Title: Instructor Discussion - Beautiful Papers - PHYS 773, September 12 - December 1, 2025

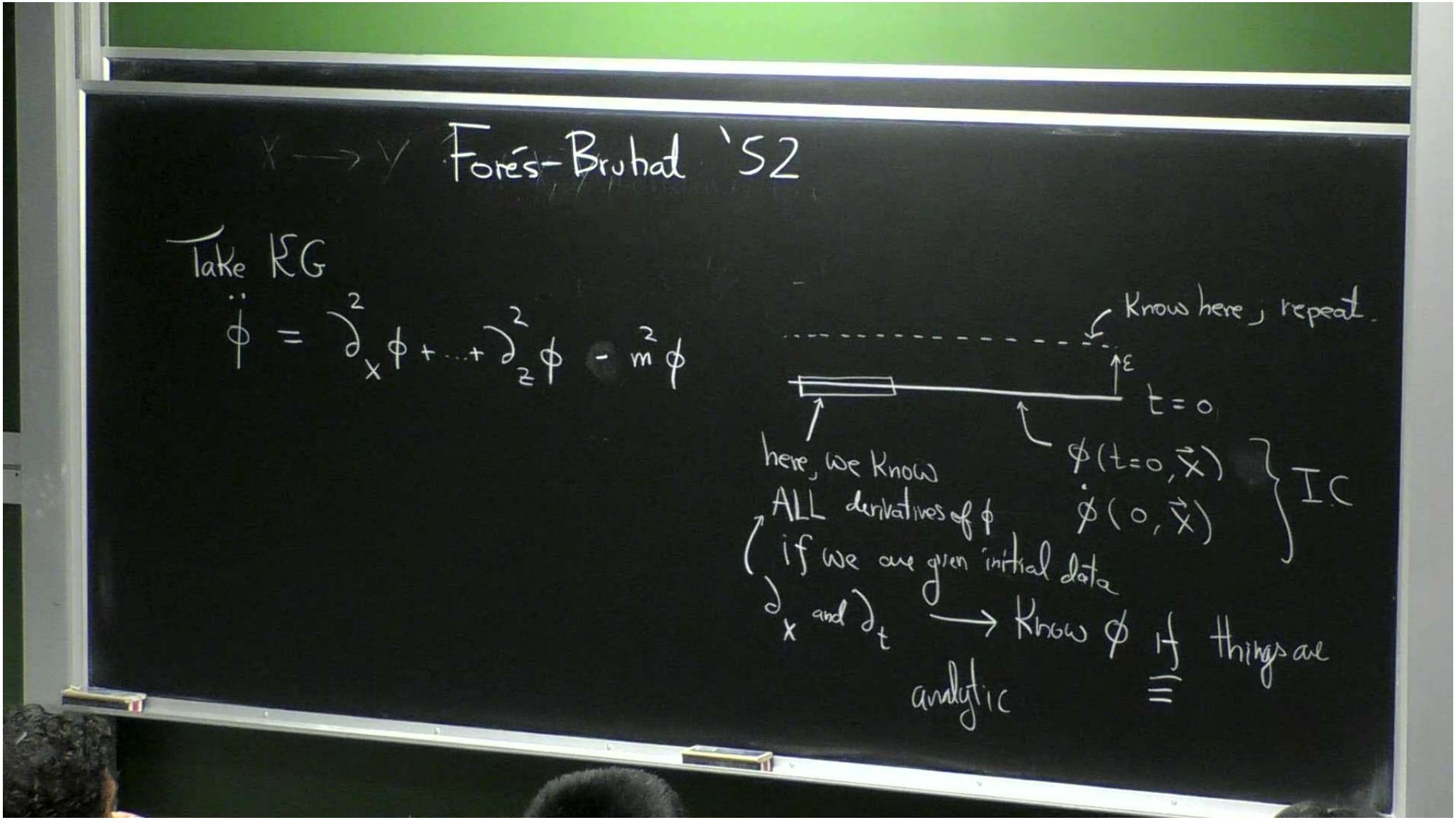
Speakers: Pedro Vieira

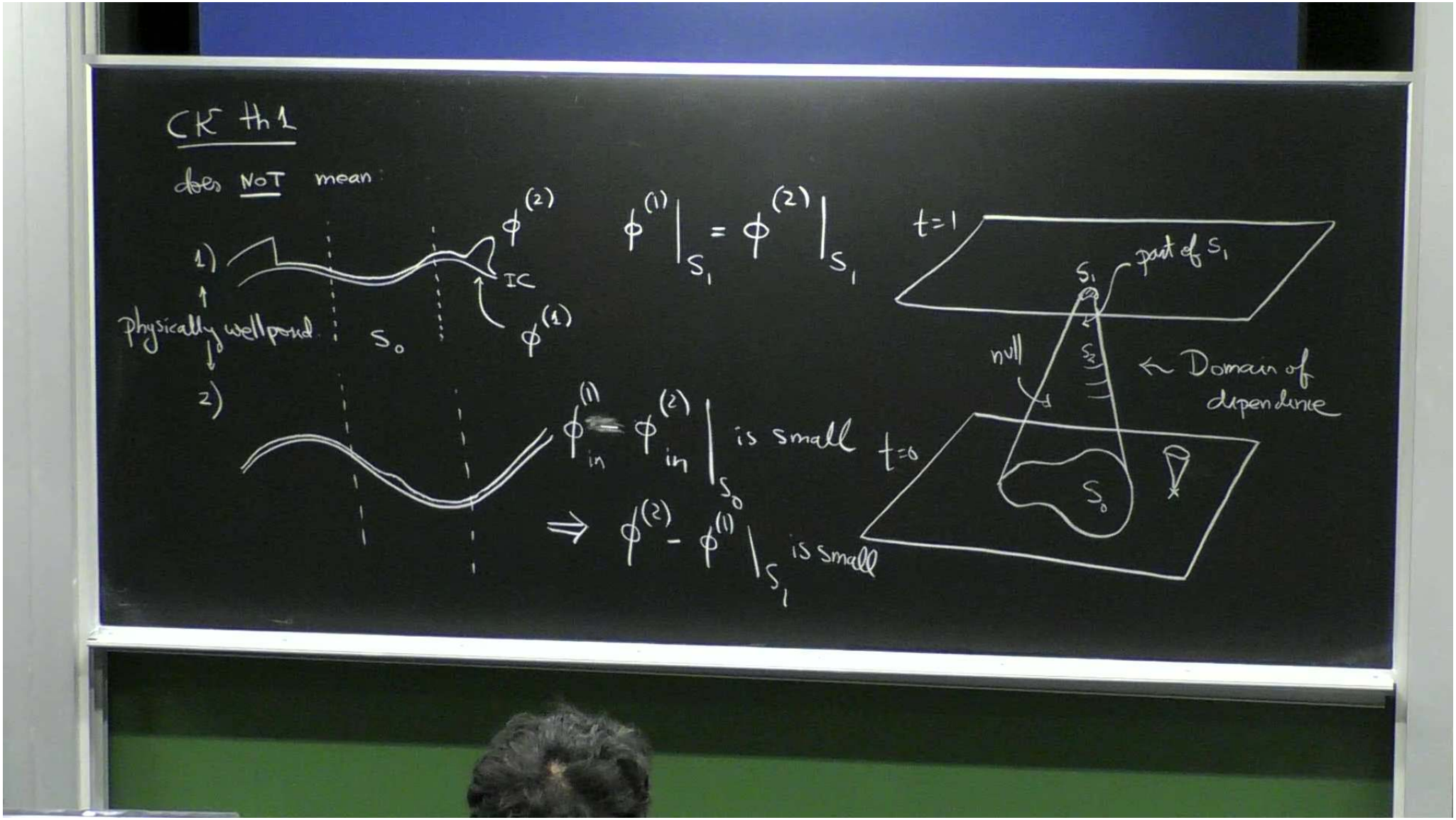
Collection/Series: Beautiful Papers, PHYS 773, September 12 - December 1, 2025

Subject: Condensed Matter, Mathematical physics, Quantum Fields and Strings, Quantum Information

Date: September 22, 2025 - 1:15 PM

URL: https://pirsa.org/25090037


Fores-Bruhat '52
Take KG
$\ddot{\phi} = \partial_x^2\phi + \ldots + \partial_z^2\phi - m^2\phi$
Know here, repeat
ε
$t=0$
here, we know
ALL derivatives of ϕ
