Title: Lecture - Quantum Theory (Core), PHYS 605

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Subject: Quantum Foundations

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Canonical Quantization

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

$$\varphi(\vec{x},t) = \underbrace{\int\frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}}_{\text{Lorentz-invariant measure}}\left(\underbrace{a(\vec{k})}_{\substack{\text{annihilation operator}\\\text{destroys particles}}}e^{-ik\cdot x} + \underbrace{a^\dagger(\vec{k})}_{\substack{\text{creation operator}\\\text{creates particles}}}e^{ik\cdot x}\right)\Bigg|_{k^0=E_{\vec{k}}}$$

$\varphi(\vec{x},t) \downarrow$ field operator

$$\int \frac{d^4k}{(2\pi)^4}\, \delta(\underbrace{k^2 - m^2}_{k^{0\,2} - \vec{k}^2 - m^2})\, \Theta(k^0) = \int \frac{d^3k}{(2\pi)^3 2E_k} \quad \text{with } k^0 = E_{\vec{k}}$$

step function

Invariant under Lorentz that preserve arrow of time

$$k^0 = \pm E_{\vec{k}}$$

$$E_{\vec{k}}^2 = \vec{k}^2 + m^2$$

$$\delta(k^{0\,2} - E_{\vec{k}}^2) = \frac{\delta(k^0 - E_{\vec{k}})}{2E_{\vec{k}}} + \frac{\delta(k^0 + E_{\vec{k}})}{2E_{\vec{k}}}$$

$$\int \delta(f(x))\, dx = \int \delta(y) \frac{dy}{|f'(x)|} \rightarrow \boxed{\delta(f(x)) = \sum_{x_0} \frac{\delta(x - x_0)}{|f'(x_0)|} \quad \text{where } f(x_0) = 0}$$

$$y = f(x)$$

1) Choose $\mathcal{L}$. We choose $\mathcal{L}_{KG}$

2) Find $\pi(\vec{x},t)$ $(+H)$

conjugate momentum to $\varphi(\vec{x},t)$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x)$$

$$H = \int (\pi \dot{\varphi} - \mathcal{L})\, d^3x$$

$$= \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right)$$

3)

3) Impose $[\,,\,]$

QM: $[q_a, p_b] = i\delta_{ab}$

$$[q_a, q_b] = 0 = [p_a, p_b]$$

QFT: $[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta(\vec{x}-\vec{y})$ Schrödinger

$$[\hat{\varphi}(\vec{x}), \hat{\varphi}(\vec{y})] = 0 = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})]$$

$a(\vec{k}),\ a^*(\vec{k}) \to a_{\vec{k}},\ a^\dagger_{\vec{k}}$

operators

$$[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3\, 2E_{\vec{k}}\, \delta(\vec{k}-\vec{p})$$

$$[a_{\vec{k}}, a_{\vec{p}}] = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{p}}]$$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\, 2E_{\vec{k}}}\, E_{\vec{k}}\left(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}}\right)$$

particles

4) Impose an ordering

$$\mathcal{L} \to \mathcal{L} + \underbrace{(\varphi\dot{\varphi} - \dot{\varphi}\varphi)}$$

$= 0$ classically

$\neq 0$ quantum

particles

particles

Vacuum State

$|0\rangle$

vacuum of KG
or ground state

$$\boxed{\begin{array}{l} a_{\vec{k}}|0\rangle = 0 \quad \text{for all } \vec{k} \\ \langle 0|0\rangle = 1 \end{array}}$$

$$H|0\rangle = E_0|0\rangle$$

$$H|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} \left(a^\dagger_{\vec{k}} \cancel{a_{\vec{k}}} + a_{\vec{k}} a^\dagger_{\vec{k}} \right)|0\rangle$$

$$= \frac{1}{2}\int \frac{d^3k}{\cancel{(2\pi)^3 2E_{\vec{k}}}} E_{\vec{k}} \cancel{(2\pi)^3 2E_{\vec{k}}} \delta(\vec{k}-\vec{k})|0\rangle$$

$$= \frac{1}{2}\int d^3k \, E_{\vec{k}} \, \delta(\vec{0})|0\rangle$$

$= \infty|0\rangle$ in non-gravitational, absolute

Infra-red (IR) (low-energy) divergence

$$(2\pi)^3 \delta(\vec{0}) = \lim_{L\to\infty} \int_{-L}^{L} d^3x \, e^{i\vec{x}\cdot\vec{p}} \Big|_{\vec{p}=0}$$

$$= \text{vol}$$

$$\rho_0 = \frac{E_0}{\text{vol}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} E_{\vec{k}}$$

