Title: Instructor Discussion - Beautiful Papers - PHYS 773, September 12 - December 1, 2025

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Collection/Series: Beautiful Papers, PHYS 773, September 12 - December 1, 2025

Subject: Condensed Matter, Mathematical physics, Quantum Fields and Strings, Quantum Information

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Wigner '32

$$P(x,p)\,dx\,dp = e^{-\beta\underbrace{\left(\frac{p^2}{2m}+V(x)\right)}_{\epsilon(p,x)}}\,dx\,dp$$

GOAL

Classically

$$\hat{O} = \hat{x}^2 \longrightarrow f_{\hat{O}}(x,p)$$

$\hat{p}^2$

$\hat{x}\hat{p}$

$$f_{\hat{x}^2}(x,p) = x^2$$

$$f_{\hat{p}^2} = p^2$$

$$f_{\hat{x}\hat{p}} = xp + i\hbar/2 \ \ldots$$

Such that

$$\langle \hat{O} \rangle = \int dx\,dp\, f_{\hat{O}}(x,p)\; W(x,p)$$

2

numbers

Wigner dist

(P in the paper)

$$\langle \psi | \hat{O} | \psi \rangle \text{ or } \mathrm{tr}(\rho \hat{O})$$

$$\text{or } \mathrm{tr}(e^{-\beta \hat{H}} \hat{O})$$

Proposal

$$W(x,p) \overset{|\psi\rangle}{\equiv} \frac{2}{\hbar} \int dy \; \psi^*(x+y)\,\psi(x-y)\, e^{2iyp/\hbar}$$

Indeed,
$$\int dp \; W(p,x) = \mathbb{C} \, |\psi(x)|^2$$
$$\int dx \; W(p,x) = \mathbb{C} \, |\hat{\psi}(p)|^2$$

GOAL ✓, $\hat{O} = f(\hat{x}) + h(\hat{p})$, map $f_{\hat{O}} = f(x) + h(p)$ ✓

GOAL: $\hat{O} = f(\hat{x}) + h(\hat{p})$, map $\hat{O} \to f(x) + h(p)$ ← GOAL

Proposal

$$W(x,p) \stackrel{|\psi\rangle}{\equiv} \frac{2}{\hbar} \int dy \; \psi^*(x+y)\,\psi(x-y)\, e^{2iyp/\hbar}$$

* for a mixed state

$\rho = \sum w_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|$, $\quad W \to \sum w_\lambda W_\lambda$

* W can be < 0 ! ⇝ not a prob.

$\hat{p}^2$ ① $f_{\hat{p}^2} = p^2$

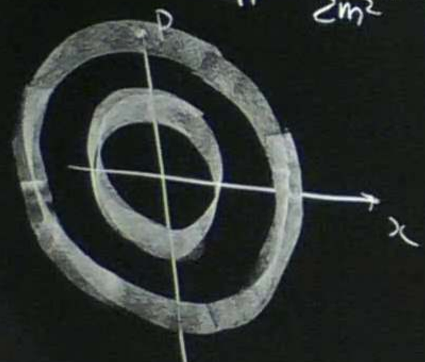
Example

Laguerre pol

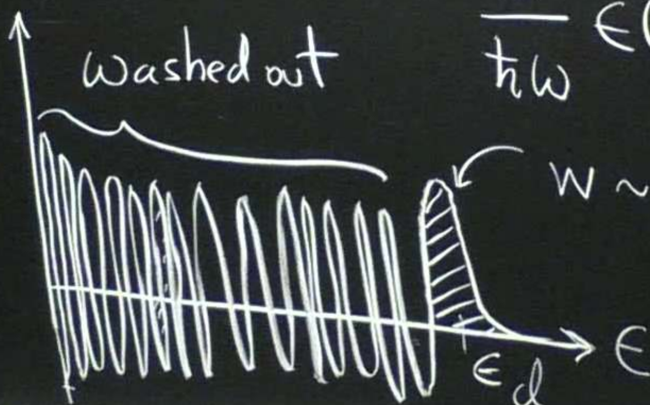
$$W_{n\text{-th excited state of HO}} = \frac{(-1)^n}{\pi\hbar} e^{-\xi} L_n(2\xi)$$