if we are given initial data
∂_x and ∂_t
$\phi(t=0,\vec{x})$
$\dot{\phi}(0,\vec{x})$
IC
Know ϕ if things are
analytic


CK th 1
does NoT mean
1)
Physically well posed
2)
S_0
IC
$\phi^{(2)}$
$\phi^{(1)}$
$\phi^{(1)}|_{S_1} = \phi^{(2)}|_{S_1}$
$\phi^{(1)}_{in} = \phi^{(2)}_{in}\Big|_{S_0}$ is small
$\Rightarrow \phi^{(2)} - \phi^{(1)}\Big|_{S_1}$ is small
t=1
t=0
S_1
part of S_1
null
S_2
← Domain of dependence
S_0

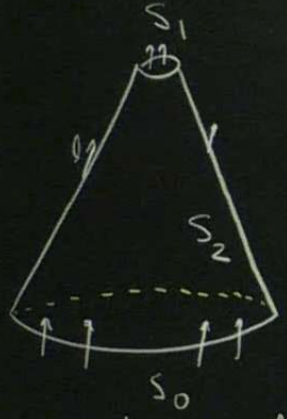
KG is well posed

$$\partial_a\phi\,\partial_b\phi - \frac{\eta_{ab}}{2}\left(\partial_c\phi\,\partial_c\phi + m^2\phi\right)$$

