, G|}$

Thm 13 $\Rightarrow$ $I(z) \sim \sum_{k \geq 0} c_k z^{-k}$ and $c_k = \sum_{\substack{G \in \mathcal{J}^{u} \mid \deg \in D \\ \chi(G) = -k}} \frac{1}{|\mathrm{Aut}\, G|}$

$-\frac{y^2}{2} = -\frac{x^2}{2} + \rho(x)$ solve for $x(y) \in \mathbb{Q}[[y]]$

$\rightsquigarrow c_k = (2k-1)!! \, [y^{2k}] \frac{\partial x(y)}{\partial y}$

Prop. 6.3.1 (Stanely EC Vol2)

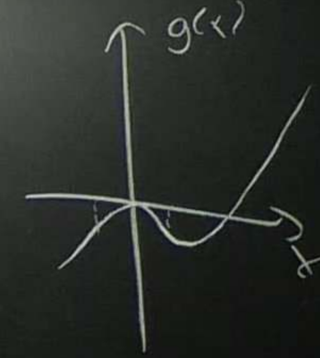
Every algebraic power series is D-finite.

$\Rightarrow$ coefficients of $x(y)$ fulfill a linear recursion eq.

$\Rightarrow$ c_k can be computed via similar recursions.

Example: (See Assignment 1.1)

$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\varepsilon}^{\varepsilon} \exp\left(z\left(-\frac{x^2}{2}+\frac{x^3}{3!}\right)\right) dx$$



E.g. $\varepsilon = \frac{1}{10}$

$$\Rightarrow I(z) = \sum_{k=0}^{R-1} c_k z^{-k} + O(z^{-R}) \quad \text{for all } R \geq 0$$

where $c_k = \sum_{G \in \mathcal{G}^{4}|_{\deg=3},\ \chi(G)=-k} \frac{1}{|\mathrm{Aut}\, G|}$

$\int_{\gamma} D\varphi\, DA_{+}\, DA_{-}\, e^{\cdots}$

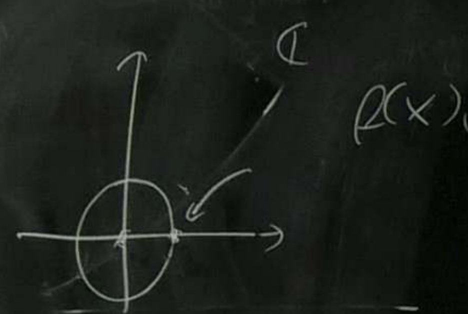
$\int D\varphi_{+} D\bar{\varphi}_{-} DA_{+} D\bar{A}_{+} DA_{-} D\bar{A}_{-}\, e^{\frac{i}{\hbar}(S-\bar{S})}$

