Title: Lecture - Classical Physics, PHYS 612

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Collection/Series: Classical Physics (Core), PHYS 612, September 2 - October 7, 2025

Subject: Mathematical physics, Other

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$$\begin{cases} \nabla_\mu F^{\mu\nu} = -4\pi J^\nu \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$$

$$\left.\begin{matrix} \nabla_{[\mu} F_{\nu\rho]} = 0 \\ F_{\mu\nu} + F_{\nu\mu} = 0 \end{matrix}\right]$$

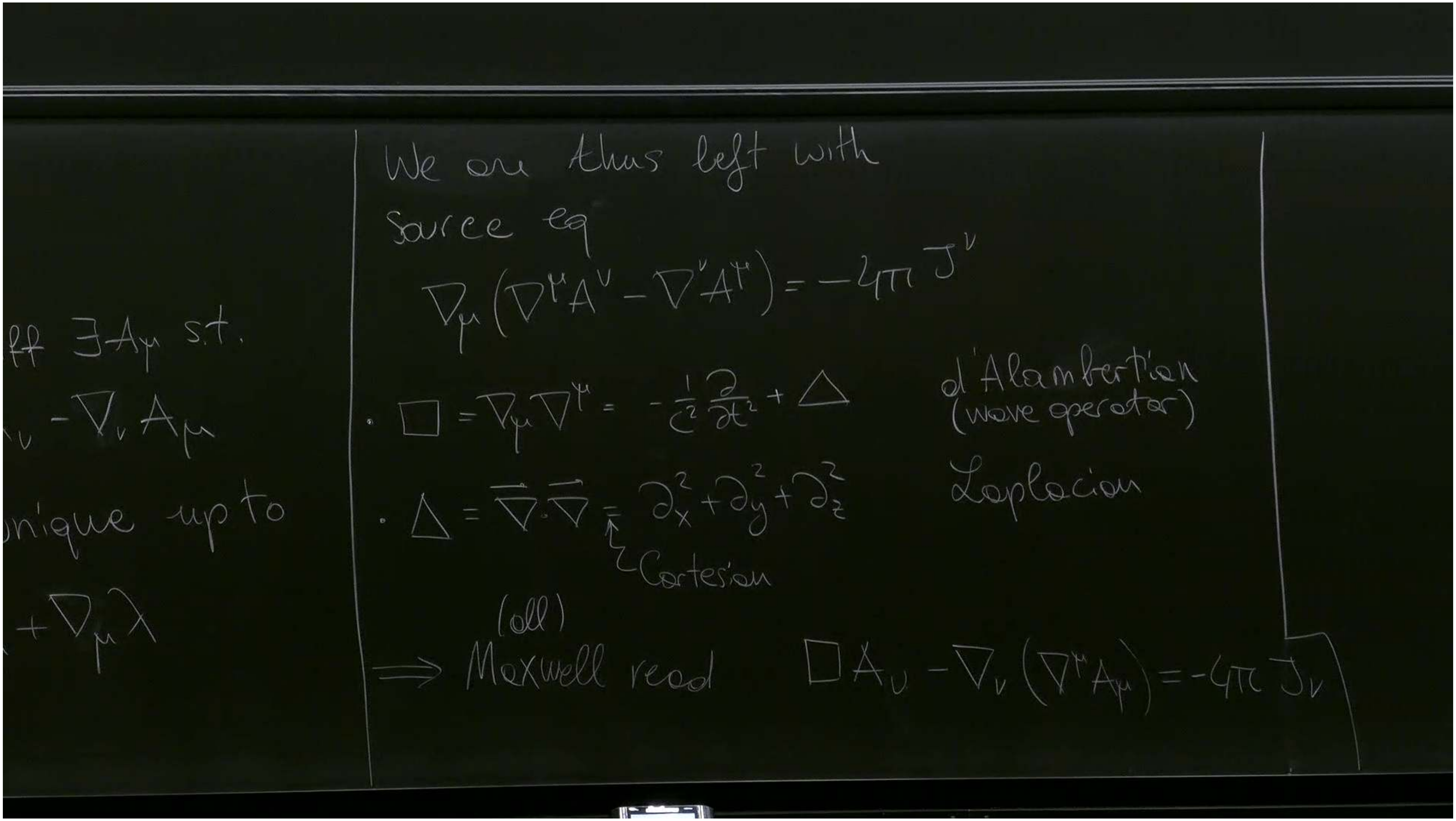
in $\mathbb{R}^4$ satisfied iff $\exists A_\mu$ s.t.

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

with A_μ unique up to

$$A_\mu \longmapsto A_\mu + \nabla_\mu \lambda$$

("gauge freedom")



ff $\exists A_\mu$ s.t.

$_\nu - \nabla_\nu A_\mu$

nique up to

$+\nabla_\mu \lambda$

We are thus left with
source eq

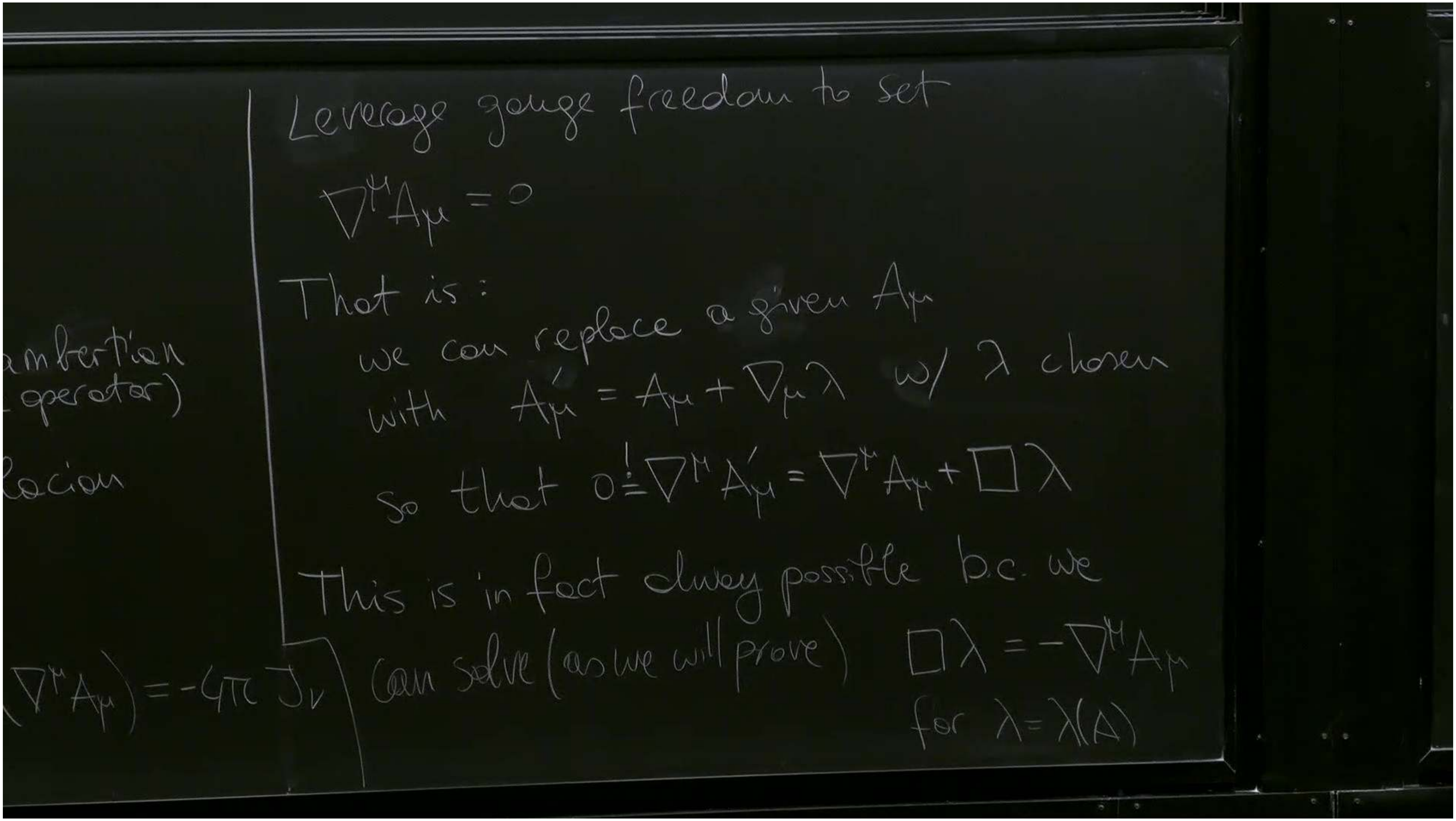
$$\nabla_\mu\left(\nabla^\mu A^\nu - \nabla^\nu A^\mu\right) = -4\pi J^\nu$$