$(\partial/\partial t)_a$

$$\int_{S_1} T_{ab}\,\xi^a\xi^b + \int_{S_2} T_{ab}\,l^a\xi^b = \int_{S_0} T_{ab}\,\xi^a\xi^b$$

T_{00} >0 $\perp$ surface >0 >0



- Analyticity not used!
- Signs matter!

$$\int_{S_1}\left(\dot\phi^2 + |\vec\nabla\phi|^2 + m^2\phi^2\right) \le \int_{S_0} \text{same}$$

for any sol of KG, in particular for $\chi \equiv \phi^{(1)} - \phi^{(2)}$

if $\chi = 0$ @ $t=0$, in S_0

$\Rightarrow \chi = 0$ @ S_1 ∇_0

Th 1 $\ddot{\phi}_i = F_i(t, x^a, \phi_j, \partial_t \phi_j, \partial_a \phi_j, \partial_a \partial_b \phi_j)$ is good to evolve things

if they are analytic [More in Ellis Hawking / Wald books] ↖ CK (Cauchy-Kowalevski)

Th 2 $g^{ab} \partial_a \partial_b \phi_i + A_{ij}^a \partial_a \phi_j + B_{ij} \phi_j + C_i = 0$ is well posed like KG

g^{ab} (+ − − −) signature; A, B, C can be x-dependent

$(\phi^{(1)}_{S_1} = \phi^{(2)}_{S_1}$ if $\phi^{(1)}_{S_0} = \phi^{(2)}_{S_0}$ etc$)$

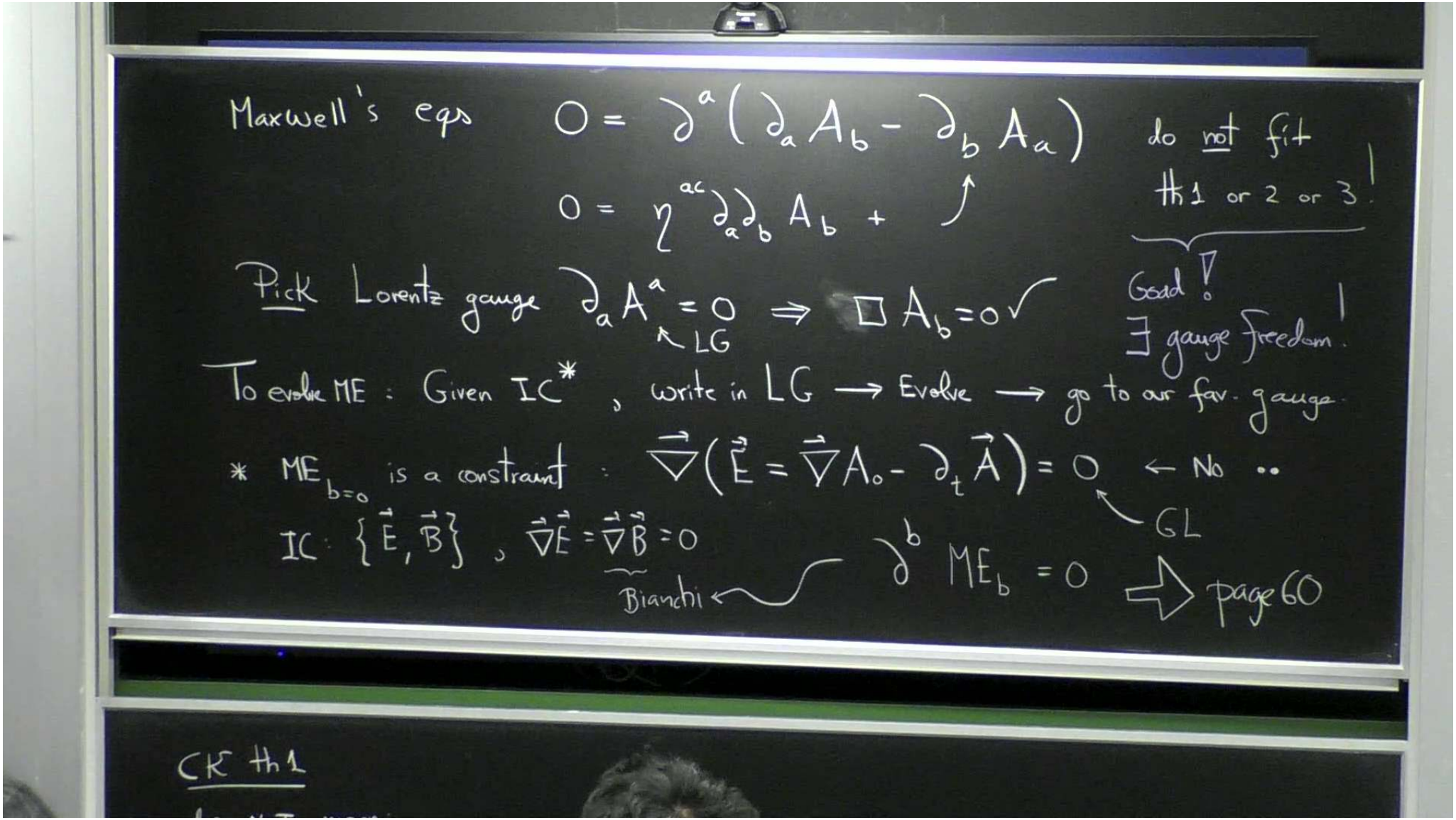
Th 3 $g^{ab}(x, \partial_a \phi_j^{n-1}, \phi_j^{n-1}) \partial_a \partial_b \phi_i^{n} = F_i(x, \phi_j^{n-1}, \partial_a \phi_j^{n-1})$