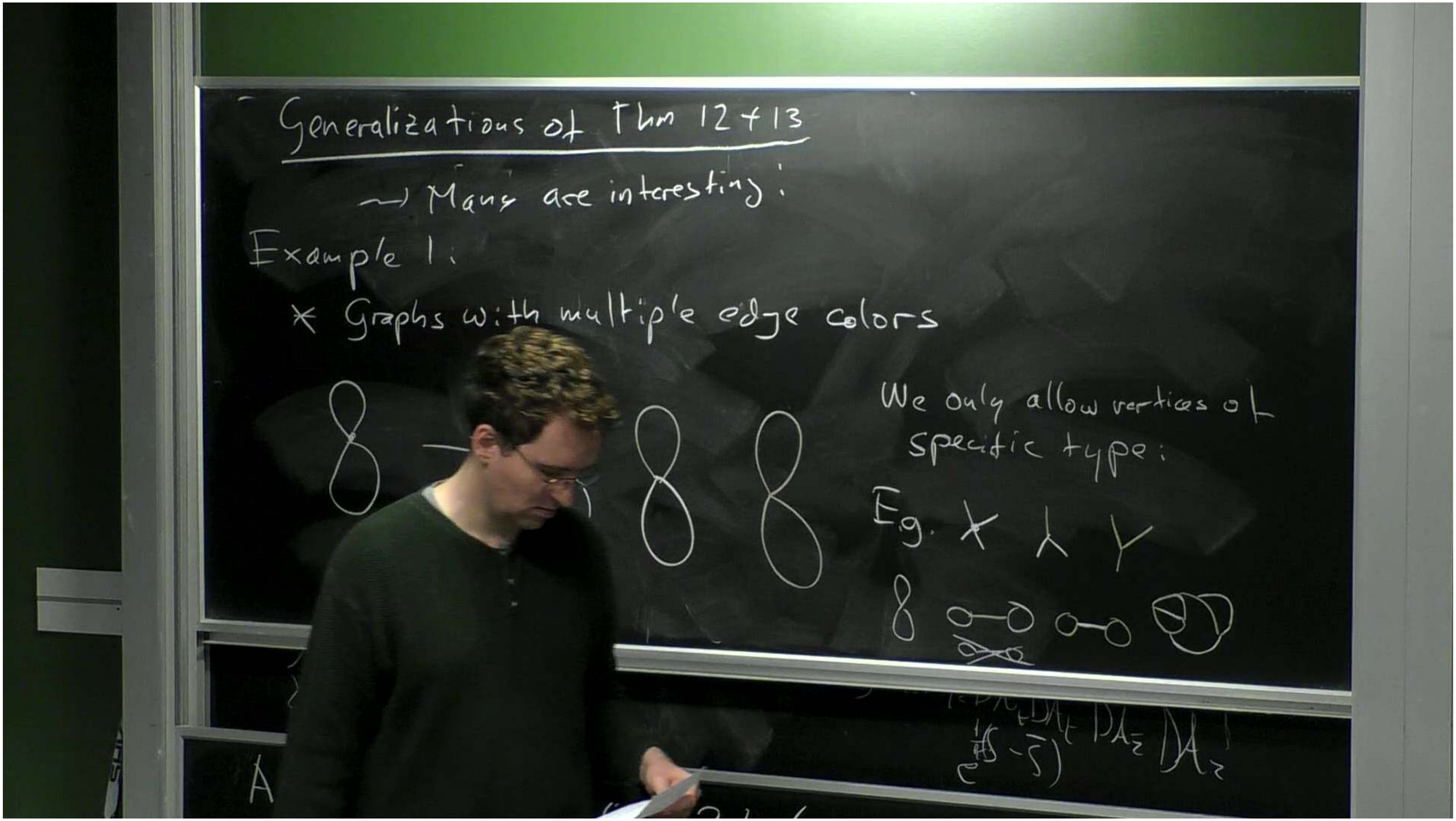
Assignment

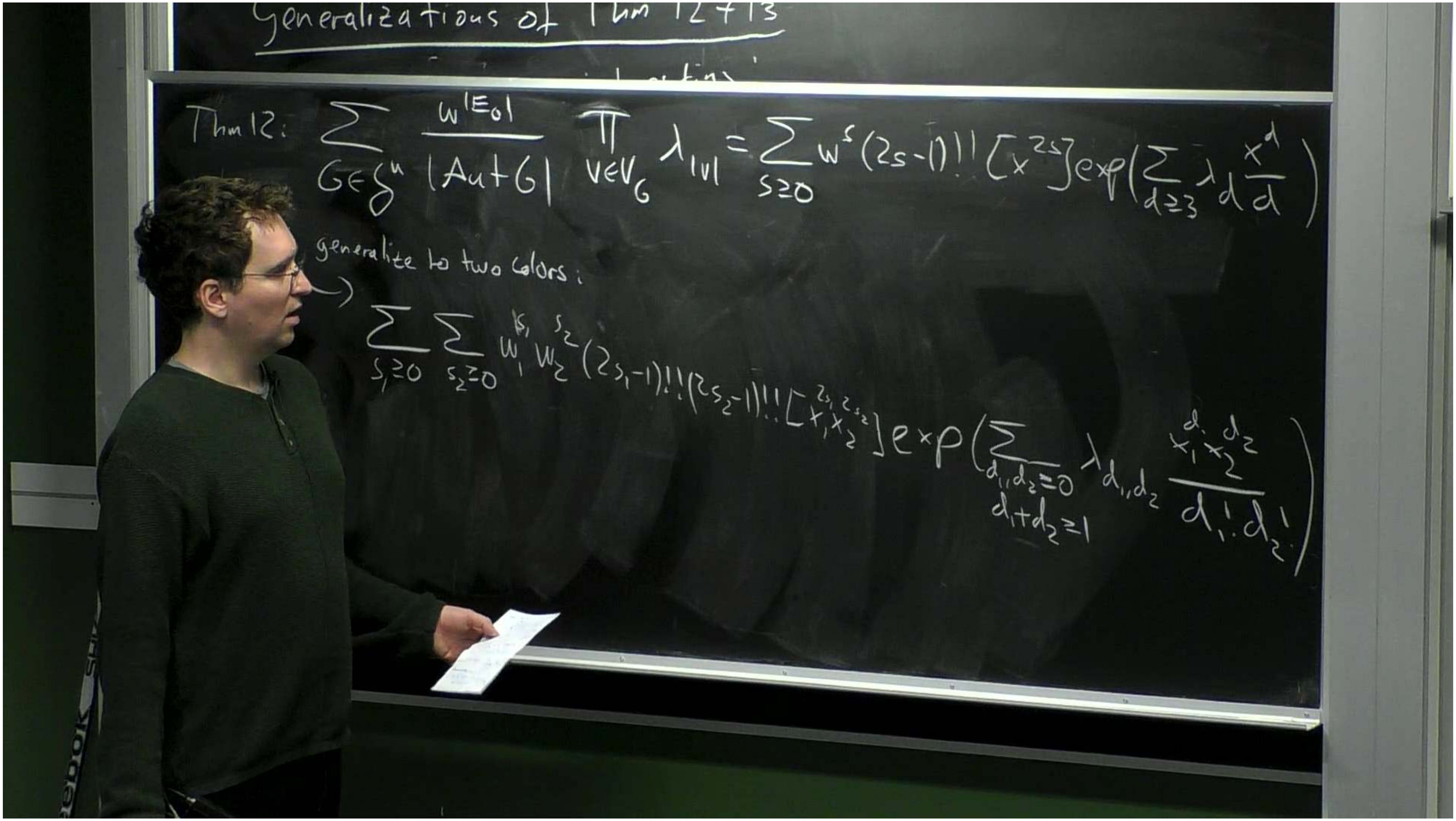
Thm 13 $\Rightarrow$ $I(z) \sim \sum_{k \geq 0} c_k z^{-k}$ and $c_k = \sum_{\substack{G \in \mathcal{G}^4(\deg \in D) \\ \chi(G) = -k}} \frac{1}{|\mathrm{Aut}\, G|}$

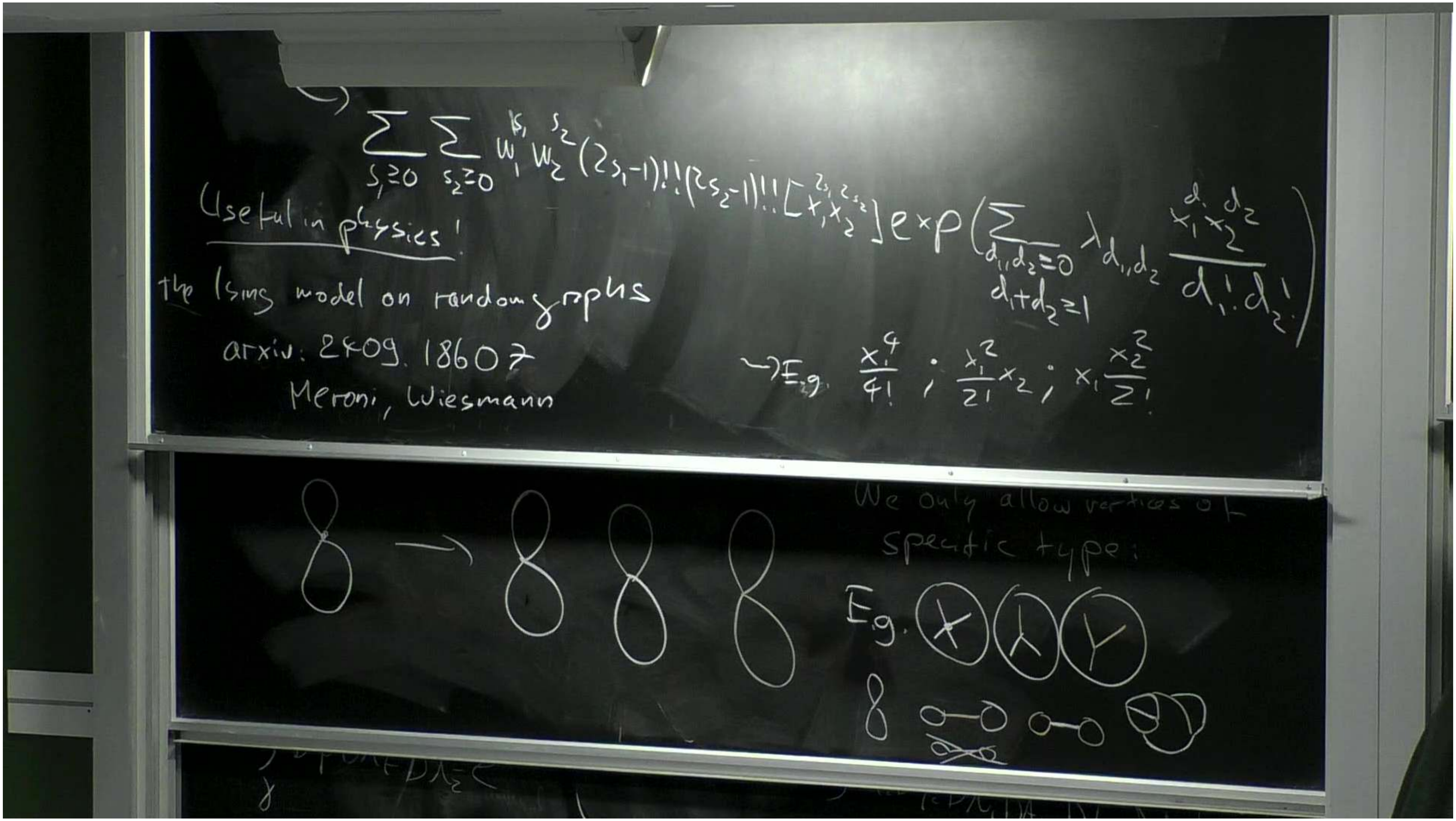
$f(x) = \sum_{n \geq 0} f_n x^n$, If $|f_n| \leq \frac{C}{r^n}$, then $f(x)$ exists for $|x| < r$.