- $\Box = \nabla_\mu \nabla^\mu = -\frac{1}{c^2}\frac{\partial}{\partial t^2} + \triangle$ d'Alambertian (wave operator)
- $\triangle = \vec{\nabla}\cdot\vec{\nabla} \underset{\uparrow}{=} \partial_x^2 + \partial_y^2 + \partial_z^2$ Laplacian

Cartesian

(all)

$\Longrightarrow$ Maxwell read $\Box A_\nu - \nabla_\nu\left(\nabla^\mu A_\mu\right) = -4\pi J_\nu$



ambertian

operator)

lacian

$(\nabla^\mu A_\mu) = -4\pi J_\nu$

Leverage gauge freedom to set

$$\nabla^\mu A_\mu = 0$$

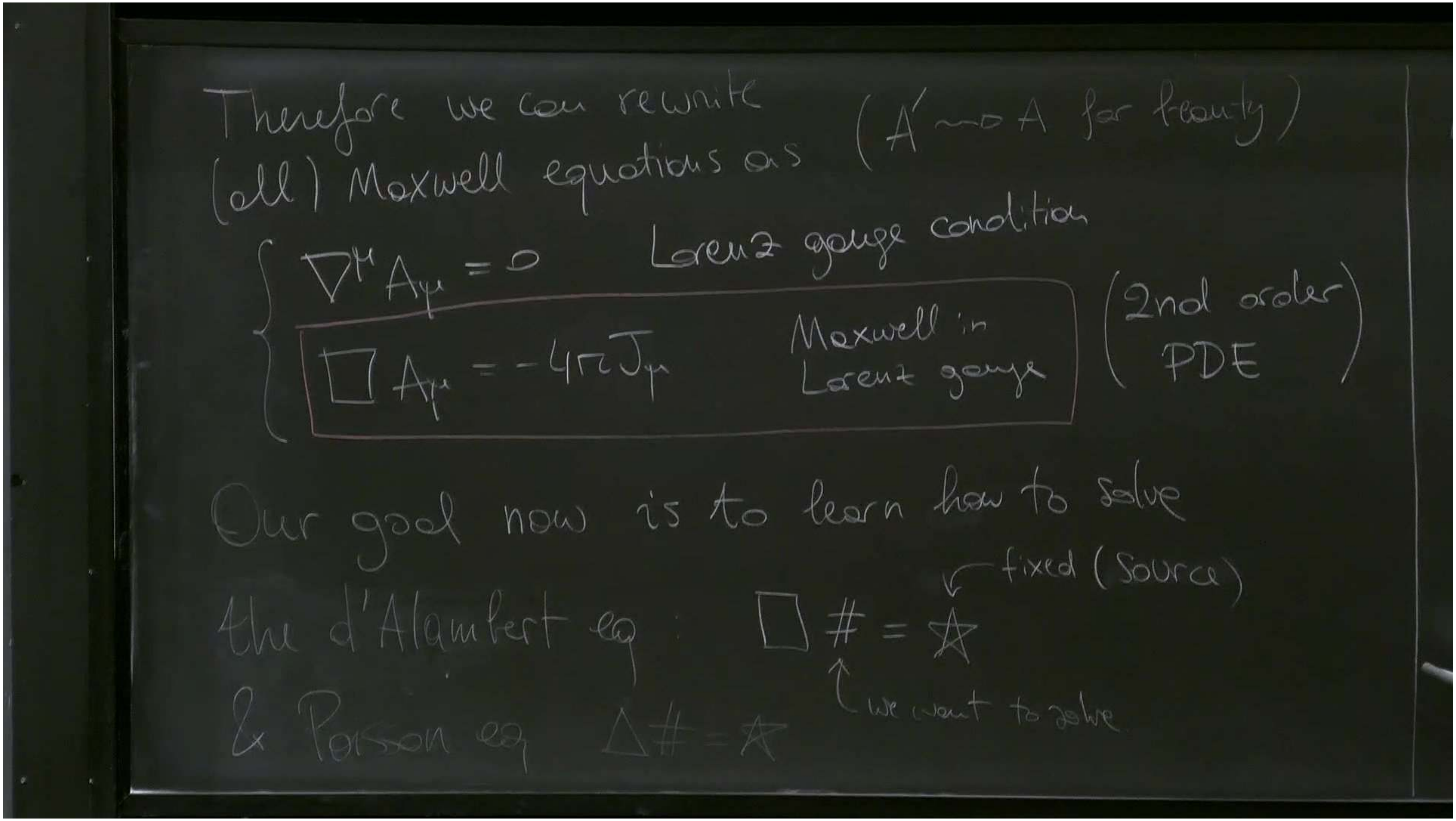
That is:

we can replace a given A_μ

with $A'_\mu = A_\mu + \nabla_\mu \lambda$ w/ λ chosen

so that $0 \overset{!}{=} \nabla^\mu A'_\mu = \nabla^\mu A_\mu + \Box \lambda$

This is in fact always possible b.c. we can solve (as we will prove) $\Box \lambda = -\nabla^\mu A_\mu$ for $\lambda = \lambda(A)$



Therefore we can rewrite (all) Maxwell equations as ($A' \to A$ for beauty)

$$\begin{cases} \nabla^{\mu} A_{\mu} = 0 & \text{Lorenz gauge condition} \\ \boxed{\Box A_{\mu} = -4\pi J_{\mu} \quad \text{Maxwell in Lorenz gauge}} \end{cases} \quad \left(\begin{matrix}\text{2nd order} \\ \text{PDE}\end{matrix}\right)$$

Our goal now is to learn how to solve

the d'Alambert eq: $\Box \# = \star$

fixed (source)

we want to solve

& Poisson eq $\Delta \# = \star$

as ($A' \rightsquigarrow A$ for beauty)

a gauge condition

$$\boxed{\text{Maxwell in Lorentz gauge}} \quad \left(\begin{matrix}\text{four}\\ \text{2nd order}\\ \text{PDEs}\end{matrix}\right)$$

o learn how to solve

fixed (Source)