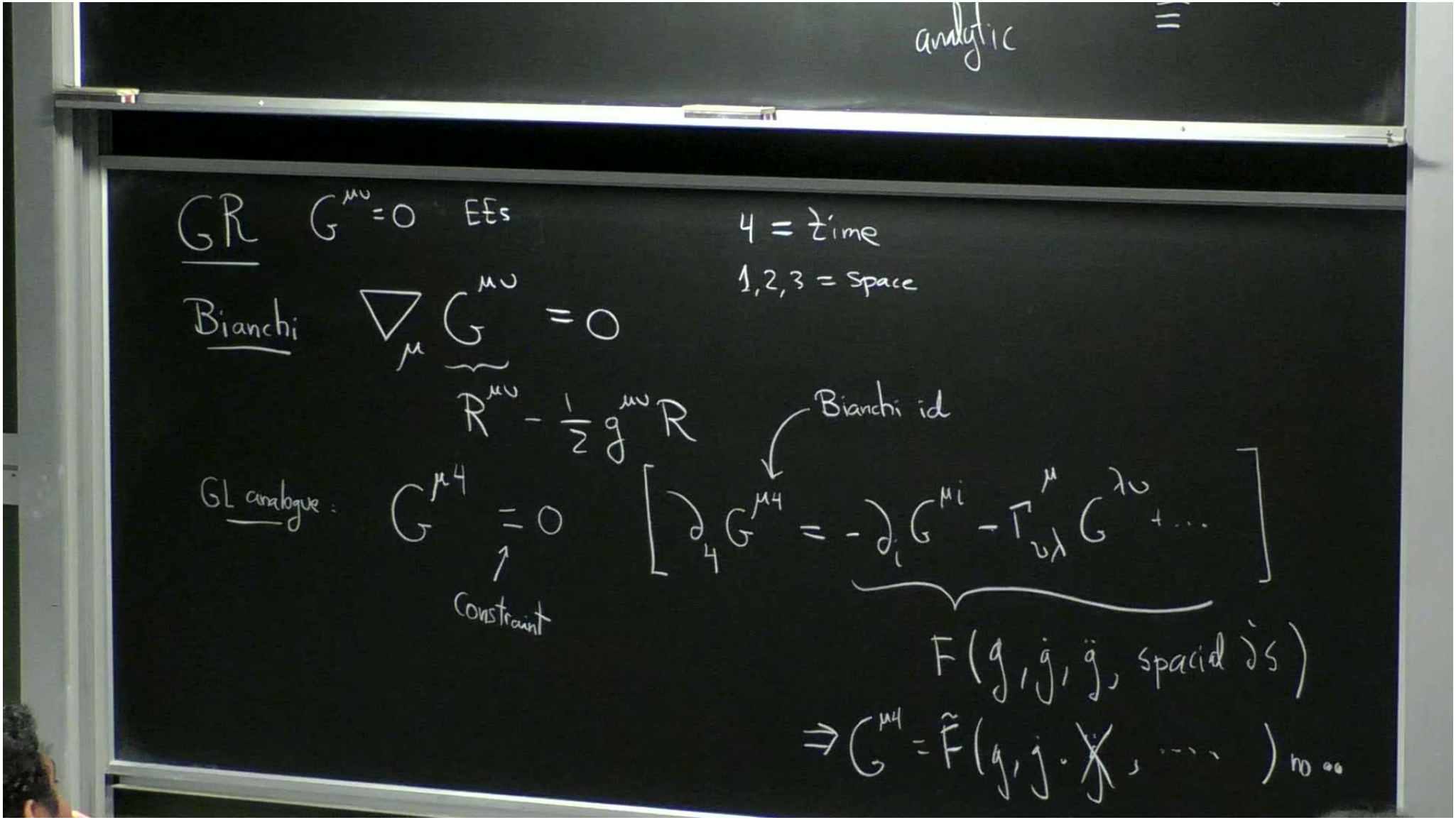
KEY: Iterations!

also in chapter 3g of the paper, see also books

Lemma: "There exists one and only one space-time given initial data $g_{\mu\nu}$ and $\dot{g}_{\mu\nu}$ on Cauchy surf S"

HARD! Evolve eqs in time yet $\mathcal{M}$ space-time, $\mathcal{M}$ is etc before evolving ...