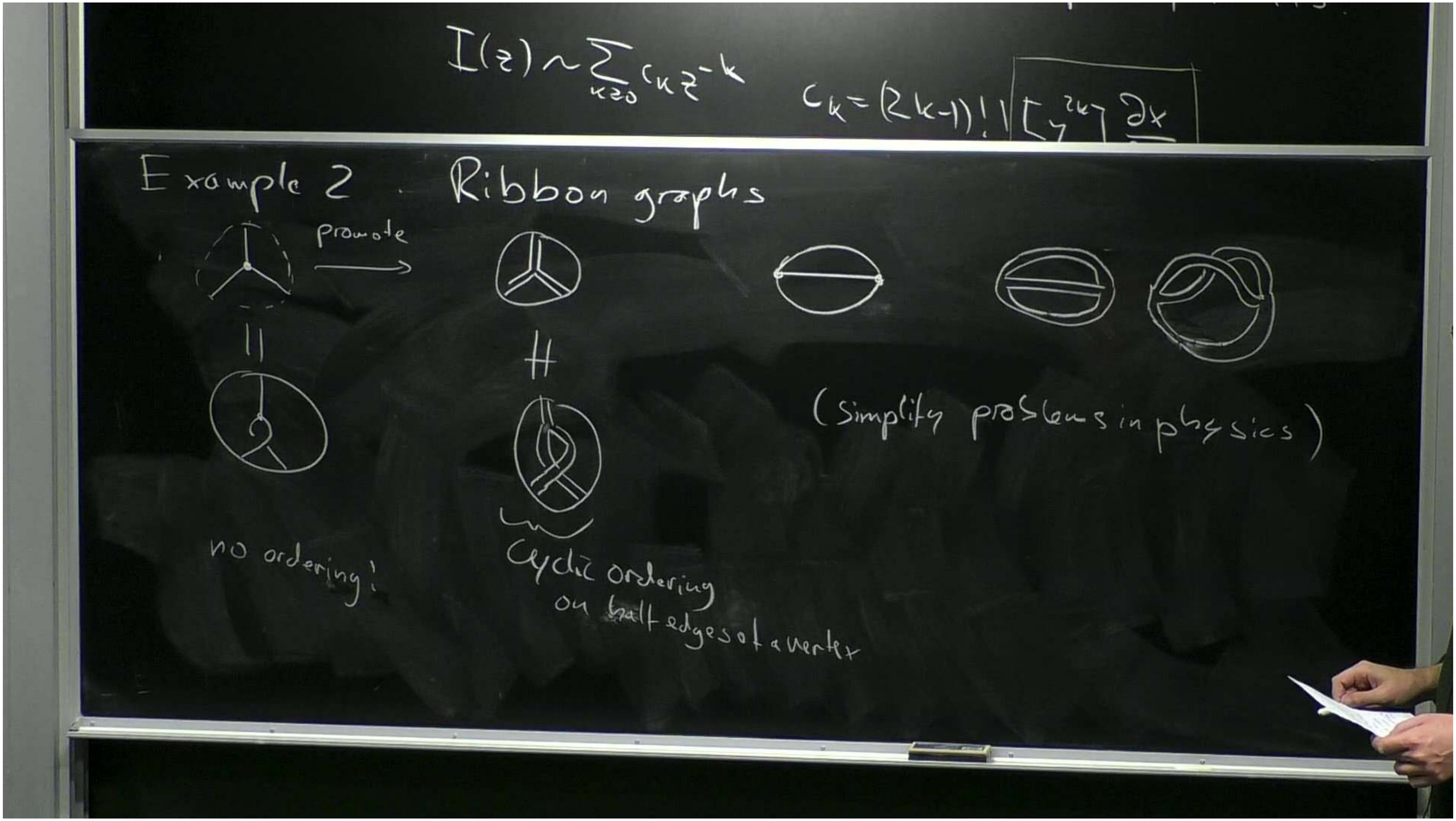
Darboux's Theorem




Generalizations of Thm 12 + 13
Many are interesting!
Example 1:
* Graphs with multiple edge colors
We only allow vertices of specific type:
E.g.


Generalizations of Thm 12 + 13
Thm 12: $\sum_{G\in \mathcal{G}^u} \frac{w^{|E_G|}}{|\mathrm{Aut}\,G|}\prod_{v\in V_G}\lambda_{|v|} = \sum_{s\ge 0} w^s (2s-1)!!\,[x^{2s}]\exp\left(\sum_{d\ge 3}\lambda_d \frac{x^d}{d}\right)$
generalize to two colors:
$\sum_{s_1\ge 0}\sum_{s_2\ge 0} w_1^{s_1} w_2^{s_2}(2s_1-1)!!(2s_2-1)!![x_1^{2s_1}x_2^{2s_2}]\exp\left(\sum_{\substack{d_1,d_2\ge 0\\ d_1+d_2\ge 1}}\lambda_{d_1,d_2}\frac{x_1^{d_1}x_2^{d_2}}{d_1!\,d_2!}\right)$


$\sum_{s_1 \geq 0} \sum_{s_2 \geq 0} w_1^{s_1} w_2^{s_2} (2s_1-1)!!(2s_2-1)!! [x_1^{2s_1} x_2^{2s_2}] \exp\left(\sum_{\substack{d_1,d_2=0 \\ d_1+d_2=1}} \lambda_{d_1,d_2} \frac{x_1^{d_1} x_2^{d_2}}{d_1! \, d_2!}\right)$
Useful in physics!
the Ising model on random graphs
arxiv: 2409.18607
Meroni, Wiesmann
$\rightsquigarrow$ E.g. $\frac{x_1^4}{4!}$; $\frac{x_1^2}{2!} x_2$; $x_1 \frac{x_2^2}{2!}$
We only allow vertices of specific type:
E.g.


I(z) ~ Σ_{k≥0} c_k z^{-k}
c_k = (2k-1)!! [y^{2k}] ∂x
Example 2. Ribbon graphs
promote
no ordering!
Cyclic ordering
on half edges of a vertex
(Simplify problems in physics)

$G = (V, E)$ $\quad$ V, E are set-partitions of H

Ribbon graphs:

* E is a set-partition of H into blocks of size.
* V is a permutation of H.

$\qquad\qquad \|$

Set-partition + cyclic order on each block computed via similar recursions.

Great trick is to upgrade the Gaussian:

$$\int e^{-\frac{x^2}{2}} x^{2n} \, dx \longrightarrow \int \exp(-\mathrm{Tr}(M^2)) \, \mathrm{Tr}(M^{2k}) \prod_{i\le j} dM_{ij}$$

Symmetric
$N \times N$ matrices

* matrix models
* random matrices
* map combinatorics

Lando-Zvonkin (book)

Eynard 2011 Formal matrix integrals...

$\rightsquigarrow$ project!?