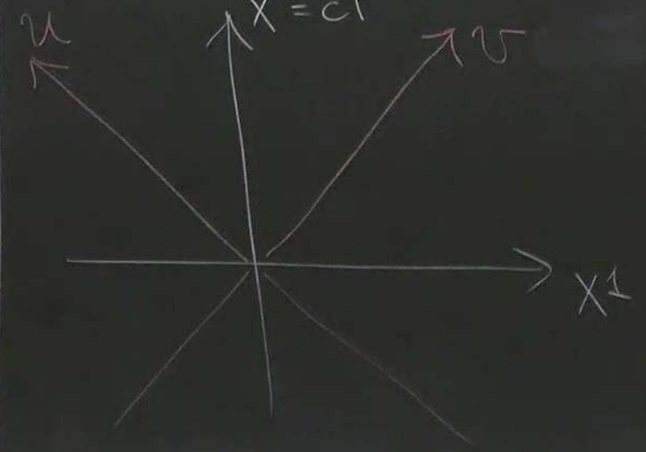
$$\Box \, \# = \star$$

we want to solve

Let's start with problem in 1+1 dim with $J_\mu = 0$

$$\Box \phi = 0$$

$$(-\partial_0^2 + \partial_1^2)\phi = 0$$



$$\begin{cases} v = x^0 + x^1 \\ u = x^0 - x^1 \end{cases}$$

light cone coords

$^{0} + x^{1}$

$x^{0} - x^{1}$

t cone

rds

In these coords $\Box = -\partial_u \partial_v$

$$0 = \Box\phi = -\partial_u \partial_v \phi \quad \text{iff} \quad \boxed{\partial_u \phi = 0} \text{ or } \boxed{\partial_v \phi = 0}$$

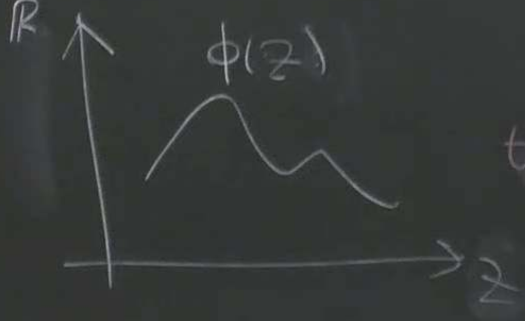
$$\phi = \phi(u) = \phi(ct - x)$$

$$\phi = \phi(v) = \phi(ct + x)$$

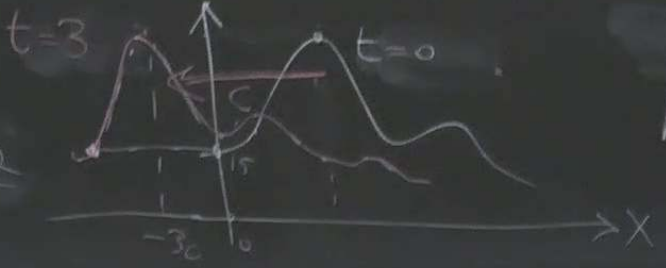
Any such f° are sol^{n}!

$$15 = \phi(0) = \phi(t=0, x=0) = \phi(t=3, x=-3c)$$

Let



$\phi(\underline{v}) = \phi(ct + x)$ is



LEFT

MOVER

More generally (arbitrary d)

$\Box\phi = 0$, assume ϕ can be Fourier transformed

$$\Box\phi(x^\mu) = \int_{\mathbb{R}^4} d^4k \, \Box e^{i k_\mu \underline{\underline{x}}^\mu} \, \hat{\phi}(k^\mu) \, , \qquad k_\mu := (\omega, \vec{k})$$

$-\partial_t^2 + \Delta$

$$= \int_{\mathbb{R}^3} d^3\vec{k} \int_{\mathbb{R}} d\omega \, \Box e^{-i\omega \underline{\underline{t}} + i\vec{k}\underline{\underline{\vec{x}}}} \, \hat{\phi}(\omega, \vec{k})$$

$$= \int d^3\vec{k} \int d\omega \, \left(-\omega^2 + \vec{k}^2\right) e^{-i\omega t + i\vec{k}\vec{x}} \, \hat{\phi}(\omega, \vec{k}) \overset{!}{=} 0$$

$\rightarrow \hat{\phi}(\omega$

suppor

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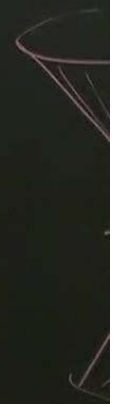
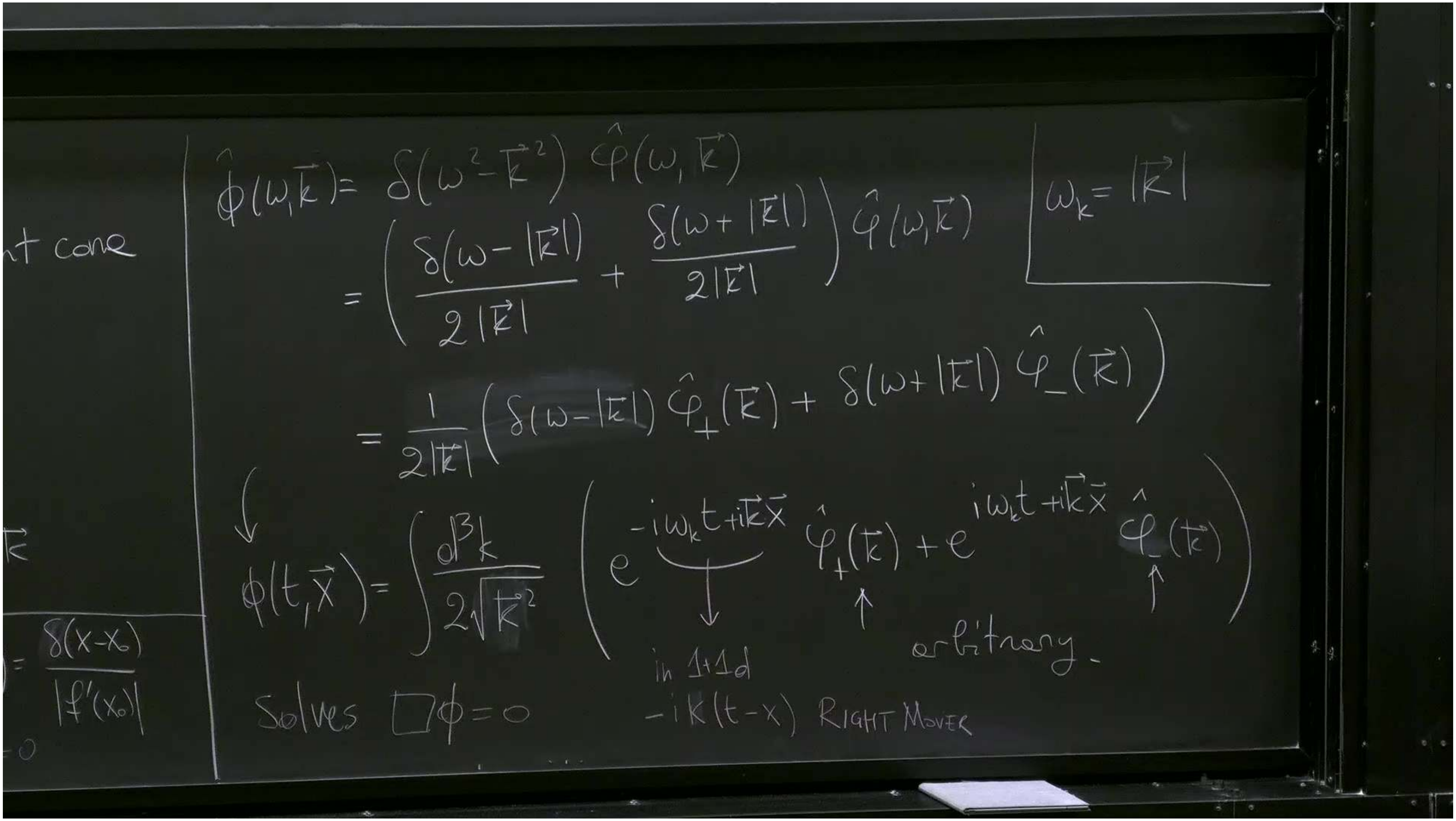