$$\epsilon(x,p) = \frac{p^2}{2m^2} + \frac{1}{2} m\omega^2 x^2$$

$$\frac{2}{\hbar\omega}\epsilon(x,p)$$



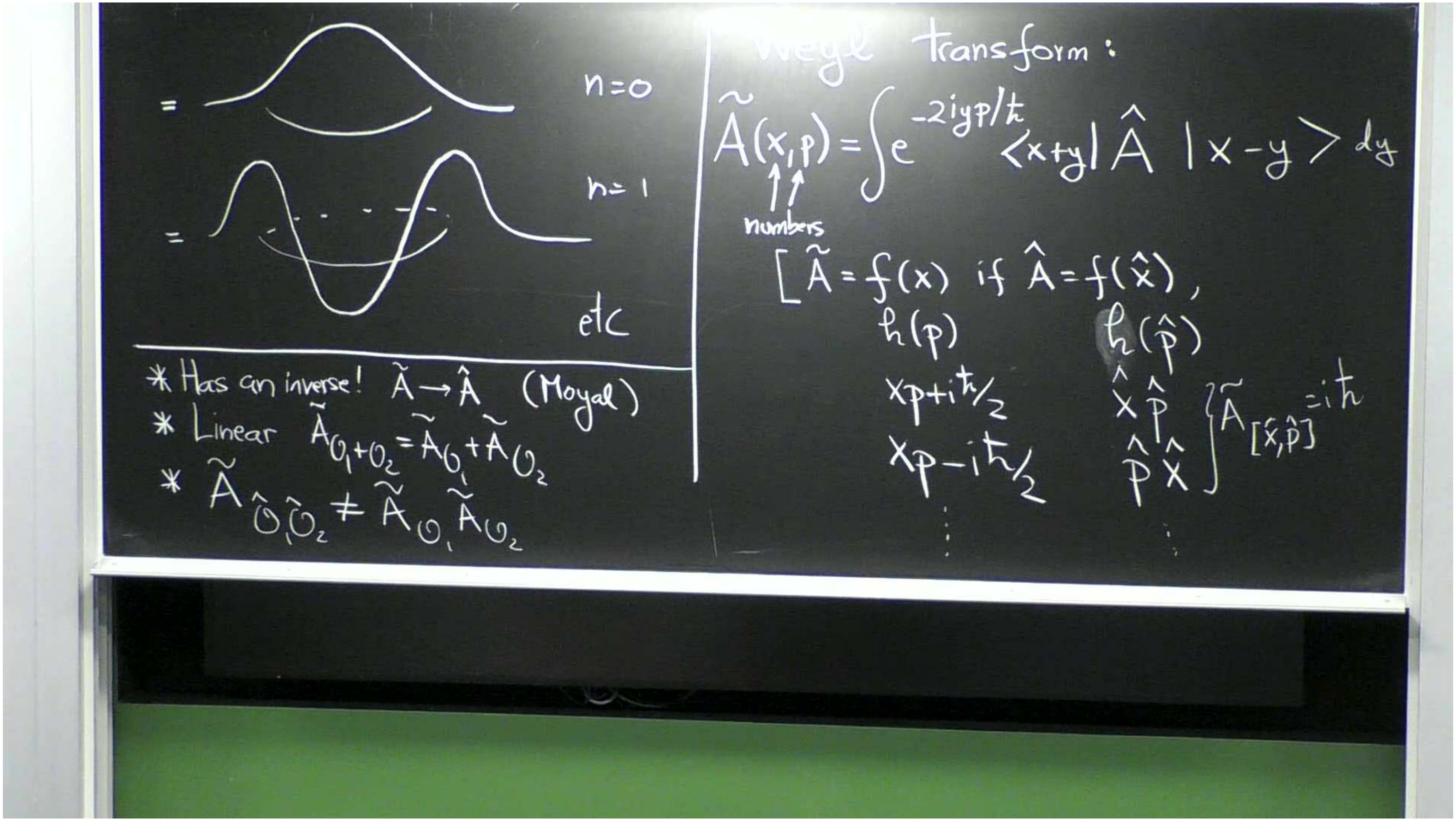
$=$ $n \gg 1$ n odd



$$W \sim \delta(\epsilon - E_{cl})$$

SAME AS DIST IN CL LIM!