Maxwell's eqs $0 = \partial^a(\partial_a A_b - \partial_b A_a)$
$0 = \eta^{ac}\partial_a\partial_b A_b + \ldots$
do not fit
#1 or 2 or 3!
Good!
∃ gauge freedom!
Pick Lorentz gauge $\partial_a A^a = 0$ (LG) $\Rightarrow \Box A_b = 0$ ✓
To evolve ME: Given IC*, write in LG → Evolve → go to our fav. gauge.
* $ME_{b=0}$ is a constraint: $\vec{\nabla}(\vec{E} = \vec{\nabla}A_0 - \partial_t \vec{A}) = 0$ ← No ..
GL
IC: $\{\vec{E}, \vec{B}\}$, $\vec{\nabla}\vec{E} = \vec{\nabla}\vec{B} = 0$
Bianchi
$\partial^b ME_b = 0$
page 60
CK th 1


analytic
GR $G^{\mu\nu}=0$ EEs
4 = time
1,2,3 = space
Bianchi $\nabla_\mu G^{\mu\nu}=0$
$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$
Bianchi id
GL analogue: $G^{\mu 4}=0$
Constraint
$\left[\partial_4 G^{\mu 4} = -\partial_i G^{\mu i} - \Gamma^{\mu}_{\nu\lambda} G^{\lambda\nu} + \dots\right]$
$F(g, \dot{g}, \ddot{g}, \text{spacial } \partial s)$
$\Rightarrow G^{\mu 4} = \tilde{F}(g, \dot{g}, \cancel{\ddot{g}}, \dots)$ no $\ddot{g}$

- Analyticity not used
- Signs matter!

$$\int_{S_1} \left(\phi + |\nabla\phi| + m\phi\right) \lesssim \int_{S_0} \quad \text{same}$$

for $\chi \equiv \phi - \phi$

if $\chi = 0$ @ $t=0$, in S_0

$\Rightarrow \chi = 0$ @ S_1 ?

LG $\longrightarrow$ $0 = F^\lambda \equiv \frac{1}{\sqrt{-g}} \frac{\partial(g^{\mu\nu}\sqrt{-g})}{\partial x^\mu}$