nt cone

$\vec k$

$= \frac{\delta(x-x_0)}{|f'(x_0)|}$

$=0$

$$\hat\phi(\omega,\vec k) = \delta(\omega^2-\vec k^2)\,\hat\varphi(\omega,\vec k)$$

$$= \left(\frac{\delta(\omega-|\vec k|)}{2|\vec k|} + \frac{\delta(\omega+|\vec k|)}{2|\vec k|}\right)\hat\varphi(\omega,\vec k)$$

$$= \frac{1}{2|\vec k|}\left(\delta(\omega-|\vec k|)\,\hat\varphi_+(\vec k) + \delta(\omega+|\vec k|)\,\hat\varphi_-(\vec k)\right)$$

$$\omega_k = |\vec k|$$

$$\phi(t,\vec x) = \int \frac{d^3k}{2\sqrt{\vec k^2}}\left(e^{\underbrace{-i\omega_k t + i\vec k\vec x}_{\text{in } 1+1d:\ -ik(t-x)\ \text{RIGHT MOVER}}}\underset{\uparrow}{\hat\varphi_+(\vec k)} + e^{i\omega_k t + i\vec k\vec x}\underset{\uparrow}{\hat\varphi_-(\vec k)}\right)$$

arbitrary.

Solves $\Box\phi = 0$

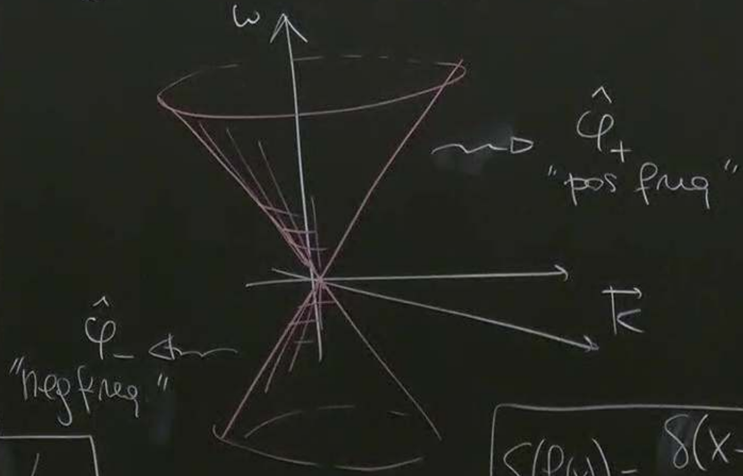
ousformed

$$k_\mu := (\omega, \vec{k})$$

$$\vec{x} \;\overset{\sim}{=}\; \hat{\phi}(\omega, \vec{k})$$

$$\vec{k}^2 \Big)\, e^{-i\omega t + i\vec{k}\vec{x}}\, \hat{\phi}(\omega, \vec{k}) \overset{!}{=} 0$$

→ $\hat{\phi}(\omega, \vec{k})$ must be supported on the (momentum space) light cone



$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}$$

$$f(x_0) = 0$$

$$\hat{\phi}(\omega, \vec{k}) = \delta(\omega^2 - \vec{k}$$

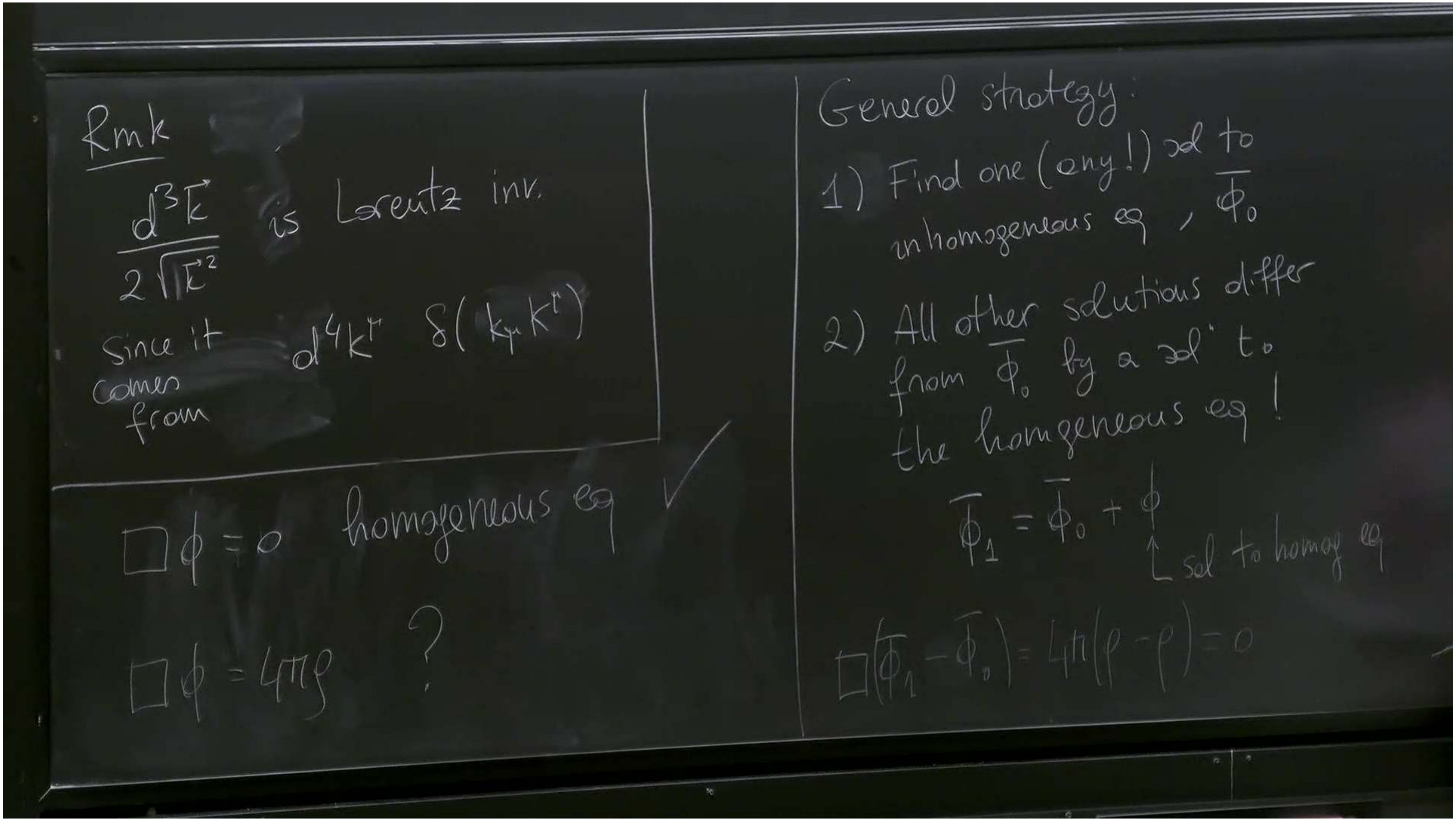
$$= \Big(\frac{\delta(\omega -}{2|\vec{k}}$$

$$= \frac{1}{2|\vec{k}|} \Big(\delta$$

↓

$$\phi(t, \vec{x}) = \int \frac{d^3k}{2\sqrt{\vec{k}}}$$

Solves $\Box \phi = 0$



Rmk

$\frac{d^3\vec{k}}{2\sqrt{|\vec{k}|^2}}$ is Lorentz inv.