energy density

spherical coordinates

$\sim |\vec{k}|$ for $|\vec{k}| \gg m$

$$\underset{|k|\gg m}{\sim} \frac{4\pi}{2} \int \frac{|k|^3 \, d|k|}{(2\pi)^3}$$

UV ultraviolet or high-energy divergence

Elimate by normal-ordering operators

$$:H: = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}$$

"by hand" move $a^\dagger_{\vec{k}}$ to left
$a_{\vec{k}}$ to right

$$:H:|0\rangle = 0|0\rangle$$

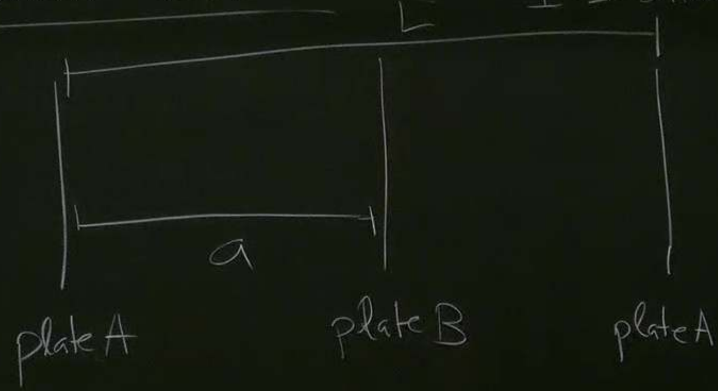
$$:a_{\vec{k}} a^\dagger_{\vec{p}}: = a^\dagger_{\vec{k}} a_{\vec{p}}$$

$$\left(:H: = H - \langle 0|H|0\rangle \right)$$

Casimir Effect

$1+1$ dimensional periodic spacetime w/ period L

eliminate IR diverge



free real scalar field φ with $m=0 \to E_k = k = \omega$

plates impose $\varphi = 0$

$$E_0(a) = E(a) + E(L-a)$$

goal: compute force between plates

$$F = -\frac{d}{da} E_0(a)$$

$= \infty|0\rangle$ in non-gravitational, absolute energy scales irrelevant

$\rightarrow E_{\vec{k}} = k = \omega$

$a)$

$$E(r) = \frac{1}{2}\sum_n \omega_n \qquad \omega_n = \frac{n\pi}{r}$$

$$F = -\frac{d}{da}\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{n\pi}{a} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{n\pi}{L-a}\right)$$

$$= -\frac{d}{da}\left(\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{L-a}\right)\underbrace{\sum_{n=1}^{\infty} n}_{\infty}\right)$$

Mistake!

high frequency modes are unaffected by plates $\omega > \pi\Lambda$

cutoff

regulator

high frequency modes do not interact

$$E(r) = \frac{1}{2}\sum_{n=1}^{n_{max}} \omega_n + \cancelto{0}{\text{constant}}$$

$$n_{max} = \lfloor \Lambda r \rfloor$$

floor

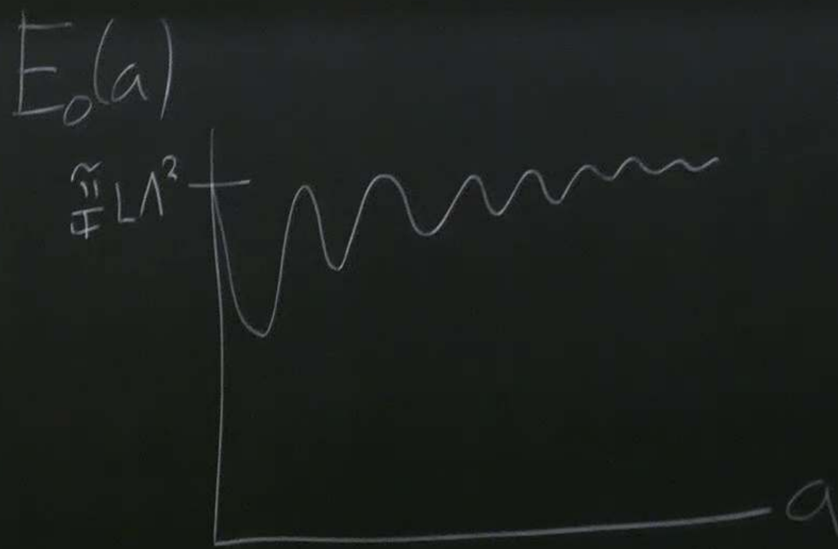
$$= \frac{\pi}{2}\,\frac{n_{max}(n_{max}+1)}{2r}$$