gauge $x^a \to f^a(x)$

4 gauge dof

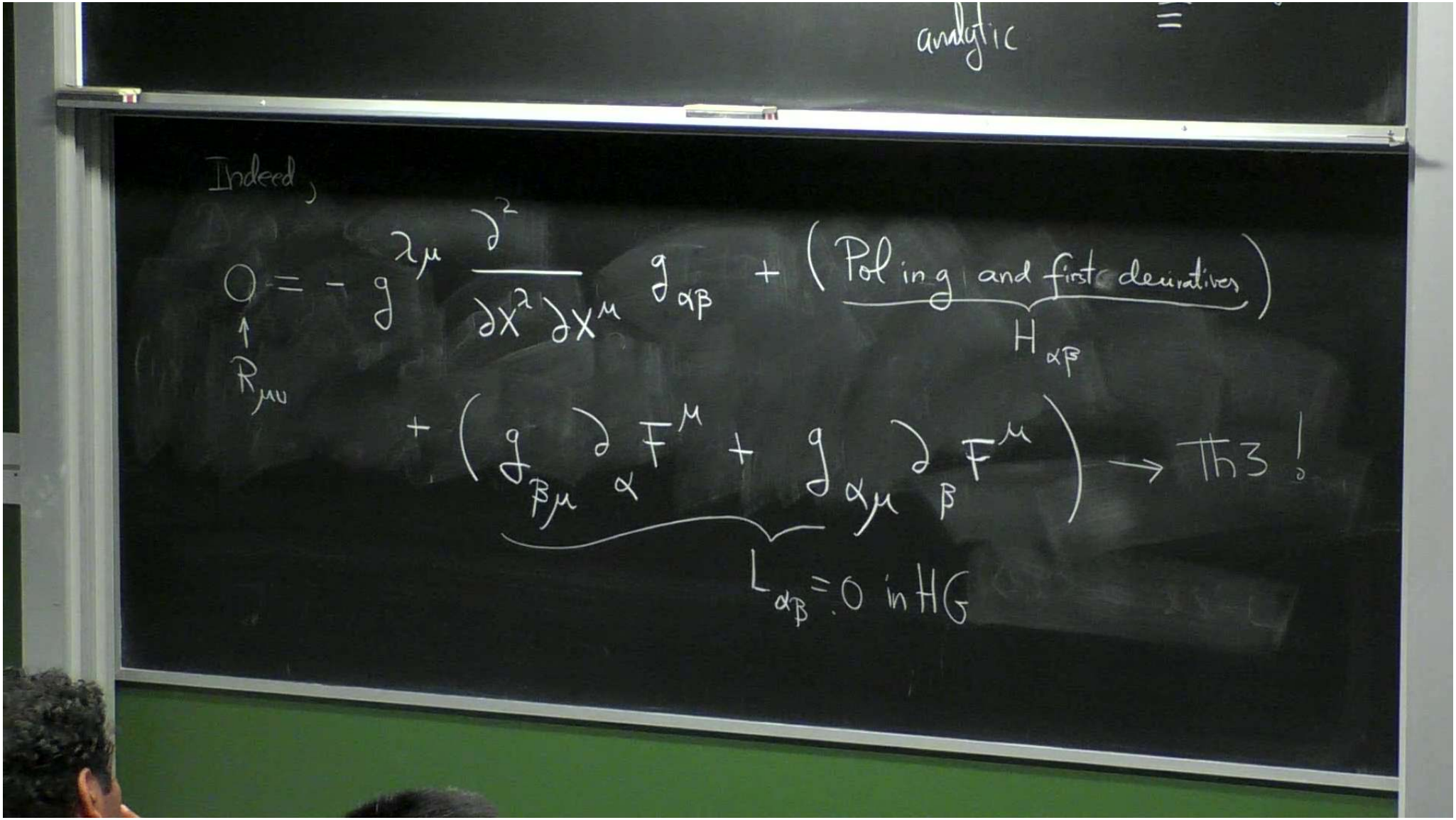
$= g^{\mu\nu}\Gamma^\lambda_{\mu\nu}$

$= \Box x^\lambda$ ← Equiv. Principle friendly coord

Isothermal or Harmonic coordinates or gauge (HG)

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\alpha\left(g^{\alpha\beta}\sqrt{-g}\,\partial_\beta\phi\right)$$

CLAIM: in HG, EE are well posed.


analytic
Indeed,
$0 = -g^{\lambda\mu}\frac{\partial^2}{\partial x^\lambda \partial x^\mu} g_{\alpha\beta} + (\text{Poling and first derivatives})$
$R_{\mu\nu}$
$H_{\alpha\beta}$
$+ \left(g_{\beta\mu}\partial_\alpha F^\mu + g_{\alpha\mu}\partial_\beta F^\mu \right) \rightarrow$ Th3 !
$L_{\alpha\beta} = 0$ in HG

- Analyticity not used
- Signs matter!

$$\int_{S_1} \left(\phi + |\nabla \phi| + m \phi \right) \lesssim \int_{S_0} \quad \text{same}$$

for $\chi = \phi - \phi'$

if $\chi = 0$ @ $t=0$, in S_0

$\Rightarrow \chi = 0$ @ S_1 ∇

Need

1) HG is preserved by $G_{\mu\nu} = 0$ evolution
2) We can always go to HG if given $(\mathcal{U}, g)$ not in HG
3) Given IC not in HG ⇝ can find IC in HG

} ⇒ Existence and Uniqueness

Maxwell: $\Box A_b = 0 \Rightarrow \Box(\partial^b A_b) = 0$

$\partial A = 0 \ \forall t$

at $t=0$ $\partial^b A_b = 0$ ⇒ $\partial A = \dot{\partial A} = 0$

need this!

GL

$$-\frac{\partial^2}{\partial t^2} A^0 + \vec{\nabla}^2 A_0 - \partial_t \left(\vec{\nabla} \vec{A} + \frac{\partial A^0}{\partial t} \right), \vec{\nabla} \left(\vec{\nabla} A_0 - \partial_t \vec{A} \right) = 0$$

IC: $\{E, B\}$, $\nabla E = \nabla B = 0$ $\Rightarrow$ $\partial^0 ME_\lambda = 0$ ↗ Page 60

1) Similar in GR:

$$0 = G^{42} \Rightarrow 0 = \Big(\underbrace{\text{something} = 0 \text{ by EOM}}_{*} + \frac{1}{2} g^{44} g_{\lambda\alpha} \underbrace{\left(\dot{F}^{\alpha}\right) = 0}_{=0 \text{ at } t=0} \Big)$$