Since it comes from $d^4k^\mu \; \delta(k_\mu k^\mu)$

$\Box\phi = 0$ homogeneous eq ✓

$\Box\phi = 4\pi\rho$?

General strategy:

1) Find one (any!) sol to inhomogeneous eq, $\bar{\Phi}_0$

2) All other solutions differ from $\bar{\Phi}_0$ by a sol^n to the homogeneous eq!

$\bar{\Phi}_1 = \bar{\Phi}_0 + \underset{\uparrow \text{ sol to homog eq}}{\phi}$

$\Box(\bar{\Phi}_1 - \bar{\Phi}_0) = 4\pi(\rho - \rho) = 0$

3) Fix initial or boundary conds by adding to ϕ_0 the appropriate homog sol ϕ.

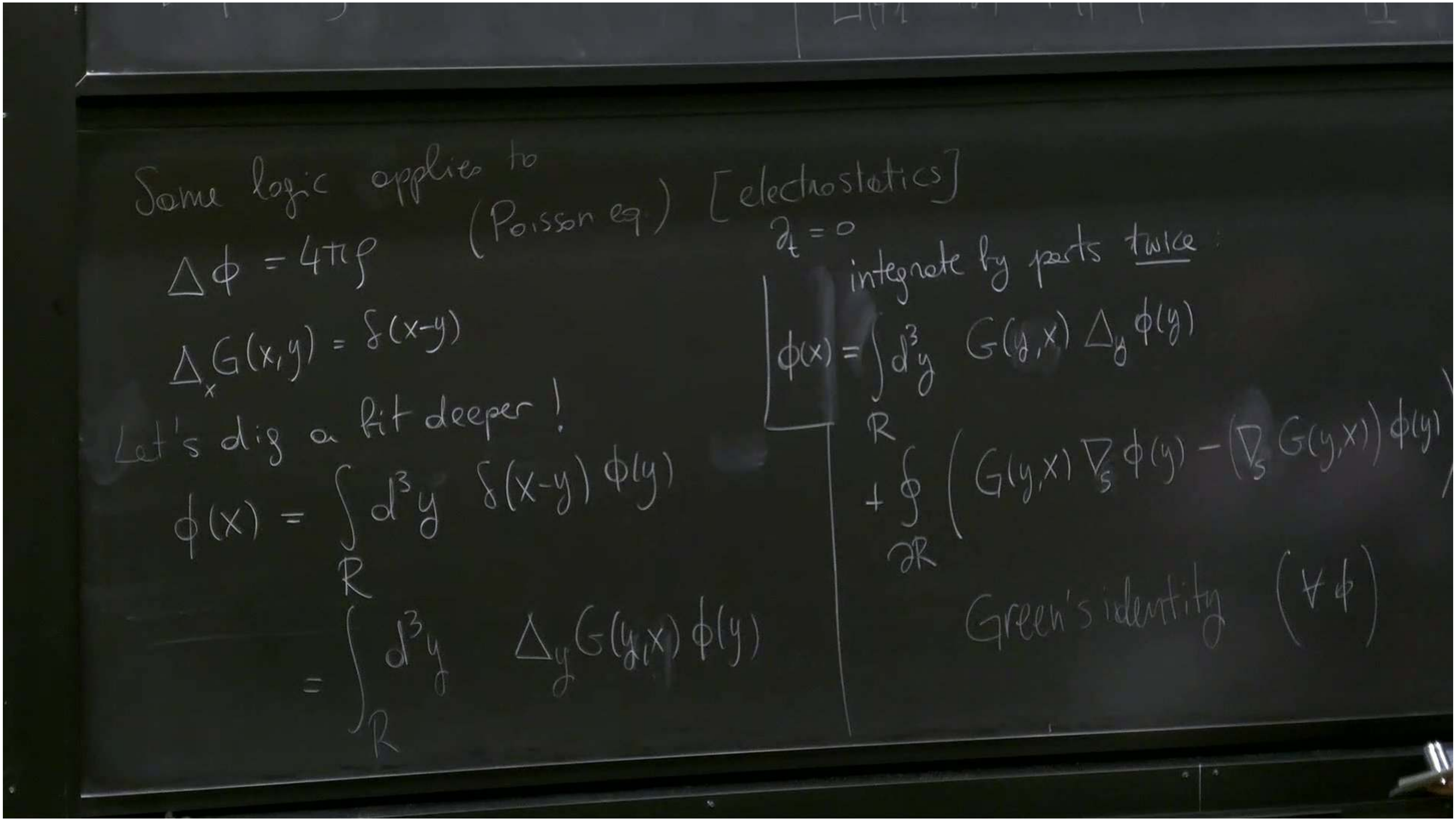
I can reduce step 1 to the following: [y "fixed"]

$$\boxed{\Box_x G(x,y) = \delta(x-y)}$$

Green's function - $\phi(x) = 4\pi \int d^4y \, G(x,y)\rho(y)$ is a sol^n of $\Box\phi = 4\pi\rho$

Since

Pf $\Box\phi(x) = 4\pi \int d^4y \left[\Box_x G(x,y)\right]\rho(y) = 4\pi \int d^4y \, \delta(x-y)\rho(y) = 4\pi\rho(x)$ ▨



Same logic applies to (Poisson eq.) [electrostatics] $\partial_t = 0$

$$\Delta\phi = 4\pi\rho$$

$$\Delta_x G(x,y) = \delta(x-y)$$

Let's dig a bit deeper!