$$W_{\psi_{WKB}} = \mathrm{osc} + \delta(\ldots)$$


$n=0$
$n=1$
etc
* Has an inverse! $\tilde{A} \to \hat{A}$ (Moyal)
* Linear $\tilde{A}_{O_1+O_2} = \tilde{A}_{O_1} + \tilde{A}_{O_2}$
* $\tilde{A}_{\hat{O}_1\hat{O}_2} \neq \tilde{A}_{O_1}\tilde{A}_{O_2}$
Weyl transform:
$\tilde{A}(x,p) = \int e^{-2iyp/\hbar} \langle x+y|\hat{A}|x-y\rangle\, dy$
numbers
$[\tilde{A} = f(x)$ if $\hat{A} = f(\hat{x})$,
$h(p)$ $h(\hat{p})$
$xp + i\hbar/2$ $\hat{x}\hat{p}$
$xp - i\hbar/2$ $\hat{p}\hat{x}$
$\tilde{A}_{[\hat{x},\hat{p}]} = i\hbar$

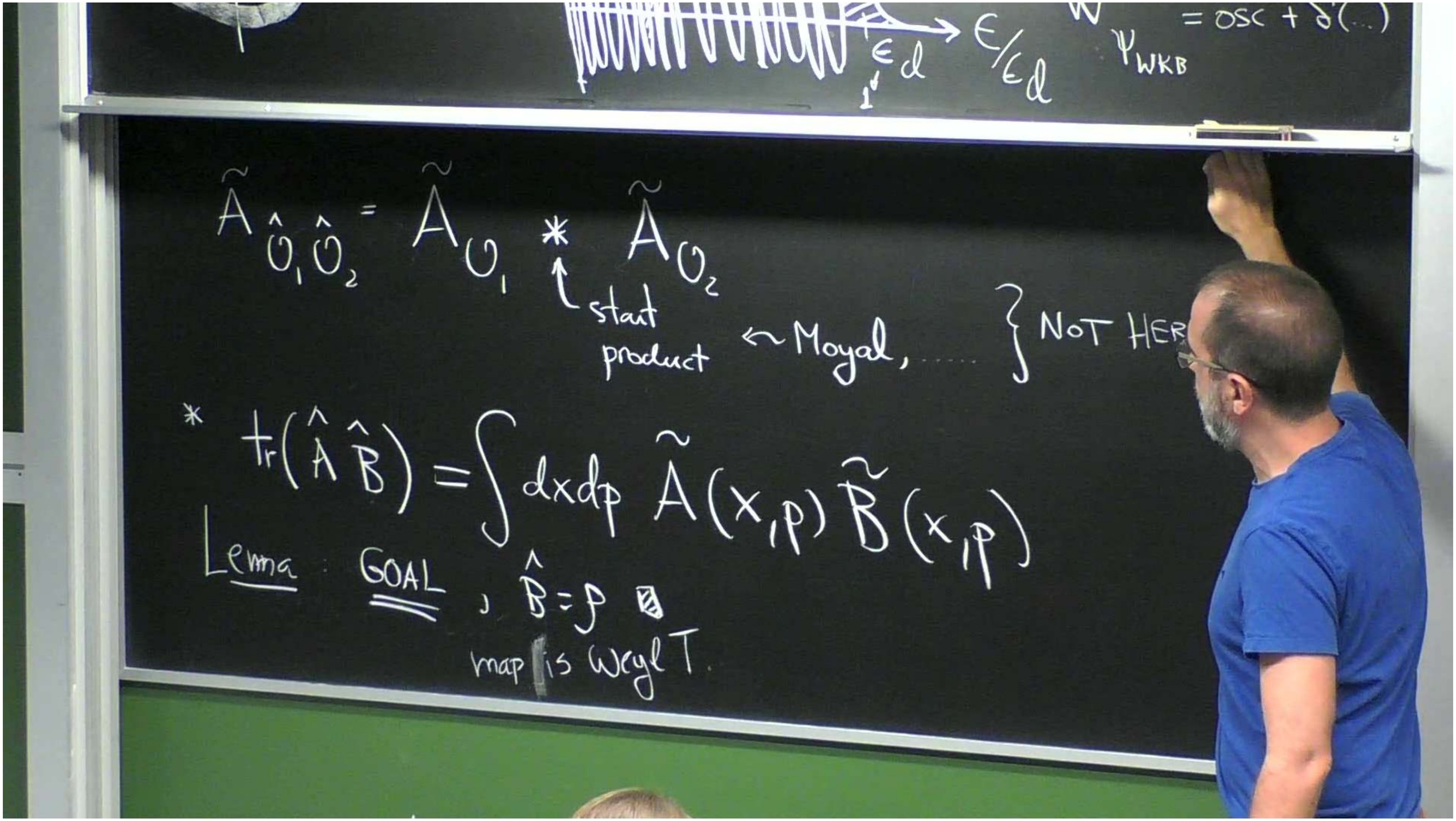
$\epsilon_d \quad \epsilon / \epsilon_d \quad \psi_{WKB} \quad = \mathrm{osc} + \delta(\ldots)$