algebra

$$E(r) = \frac{\pi}{4} L \Lambda^2 + \frac{\pi}{4a} x(1-x)$$

fractional part

$$x = \Lambda a - \lfloor \Lambda a \rfloor$$



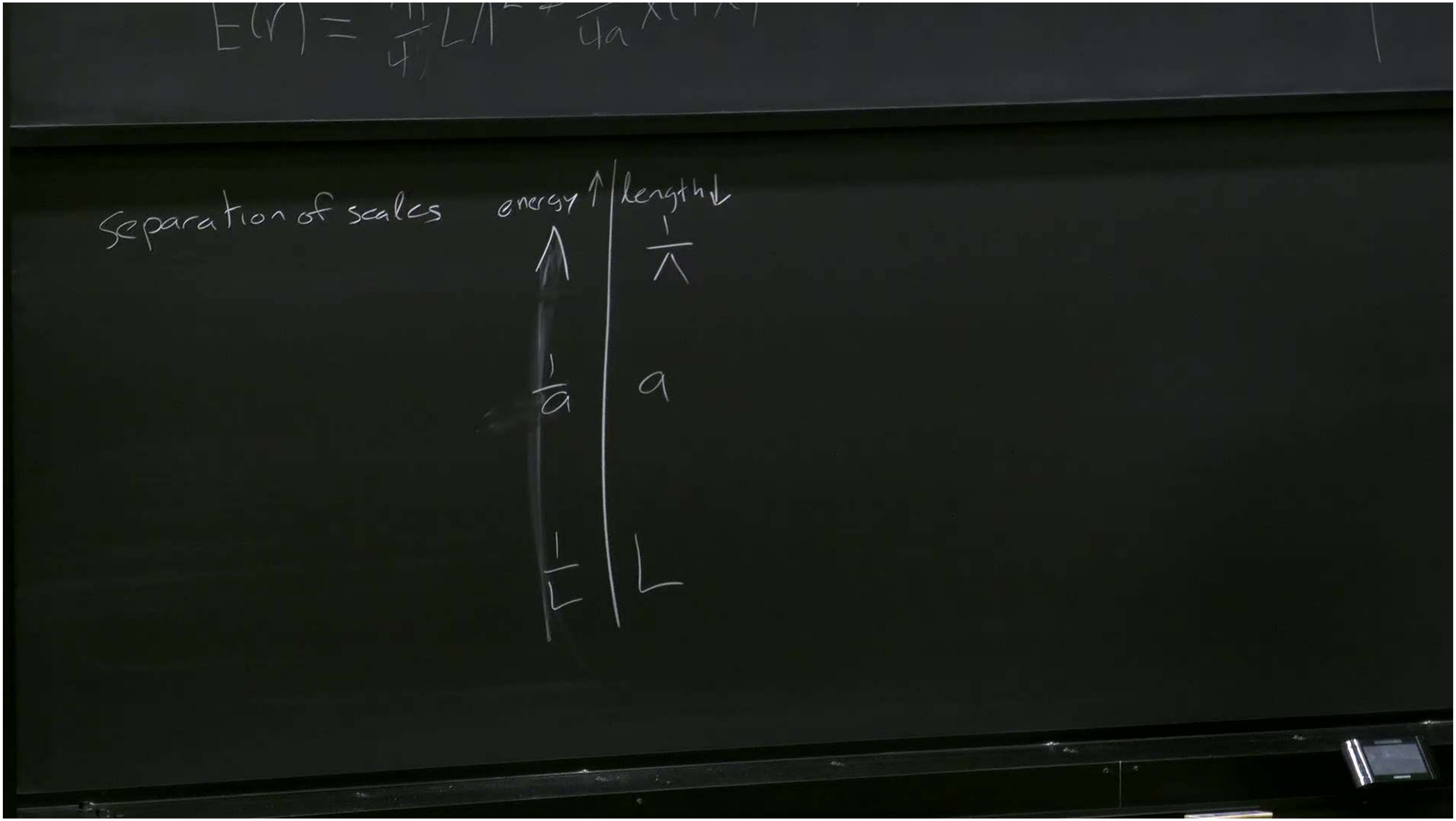
average over oscillations $\int_0^1 dx\, x(1-x) = \frac{1}{6}$

$$E_0(a) = \frac{\tilde{\pi}}{4} L\Lambda^2 - \frac{\tilde{\pi}}{24a}$$

$$F = -\frac{d}{da} E_0(a)$$

$$\boxed{F = -\frac{\tilde{\pi}\hbar}{24a^2}}$$

finite, measurable attractive force


separation of scales
energy ↑
length ↓
1/Λ
1/a
a
1/L
L

Particles

$|\vec{k}\rangle = a_k^\dagger|0\rangle$ is a one-particle state with energy $E_{\vec{k}}$ + momentum $\vec{k}$

$\langle\vec{k}|\vec{p}\rangle = \langle 0|a_{\vec{k}}a_{\vec{p}}^\dagger|0\rangle$
$= (2\pi)^3 2E_{\vec{k}}\,\delta(\vec{p}-\vec{k})$

not unit normalized

2-particle state

$|\vec{k}_1,\vec{k}_2\rangle = a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger|0\rangle$

$[\,,\,]=0 \rightarrow$ bosons