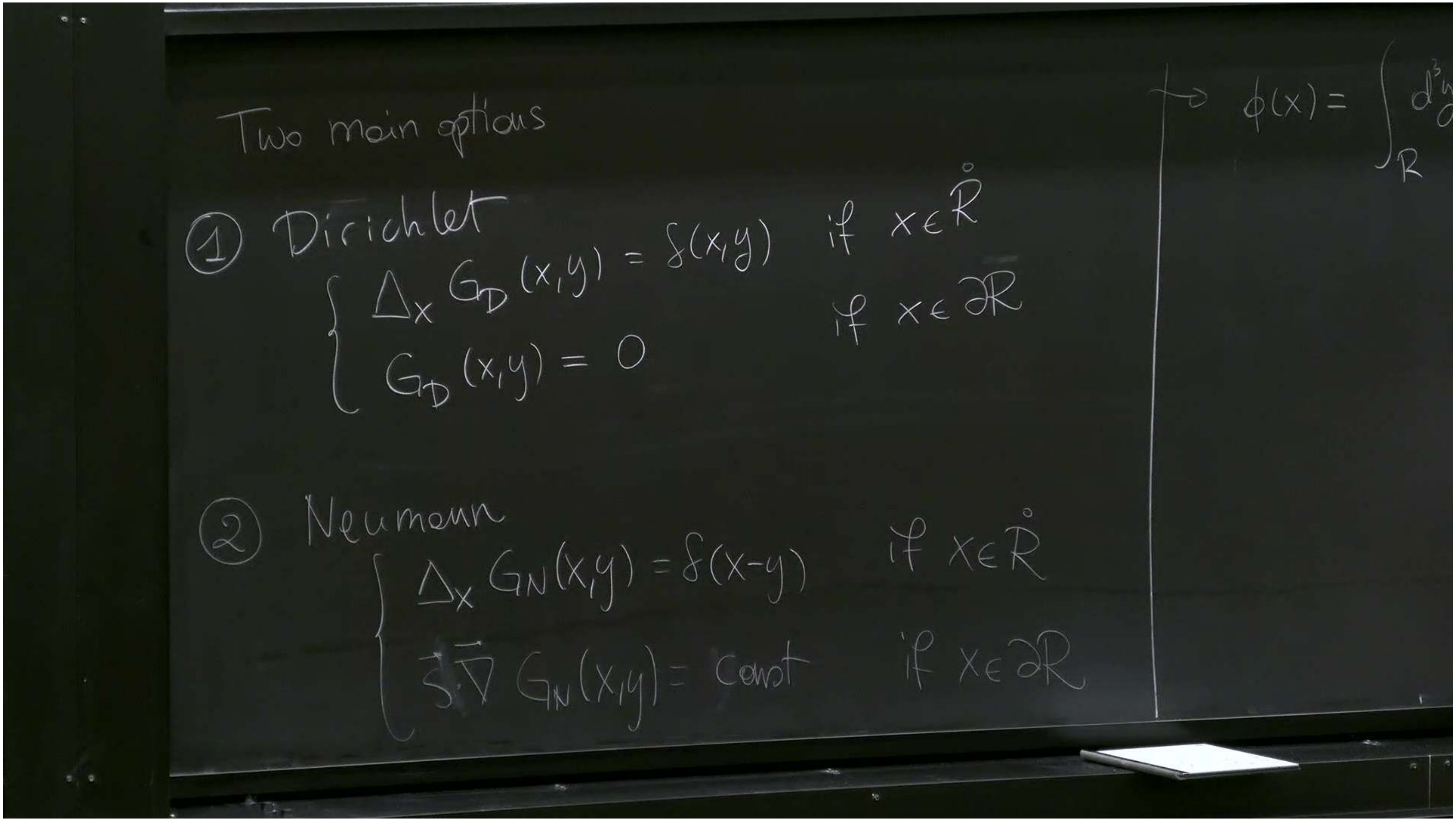
$$\phi(x) = \int_R d^3y \; \delta(x-y)\,\phi(y)$$

$$= \int_R d^3y \; \Delta_y G(y,x)\,\phi(y)$$

integrate by parts twice:

$$\boxed{\phi(x)} = \int_R d^3y \; G(y,x)\,\Delta_y \phi(y) + \oint_{\partial R} \left(G(y,x)\,\nabla_s \phi(y) - (\nabla_s G(y,x))\,\phi(y) \right)$$

Green's identity $(\forall \phi)$



Two main options

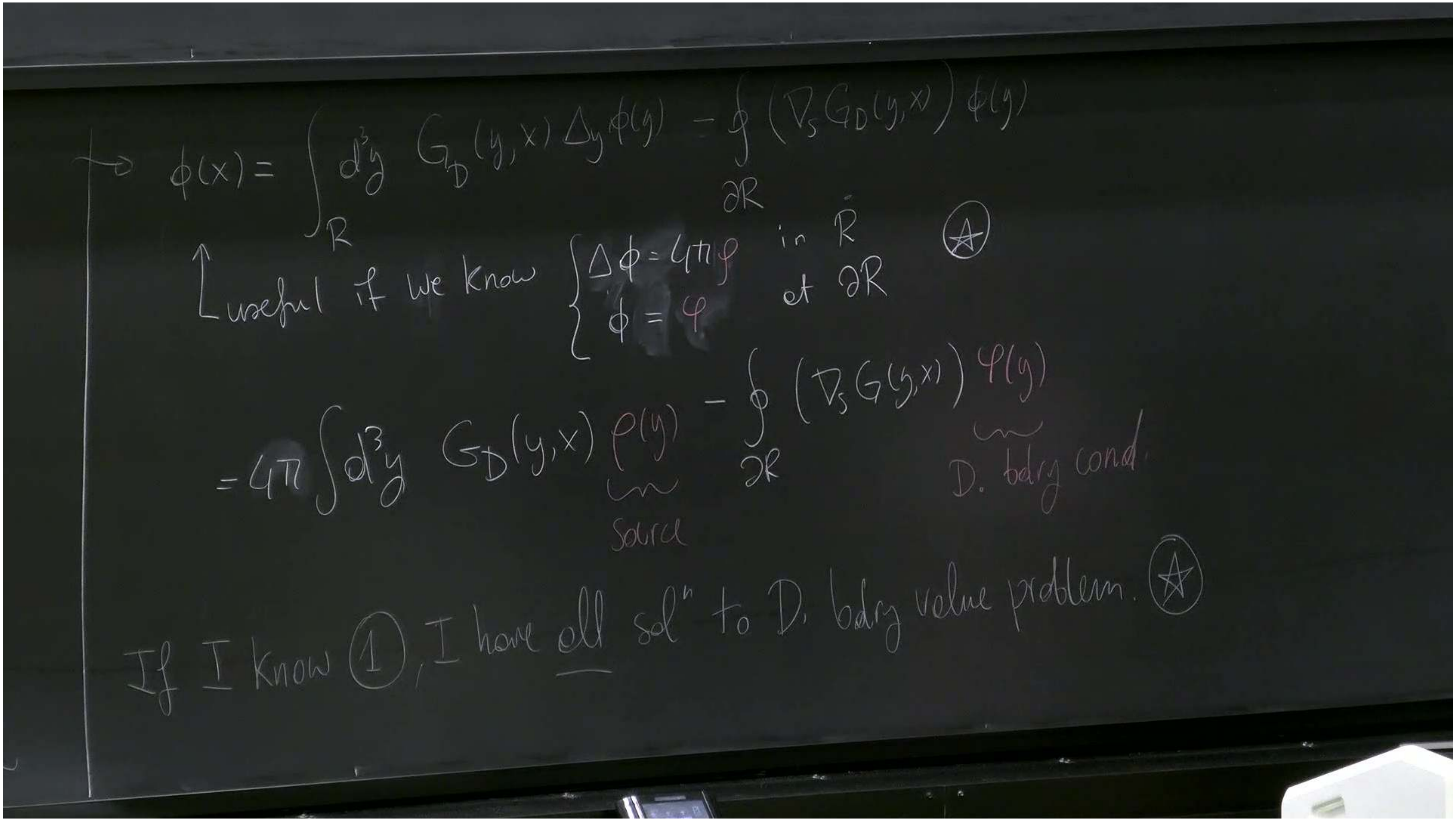
① Dirichlet

$$\begin{cases} \Delta_x G_D(x,y) = \delta(x,y) & \text{if } x \in \mathring{R} \\ G_D(x,y) = 0 & \text{if } x \in \partial R \end{cases}$$