$$\tilde{A}_{\hat{O}_1 \hat{O}_2} = \tilde{A}_{O_1} * \tilde{A}_{O_2}$$

star product ← Moyal, ... } NOT HERE.

$$* \quad \mathrm{tr}(\hat{A}\hat{B}) = \int dx\,dp\; \tilde{A}(x,p)\,\tilde{B}(x,p)$$

Lemma : GOAL , $\hat{B} = \rho$ ∎



ε/εd
ψWKB
= osc + δ(...)
Ã Ô1Ô2 = Ã O1 * Ã O2
star product ← Moyal, ... } NOT HER
* tr(Â B̂) = ∫ dxdp Ã(x,p) B̃(x,p)
Lemma : GOAL , B̂ = ρ
map is Weyl T.

GOAL: $\hat{O} = f(\hat{x}) + h(\hat{p})$, map f; $\hat{O} \to f(x) + h(p)$ GOAL

Time Evolution

Ψ obey SE $\Rightarrow$ W obeys ..... ↙ $\hbar$ is here!

$$\underbrace{\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial W}{\partial p}}_{0 \,=\, \frac{\partial W}{\partial t} + \{H, W\} \,=\, \frac{dW}{dt} \,=\, \text{classical Liouville}} = \underbrace{\sum_{k=3,5,7\ldots} \frac{(\hbar/2i)^{k-1}}{k!} \frac{\partial^k V}{\partial x^k} \frac{\partial^k W}{\partial p^k}}_{\text{"Quantum viscosity"}}$$

cl L.E.

$$\frac{\partial W}{\partial t} + \frac{1}{i\hbar}\left(H * W - W * H\right) = 0$$

$* \ \tilde{A}_{\hat{O}_1 \hat{O}_2} \neq \tilde{A}_{\hat{O}_1} \tilde{A}_{\hat{O}_2}$

Thermal State

OP, Number

$$W(x,p) = \int dy \, \langle x+y | \underbrace{e^{i\hat{x}p\hbar} \rho(\hat{x},\hat{p}) e^{-i\hat{x}p\hbar}}_{e^{-\beta \underbrace{e^{i\hat{x}p\hbar} H(\hat{x},\hat{p}) e^{-i\hat{x}p\hbar}}_{\epsilon(x,p) + \underbrace{\left[i\frac{\hbar p}{m}\frac{\partial}{\partial x} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \right]}_{\mu \leftarrow \hbar \text{ is here}}}}} | x-y \rangle$$