Bianchi $\Rightarrow$

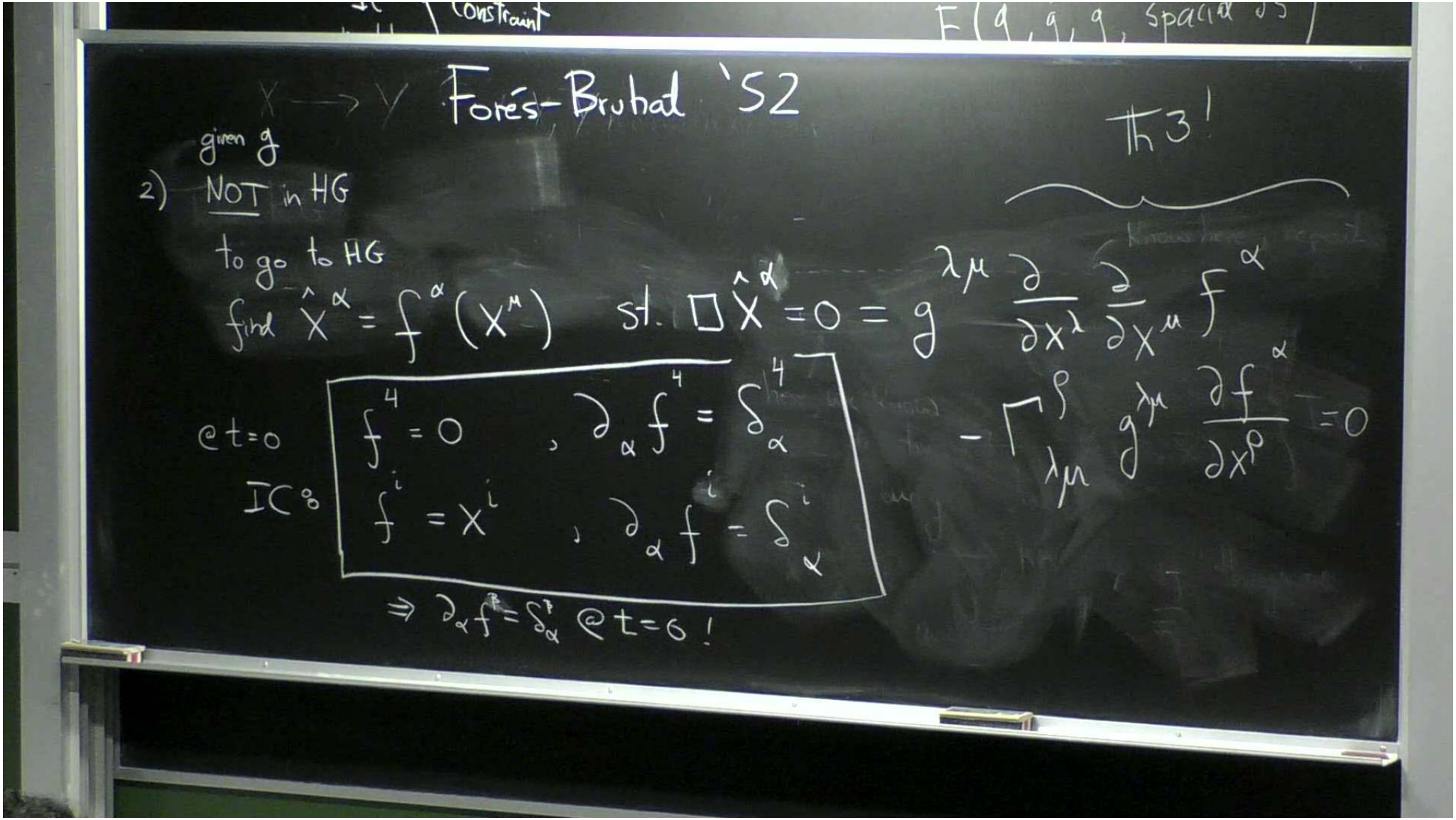
$$\Box F^{\mu} = \text{lin comb of } \partial F$$