② Neumann

$$\begin{cases} \Delta_x G_N(x,y) = \delta(x-y) & \text{if } x \in \mathring{R} \\ \vec{S} \cdot \vec{\nabla} G_N(x,y) = \text{Const} & \text{if } x \in \partial R \end{cases}$$

$$\to \phi(x) = \int_R d^3$$

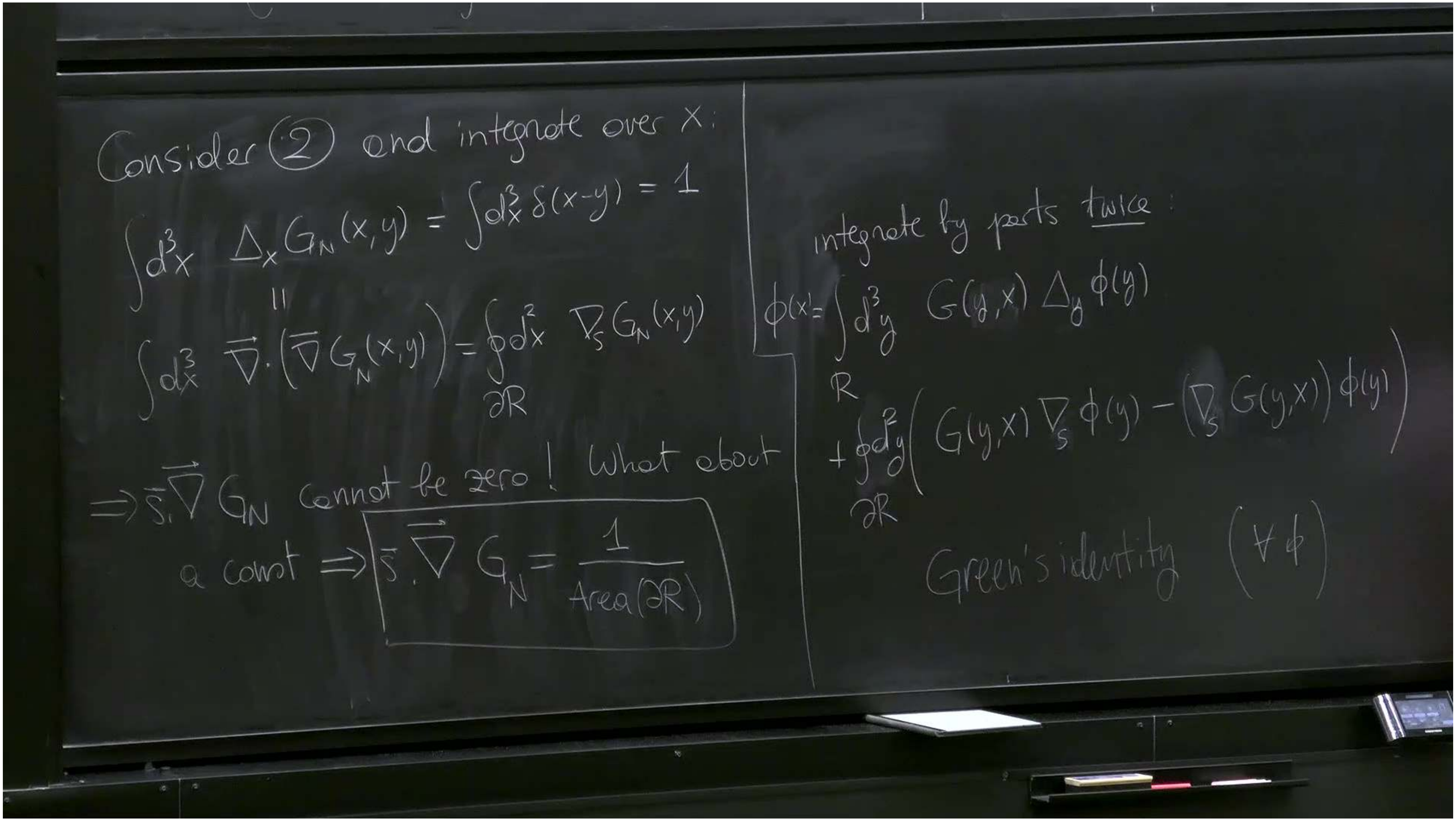


$$\to \phi(x) = \int_R d^3y \; G_D(y,x)\,\Delta_y \phi(y) - \oint_{\partial R} \left(\nabla_s G_D(y,x)\right) \phi(y)$$

$\uparrow$ useful if we know $\begin{cases} \Delta\phi = 4\pi\rho & \text{in } \mathring{R} \\ \phi = \varphi & \text{at } \partial R \end{cases}$ $\circledast$

$$= 4\pi \int d^3y \; G_D(y,x) \underbrace{\rho(y)}_{\text{source}} - \oint_{\partial R} \left(\nabla_s G(y,x)\right) \underbrace{\varphi(y)}_{\text{D. bdry cond.}}$$

If I know ①, I have all sol^n to D. bdry value problem. $\circledast$



Consider ② and integrate over x:

$$\int d^3_x \; \Delta_x G_N(x,y) = \int d^3_x \, \delta(x-y) = 1$$

$$\parallel$$

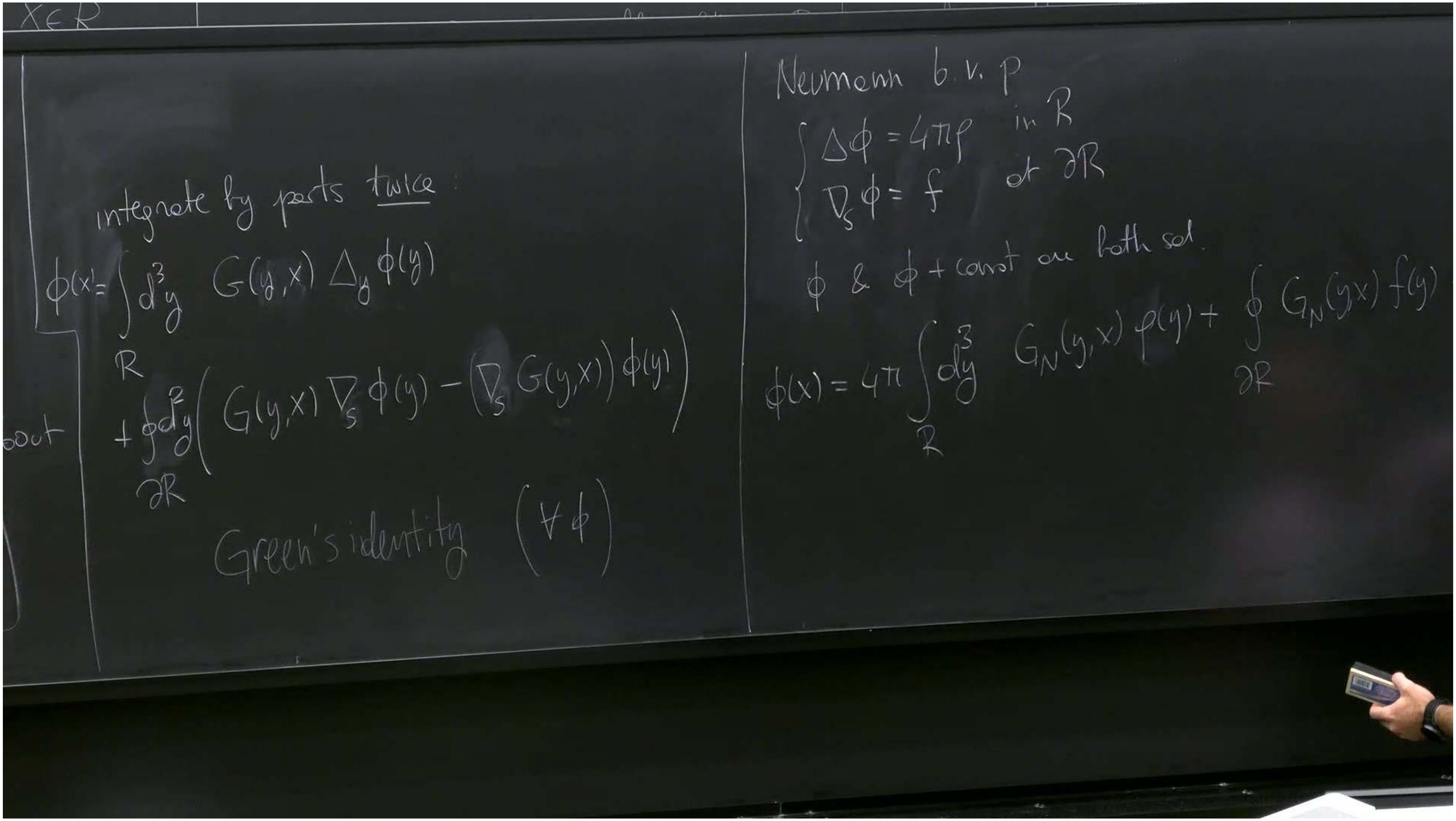
$$\int d^3_x \; \vec{\nabla}\cdot\left(\vec{\nabla} G_N(x,y)\right) = \oint_{\partial R} d^2_x \; \nabla_s G_N(x,y)$$