$$= e^{-\beta \epsilon(x,p)}$$

numbers

$$* \quad \hat{A}_{\hat{O}_1 \hat{O}_2} \neq \tilde{A}_{O_1} \tilde{A}_{O_2}$$

Thermal State

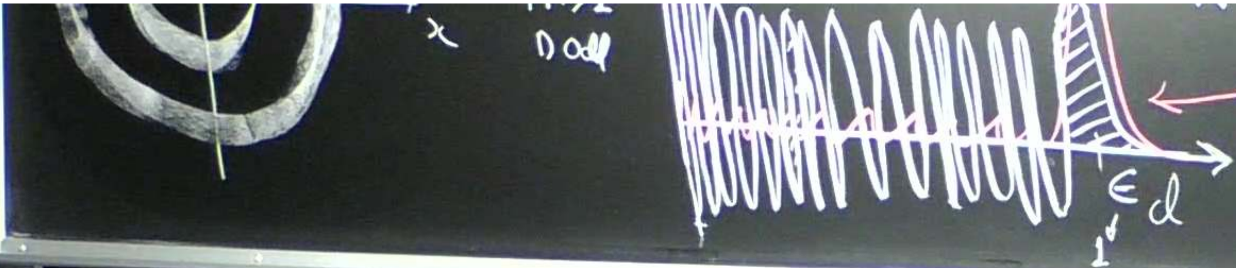
$$W(x,p) = \int dy \, \langle x+y | \, e^{i \overset{\text{OP}}{\hat{x}} \overset{\text{Number}}{p} / \hbar} \, \underbrace{\rho(\hat{x},\hat{p}) \, e^{-i \hat{x} p/\hbar}}_{} \, | x-y \rangle$$

$$\parallel$$

$$e^{-\beta \epsilon(x,p)} \left[1 + \hbar^2 w_2 + \hbar^4 w_4 + \dots \right] \qquad w_2(x,p) \text{ numbers!}$$

$$e^{-\beta \, \underbrace{e^{i \hat{x} p/\hbar} H(\hat{x},\hat{p}) e^{-i \hat{x} p/\hbar}}_{\epsilon(x,p) + \underbrace{\left[i \frac{\hbar p}{m} \frac{\partial}{\partial x} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right]}_{\hbar \text{ is here}}}}$$

numbers



ϵ_d ϵ/ϵ_d

W $\Psi_{WKB} = osc + \delta(\ldots)$

Wigner 32

$$P(x,p)\,dx\,dp = e^{-\beta\left(\underbrace{\frac{p^2}{2m}+V(x)}_{\epsilon(p,x)}\right)}\,dx\,dp$$

Classically ⟶ QMly? Answers ⇒

$$w_2(x,p) = -\frac{\beta^2}{8m}V''(x)\left[1-\frac{\beta}{3m}p^2\right]-\frac{\beta^3}{24m}V'(x)^2$$

up to 2 V's and 2 ∂'s ⟵ EFT in QM...

$$\frac{m\omega^2}{2}\langle x^2\rangle_{HO} = \underbrace{\frac{KT}{2}}_{cl.} + \frac{\hbar\,\omega^2}{24KT} + \cdots\cdots = \frac{\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\frac{1}{2m}\langle p^2\rangle_{HO} = \text{same}$$

GOAL ; $\hat{O} = f(\hat{x}) + h(\hat{p})$; map $\hat{O} = f(x) + h(p)$ GOAL

<u>Time Evolution</u> Ψ obey SE $\Rightarrow$ W obeys ..... $\hbar$ is here!

$$\underbrace{\overbrace{\frac{\partial W}{\partial t}}^{=0 \text{ in thermal}} + \frac{P}{m}\frac{\partial W}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial W}{\partial p}} = \underbrace{\sum_{K=3,5,7\ldots} \frac{(\hbar/2i)^{K-1}}{K!} \frac{\partial^K V}{\partial x^K} \frac{\partial^K W}{\partial p^K}}_{\text{"Quantum viscosity"}}$$

$$0 = \frac{\partial W}{\partial t} + \{H, W\} = \frac{dW}{dt} = \text{classical Liouville}$$

CL L.E.

$$\frac{\partial W}{\partial t} + \frac{1}{i\hbar}\left(H * W - W * H\right) = 0$$