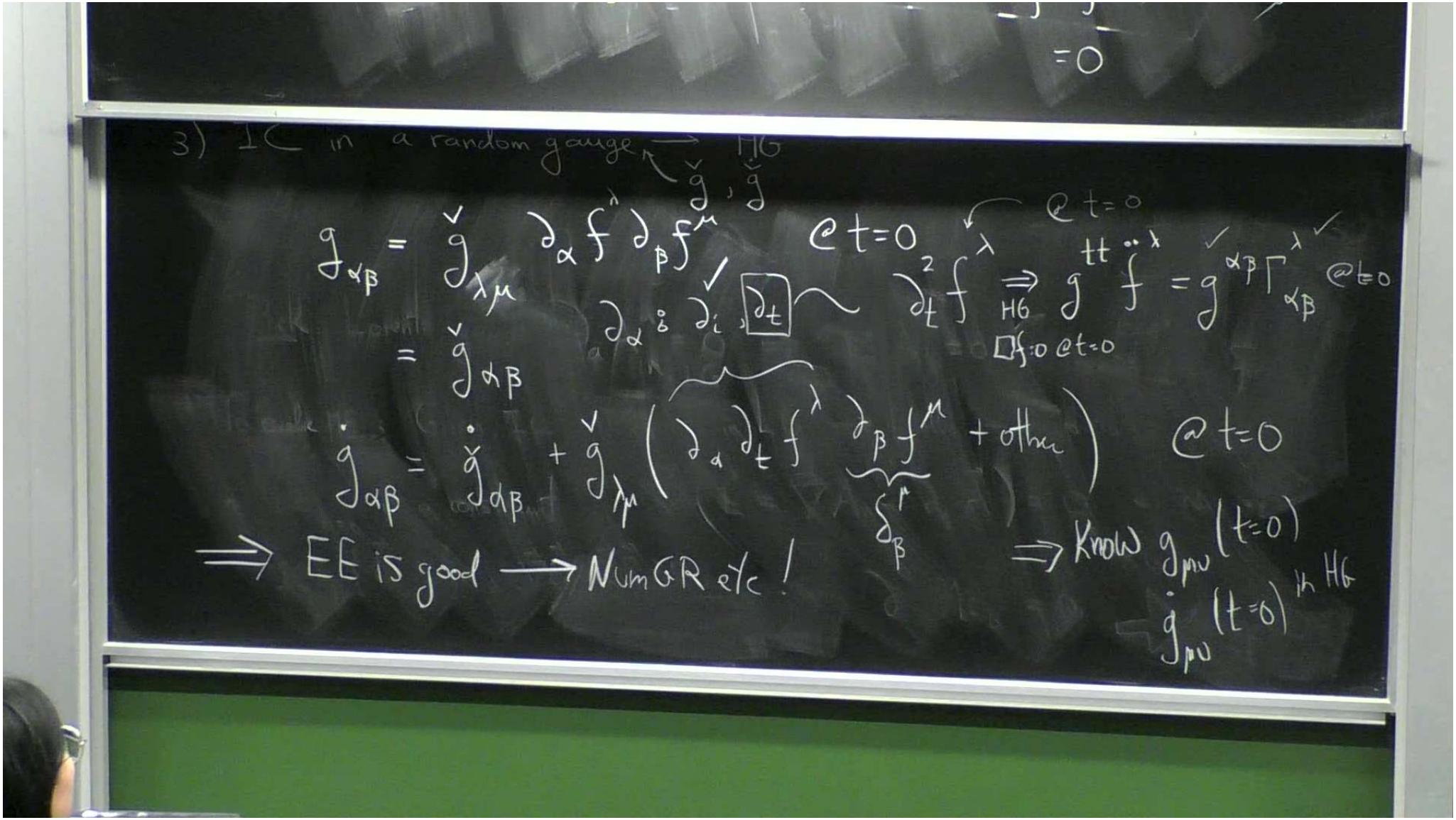
$$\left(\partial_t^2 - \partial_x^2\right) f = 0$$

$\Downarrow$ and $f(0, x) = 0$ and $\dot{f} = 0$

$f = f(x + vt) - f(x - vt)$

$= 0$


constraint
Fores-Bruhat '52
Th 3!
given g
2) NOT in HG
to go to HG
find $\hat{X}^\alpha = f^\alpha(X^\mu)$ st. $\Box \hat{X}^\alpha = 0 = g^{\lambda\mu} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} f^\alpha - \Gamma^\rho_{\lambda\mu} g^{\lambda\mu} \frac{\partial f^\alpha}{\partial x^\rho} = 0$
@ t=0
IC's
$f^4 = 0 \quad , \quad \partial_\alpha f^4 = \delta^4_\alpha$
$f^i = X^i \quad , \quad \partial_\alpha f^i = \delta^i_\alpha$
$\Rightarrow \partial_\alpha f^\beta = \delta^\beta_\alpha$ @ t=0 !


=0
3) IC in a random gauge → HG
$\check{g}, \dot{\check{g}}$
$g_{\alpha\beta} = \check{g}_{\lambda\mu}\, \partial_\alpha f^\lambda \partial_\beta f^\mu$ @t=0
$= \check{g}_{\alpha\beta}$
$\dot{g}_{\alpha\beta} = \dot{\check{g}}_{\alpha\beta} + \check{g}_{\lambda\mu}\left(\partial_\alpha \partial_t f^\lambda\, \partial_\beta f^\mu + \text{other}\right)$ @t=0
$\partial_\alpha \overset{\circ}{,} \partial_i, \boxed{\partial_t} \sim$
$\partial_t^2 f^\lambda \overset{HG}{\Rightarrow} g^{tt} \ddot{f}^\lambda = g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}$ @t=0
$\Box f = 0$ @t=0
$\Gamma^\lambda_{\alpha\beta}$
$\Rightarrow$ EE is good → Num GR etc!
$\Rightarrow$ Know $g_{\mu\nu}(t=0)$, $\dot{g}_{\mu\nu}(t=0)$ in HG