$\Rightarrow \bar{s}\cdot\vec{\nabla} G_N$ cannot be zero! What about a const $\Rightarrow \boxed{\bar{s}\cdot\vec{\nabla} G_N = \frac{1}{\text{Area}(\partial R)}}$

integrate by parts <u>twice</u>:

$$\phi(x) = \int_R d^3_y \; G(y,x)\,\Delta_y \phi(y) + \oint_{\partial R} d^2_y \left(G(y,x)\,\nabla_s \phi(y) - \left(\nabla_s G(y,x)\right)\phi(y) \right)$$

Green's identity $(\forall \phi)$



$x \in R$

integrate by parts <u>twice</u>:

$$\phi(x) = \int_R d^3y \; G(y,x)\, \Delta_y \phi(y) + \oint_{\partial R} d^2y \left(G(y,x)\, \nabla_s \phi(y) - \left(\nabla_s G(y,x)\right) \phi(y) \right)$$

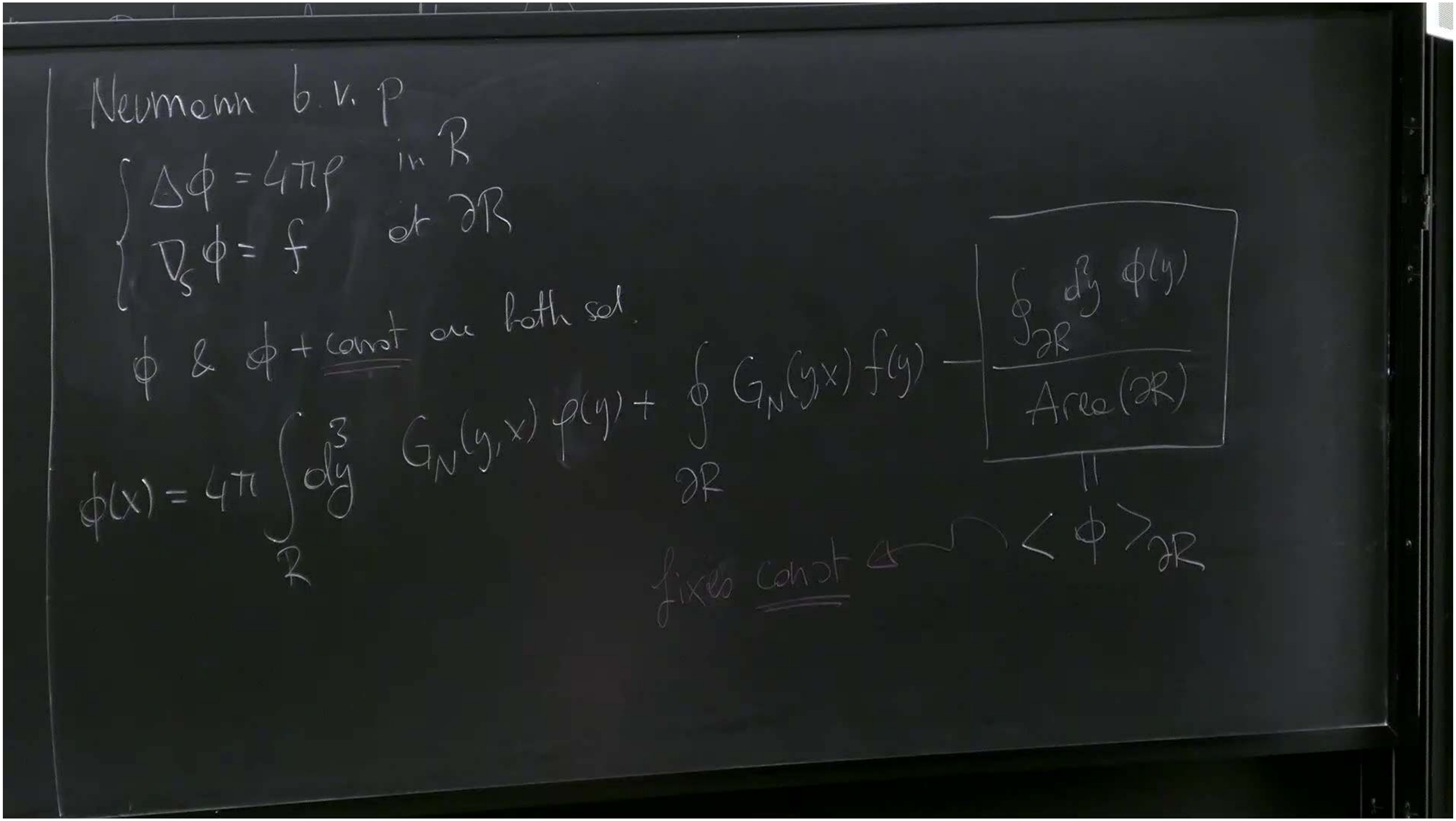
Green's identity $(\forall \phi)$

Neumann b. v. p

$$\begin{cases} \Delta\phi = 4\pi\rho & \text{in } R \\ \nabla_s \phi = f & \text{at } \partial R \end{cases}$$

ϕ & $\phi + \text{const}$ on both sol.

$$\phi(x) = 4\pi \int_R d^3y \; G_N(y,x)\, \rho(y) + \oint_{\partial R} G_N(y,x)\, f(y)$$



Neumann b.v.p

$$\begin{cases}\Delta\phi = 4\pi\rho & \text{in } R\\ D_s\phi = f & \text{at } \partial R\end{cases}$$

ϕ & $\phi + \underline{\text{const}}$ are both sol.

$$\phi(x) = 4\pi\int_R d^3y\; G_N(y,x)\,\rho(y) + \oint_{\partial R} G_N(y,x)\,f(y) - \boxed{\frac{\oint_{\partial R} d^2y\;\phi(y)}{\text{Area}(\partial R)}}$$

$$\|$$

fixes $\underline{\text{const}}$ $\leftarrow$ $\langle\phi\rangle_{\partial R}$